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# ON THE HURWITZ PROBLEM OVER AN ARBITRARY FIELD II

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## 1. Introduction

This paper contains generalizations of some results presented in part I. Here, for an arbitrary field $F$ of characteristic not two, the non-existence of normed maps $F^{4h+1} \times F^3 \to F^{4h+2}$ and $F^{4h+2} \times F^3 \to F^{4h+3}$ is proved. This extends to any $h$ the particular results given in [1] only for $h = 1$. Observe that the similar generalization of (7.1) of [1] is not achieved and it is left as an open problem (see the remarks in the last section).

As shown before, the existence of a normed map $F^p \times F^q \to F^n$ is equivalent to having a set of $q$ rectangular matrices of order $p \times n$ fulfilling certain conditions.

A first matrix of this set is characterized as the join of an identity matrix and a zero matrix. Then considering $F \subset K$, where $K$ is an algebraically closed field, a second matrix is reduced to a special canonical form over $K$, for some values of $(p, n)$. This is given in lemma (2.5) and the proof is a hard piece of work. It contains, however, significant simplifications regarding similar arguments already given in [1].

The proofs are completed in sections 3 and 4, where some very special arguments are used to establish the two above mentioned cases. In section 5 these results are used to determine the minimal $n$ needed for the existence of a normed map for $1 \leq q \leq 3$ and any $p$.

All references here are made to [1] and to simplify the writing, in order to indicate a reference we will only use a square bracket without the 1. For instance, we will write [(6.1)] for [1; (6.1)].

## 2. Orthogonal equivalence of certain matrices

Let $M$ be a rectangular $p \times n$ matrix over a field $F$ of characteristic different from two. Assume that

$$(2.1) \qquad MM^t = I_p,$$

where $M^t$ denotes the transpose of $M$ and $I_p$ is the identity matrix of order $p$. Also, assume that $M$ decomposes into

$$(2.2) \qquad M = [A, B],$$

where $A^t = -A$, is an alternate matrix of order $p$, and $B$ is a rectangular $p \times (n - p)$ matrix.

Let $C$ be the alternate matrix of order two given by

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and define the following square matrices

$$(2.3)\qquad U = \mathrm{diag}[\underbrace{C, \cdots, C}_{k \text{ times}}] \quad \text{and} \quad U_1 = \mathrm{diag}[0,\ U].$$

Then, $U$ and $U_1$ are alternate matrices or order $2k$ and $2k + 1$, respectively.

Now, consider

$$(2.4)\qquad D = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} 1 \\ D \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 1 & 0 \\ D & D \end{bmatrix}$$

where $D$ is a $2k$-column of zeros and $D$, $D_1$ and $D_2$ as shown are regarded, respectively, as $2k \times 1$, $(2k + 1) \times 1$ and $(2k + 1) \times 2$ matrices.

Let $K$ be a field containing the originally given field $F$, that is $K \supset F$. Recall that a $p \times n$ matrix $M$ over $F$ is said to be orthogonally equivalent over $K$ to a $p \times n$ matrix $N$ if there exist $P$ and $Q$, two orthogonal matrices, respectively, of orders $p$ and $n$, such that $PMQ = N$, where $P$, $Q$ and $N$ are considered as matrices over $K$. If the fields $K \supset F$ are clearly understood, then we will briefly write $M \sim N$ to indicate that $M$ over $F$ is orthogonally equivalent to $N$ over $K$.

LEMMA (2.5). *Let $M = [A, B]$ be a $p \times n$ matrix over $F$ fulfilling the conditions* (2.1), (2.2), *and suppose $K \supset F$ where $K$ is an algebraically closed field. The following results hold*:

$$(2.6)\qquad \text{if } (p, n) = (2k, 2k + 1) \text{ then } M \sim [U, D],$$

$$(2.7)\qquad \text{if } (p, n) = (2k + 1, 2k + 2) \text{ then } M \sim [U_1, D_1],$$

$$(2.8)\qquad \text{if } (p, n) = (2k + 1, 2k + 3) \text{ then } M \sim [U_1, D_2].$$

*where $U$, $U_1$, $D$, $D_1$ and $D_2$ are as in* (2.3) *and* (2.4). *Here the equivalence $PMQ = N$ can be chosen so that $P[I_p, 0]Q = [I_p, 0]$. Hence, as in* [(5.1)] *it preserves the $p \times n$ matrix $[I_p, 0]$.*

*Proof.* Clearly, $M$ can be replaced by any matrix orthogonally equivalent (over $K$) to it. Since an equivalence of $A$ can be extended to $M$ [see (5.1)], it follows that $A$ can be considered to have any of the possible canonical forms given in [(4.9)] for an alternate matrix. Hence,

$$(2.9)\qquad A = \mathrm{diag}[W^{(d_1)}, \cdots, W^{(d_s)}; W_{a_1}{}^{(q_1, q_1)}, \cdots, W_{a_r}{}^{(q_r, q_r)}]$$

where $(\lambda - a_j)^{q_j}$, $(\lambda + a_j)^{q_j}$ and $\lambda^{d_k}$, with $j = 1, \cdots, r$; $k = 1, \cdots, s$ and each $d_k$ an odd number, form a list of all the elementary divisors of $A$.

Since $A$ is an alternate matrix its rank must be even. Also, from [(3.9)] it follows that rank $A \geq 2p - n$. Therefore,

$$(2.10)\qquad \text{rank} \quad A = 2k,$$

if $(p, n)$ is as in (2.6), (2.7) or (2.8).

In order to organize the proof the following auxiliary propositions will first be established.

(2.11) *If* $(p, n)$ *is as in* (2.6) *then all the characteristic values of* $A$ *are nonzero. Hence,* $A$ *has no components of the form* $W^{(d)}$ *and all the other components* $W_{a_j}^{(q_j, q_j)}$ *in* (2.9) *are with* $a_j \neq 0$.

(2.12) *If* $(p, n)$ *is as in* (2.7) *or* (2.8) *then* $A$ *has one and only one characteristic value equal to zero. Therefore,* $A$ *has one and only one term* $W^{(1)} = [0]$ *in that part of* (2.9) *where the components* $W^{(d)}$ *appear and all the other components* $W_{a_j}^{(q_j, q_j)}$ *are with* $a_j \neq 0$.

The proof of (2.11) is almost immediate. In fact, if $(p, n)$ is as in (2.6) then (2.10) implies that $\det A \neq 0$. Hence, zero is not a characteristic value of $A$. Therefore, $A$ has in (2.9) neither $W^{(d)}$ nor $W_0^{(q,q)}$ as a component. This proves (2.11).

The proof of (2.12) is much more complex. First consider the case where $(p, n)$ is as in (2.8). Since $A$ is of odd order and each $W_a^{(q,q)}$ is of even order, it follows that $A$ must contain at least one component $W^{(d)}$ of order $d = 2e + 1$.

A permutation of the components of $A$ is given by an orthogonal similarity and it induces an orthogonal equivalence on $M = [A, B]$ [see (section 5)]. Hence, we can suppose from the beginning that

$$A = \operatorname{diag}[W^{(2e+1)}, N], \tag{2.13}$$

where $N$ denotes a suitable matrix.

Assume $2e + 1 > 1$ and let

$$w^{(2e+1)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_{2e+1} \end{bmatrix}, \qquad \text{where} \quad w_i \in K^{2e+1}.$$

From the explicit expression for $W^{(2e+1)}$ given in [(4.8)], the following relations are easily verified

$$\begin{cases} w_1 w_1^t = w_1 w_{2e+1}^t = w_{2e+1} w_{2e+1}^t = 0 \quad \text{if} \quad e > 1, \\ w_1 w_2^t = w_2 w_{2e+1}^t = 0 \quad \text{for all} \quad e, \\ w_2 w_2^t = 0 \quad \text{if} \quad e = 1. \end{cases} \tag{2.14}$$

Under the map $w_i \to (w_i, 0, \cdots, 0)$, the rows $w_i$ of $W^{(2e+1)}$ can be regarded, using the same notation, as rows of $A$ and the above relations continue to hold in $A$.

Suppose $(p, n)$ as in (2.8) and let

$$A = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_p \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & c_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ b_p & c_p \end{bmatrix}, \tag{2.15}$$

where $w_i \in K^p$ and $b_i, c_i \in K$. As in [(6.11)], the condition (2.1) becomes

$$w_i w_j^t + b_i b_j + c_i c_j = \delta_{ij}, \tag{2.16}$$

where $\delta_{ij}$ is the Kronecker delta.

There are three cases to be considered with regard to the component $W^{(d)}$ of $A$. They are, $d > 3$, $d = 3$ and $d = 1$.

Let $d = 2e + 1$ with $e > 1$. Since $w_1 w_1^t = 0$, from [(7.4)] it follows that we can assume to have $b_1 = 1$ and $c_1 = 0$. Then substitution of these values in (2.16) gives

$$w_1 w_j^t + b_j = 0 \quad \text{for} \quad j \neq 1.$$

This equality and the relations $w_1 w_2^t = w_1 w_{2e+1}^t = 0$ of (2.14), imply that

$$b_2 = b_{2e+1} = 0. \tag{2.17}$$

Also, using the relations $w_2 w_{2e+1}^t = w_{2e+1} w_{2e+1}^t = 0$ of (2.14) in (2.16), we get

$$\begin{cases} b_2 b_{2e+1} + c_2 c_{2e+1} = 0, \\ \\ b_{2e+1}^2 + c_{2e+1}^2 = 1. \end{cases} \tag{2.18}$$

Now, a substitution of (2.17) in (2.18) gives $c_2 c_{2e+1} = 0$ and $c_{2e+1}^2 = 1$. Hence, $c_2 = 0$ and, since $b_2 = 0$, again from (2.16) it follows that $w_2 w_2^t = 1$. But a direct computation in $W^{(2e+1)}$, using the expression [(4.8)], gives

$$w_2 w_2^t = \begin{cases} \dfrac{i}{2} & \text{if} \quad e = 2, \\ \\ 0 & \text{if} \quad e > 2 \end{cases}$$

and this is a contradiction. Therefore, the component $W^{(d)}$ cannot be in $A$ for $d > 3$.

Let $d = 3$ and $e = 1$. As shown in (2.14), in this case $w_2 w_2^t = 0$. Then, from [(7.4)], as before it can be assumed that $b_2 = 1$ and $c_2 = 0$.

With these values and the relations $w_1 w_2^t = w_2 w_3^t = 0$ of (2.14) in (2.16), we obtain $b_1 = b_3 = 0$. Therefore, from (2.16), we get

$$w_1 w_1^t + c_1^2 = 1 \quad \text{and} \quad w_3 w_3^t + c_3^2 = 1 \tag{2.19}$$

Using the explicit form of $W^{(3)}$, given in [(4.8)], and the known values of $b_i$ and $c_j$, the matrix $M$ becomes

$$M = [W^{(3)}, B] = \begin{bmatrix} 0 & (1+i)/2 & 0 \\ -(1+i)/2 & 0 & (-1+i)/2 \\ 0 & (1-i)/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 \\ 1 & 0 \\ 0 & c_3 \end{bmatrix} \tag{2.20}$$

The rows of $M$ must satisfy the conditions (2.16). These conditions express that each row has norm 1 and that different rows are orthogonal. Applying them to the first and third rows, they give $c_1^2 = 1 - \frac{i}{2}$ and $c_3^2 = 1 + \frac{i}{2}$. Then the product of $c_1^2$ and $c_3^2$, from these experessions becomes $c_1^2 c_3^2 = \frac{5}{4}$. On the other hand, the first and third rows being orthogonal, imply that $c_1c_3 = -\frac{1}{2}$. Hence, $c_1^2c_3^2 = \frac{1}{4}$. This is a contradiction and it rules out the case $d = 3$. Consequently, $d = 1$ and $W^{(1)} = [0]$ is a component of $A$. Then, the expression (2.13) becomes

$$A = \operatorname{diag}[0, N],$$

and since the order of $N$ is $2k$ and rank $A = 2k$, it follows that $N$ does not have zero as a characteristic value. This proves (2.12) for $(p, n)$ as in (2.8).

The proof of (2.12) for $(p, n)$ as in (2.7) is obtained as an especial case of the above proof, by taking out the elements $c_1, \cdots, c_p$. We will point out the main steps of the arguments.

From the beginning all the steps hold up to the expressions (2.15) and (2.16) where the $c_i$'s need to be deleted. Then the proof continues until (2.18) where, after taking out $c_{2e+1}$, it follows that $b_{2e+1}^2 = 1$, in direct contradiction with (2.17) where it is stated that $b_{2e+1} = 0$. Therefore, $W^{(2e+1)}$ with $e > 1$ cannot be in (2.13).

Now, let $e = 1$, hence $d = 3$. Again all the steps of the above argument remain valid after removing the $c_i$'s. Then, from (2.19), it follows that $w_1w_1^t = 1$ and the first row of (2.20) gives $w_1w_1^t = \dfrac{i}{2}$ and this is a contradiction. Hence, the component $W^{(3)}$ cannot be in $A$.

The rest of the argument goes as before and this ends the proof of (2.12) for $(p, n)$ as in (2.7). Hence, (2.12) is established.

Now we go back to the proof of lemma (2.5) and its three propositions (2.6), (2.7) and (2.8). First consider the case (2.8) and let $A$ and $B$ be as in (2.15). From (2.12) it follows that $A$ can be arranged so that $w_1 = 0$. The argument already used [see (7.4)] allows us to assume that $b_1 = 1$ and $c_1 = 0$ and from (2.16) it follows that $b_1b_j = b_j = 0$, for $j = 2, \cdots, p$. Let us reindex the rows of $A$ and $B$, to have

$$(2.21) \qquad A = \begin{bmatrix} 0 \\ w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_{p-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & c_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & c_{p-1} \end{bmatrix}.$$

To study the form of the components of $A$, suppose that $(\lambda - a)^q$, $(\lambda + a)^q$ with $a \neq 0$ and $q \geq 1$, are elementary divisors of $A$. Then the matrix $W_a^{(q,q)}$ of order $2q$, constructed in [(4.7)], is a component of $A$. Therefore, we can suppose that

$$A = \operatorname{diag}[0, W_a^{(q\cdot q)}, S],$$

where $S$ is a suitable matrix. Also, as before the rows $w_1, \cdots, w_{2q}$ of $A$ can be identified with the corresponding rows of $W_a^{(q,q)}$, where

$$W_a^{(q,q)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_{2q} \end{bmatrix}$$

Using the expression [(4.7)] of $W_a^{(q,q)}$ it is easy to verify the following relations

$$\begin{aligned} w_j w_j^t &= -a^2, \quad \text{if} \quad j = 1, \cdots, 2q, \\ w_1 w_2^t &= -2a, \quad \text{if} \quad q > 1, \\ w_1 w_{2q}^t &= 0. \end{aligned} \tag{2.22}$$

Clearly, the corresponding rows of $A$ also fulfill the conditions (2.22).

From (2.16) and (2.21) it follows that $w_j w_j^t + c_j^2 = 1$ for $j = 1, \cdots, p-1$, and using (2.22), we get

$$c_j^2 = 1 + a^2 \quad \text{for} \quad j = 1, \cdots, 2q. \tag{2.23}$$

Then, $c_1^2 = \cdots = c_{2q}^2$. From (2.16) it follows that $w_1 w_{2q}^t + c_1 c_{2q} = 0$, and from (2.22) we get that $w_1 w_{2q}^t = 0$. Hence, $c_1 c_{2q} = 0$ and this together with $c_1^2 = \cdots = c_{2q}^2$ implies that $c_1 = \cdots = c_{2q} = 0$. Therefore, $a^2 + 1 = 0$ and $ia = \pm 1$.

Now, if $q > 1$, from (2.16) it follows that $w_1 w_2^t = 0$ and (2.22) gives $w_1 w_2^t = -2a \neq 0$. This contradiction rules out the case $q > 1$.

Then, $A$ has the component

$$W_a^{(1,1)} = \begin{bmatrix} 0 & ia \\ -ia & 0 \end{bmatrix}$$

with $ia = \pm 1$, and the argument used in [(6.12)] shows that the sign can be chosen so that $ia = 1$. Thus, $W_a^{(1,1)} = C$, and by induction it follows that

$$A = \operatorname{diag}[0, C, \cdots, C].$$

Consequently, $M \sim [U_1, D_2]$, and this ends the proof of (2.8).

To proceed with the proof of lemma (2.5) consider now the case (2.7). As in the proof of (2.12), here, the proof of (2.7) is also obtained from the proof of (2.8), by taking out the elements $c_1, \cdots, c_p$. As before, we will point out the main steps of the arguments.

Set out with the expressions obtained from (2.15) and (2.16) when the $c_i$'s are deleted. That is

$$A = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_p \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_p \end{bmatrix} \tag{2.24}$$

and

(2.25) $$w_i w_j^t + b_i b_j = \delta_{ij}.$$

Then, the arguments of the proof follow without any problem until (2.21), where $A$ is as stated there, after changing the indices of its rows, and $B = D_1$. The proof continues until (2.23) where this equality becomes $0 = 1 + a^2$. From there, following the last two paragraphs of the proof of (2.8), in a straightforward form, the proof of (2.7) is completed.

Finally, let us consider the case (2.6) of lemma (2.5). Let $M = [A, B]$ where $A$ is an alternate matrix of order $p = 2k$ and $B$ is a rectangular $2k \times 1$ matrix. Using the notation (2.24), it follows that the rows of $A$ and the elements of $B$ satisfy the relations (2.25).

From (2.11), we can assume that

$$A = \operatorname{diag}[W_a^{(q,q)}, N],$$

where $a \neq 0$, and, as before, identifying the rows of $W_a^{(q,q)}$ with the corresponding rows of $A$, it follows that the first $2q$ rows of $A$ fulfill the conditions (2.22). Then, from $w_j w_j^t + b_j^2 = 1$ and $w_j w_j^t = a^2$, we get that $b_j^2 = 1 + a^2$. Hence, $b_1^2 = \cdots = b_{2q}^2$. On the other hand, $w_1 w_{2q}^t = 0$ implies that $b_1 b_{2q} = 0$. Therefore, $b_1 = \cdots = b_{2q} = 0$, $w_j w_j^t = -a^2 = 1$ for $j = 1, \cdots, 2q$ and $ia = \pm 1$.

Now, if $q > 1$, then, from (2.22), $w_1 w_2^t = -2a \neq 0$ and, from (2.25), $w_1 w_2^t = 0$. Thus, $q = 1$.

As before, the sign can be chosen so that $ia = 1$, and by induction it follows that $A = U$ and $B = D$. This ends the proof of lemma (2.5)

*Remark.* The proposition (2.8) will not be used here since it is related to the case mentioned in the introduction to be left unsettled.

## 3. Nonexistence of normed maps for $p = 4h + 1$, $q = 3$ and $n = 4h + 2$

The following generalization of [(6.1)] will be established.

THEOREM (3.1). *For any field $F$ of characteristic different from two, no normed map $F^{4h+1} \times F^3 \to F^{4h+2}$ can exist.*

*Proof.* Suppose that such map exists over some field $F \subset K$, where $K$ is an algebraically closed field. Let $M_1$, $M_2$ and $M_3$ be the $(4h + 1) \times (4h + 2)$ matrices of [(2.5)] associated with the map. Then, from [(5.1)] and (2.7), it follows that $M_1 = [I_{4h+1}, D]$, $M_2 = [U_1, D_1]$ and $M_3 = [A_3, B_3]$, can be considered as matrices over $K$, where $U_1$, $D$ and $D_1$ are as in (2.3) and (2.4), with $k = 2h$ and $p = 4h + 1$. With the usual notation, set

$$A_3 = (a_{ij}) = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_p \end{bmatrix}, B_3 = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_p \end{bmatrix} \quad \text{and} \quad U_1 = (b_{ij}),$$

where $w_i \in K^p$, $b_j \in K$ and $1 \leq i, j \leq p$. Recall that the matrices $A_3$ and $U_1$ are alternate of order $p$. Easily, it follows from (2.3) that the elements $b_{ij}$ of $U_1$ are given by

$$(3.2)\qquad \begin{cases} b_{2i,\,2i+1} = -b_{2i+1,\,2i} = 1 \quad \text{if} \quad 1 \leq i \leq 2h, \\ b_{i,r} = 0 \quad \text{otherwise.} \end{cases}$$

The matrix $M_3$ must satisfy the condition

$$M_3M_2^t + M_2M_3^t = 0,$$

and this is equivalent with [see (3.8)]

$$A_3U_1 + U_1A_3 = B_3D_1^t + D_1B_3^t.$$

Now, a direct computation gives

$$(3.3)\qquad B_3D_1^t + D_1B_3^t = \begin{matrix} 2b_1 & b_2 & \cdots & b_p \\ b_2 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_p & 0 & \cdots & 0 \end{matrix}$$

On the other hand, if $C = (c_{ij}) = A_3U_1 + U_1A_3$, then $C$ is a symmetric matrix and each

$$c_{ij} = \textstyle\sum_{r=1}^{p} (a_{ir}b_{rj} + b_{ir}a_{rj}).$$

Then, replacing the values of (3.2) in these expressions for $c_{ij}$, many terms become zero and it follows that

$$(3.4)\qquad \begin{cases} c_{1,1} = 0,\ c_{1,2j} = -a_{1,2j+1},\ c_{1,2j+1} = a_{1,2j}, \\ c_{2i,2j} = c_{2i+1,2j+1} = a_{2i+1,2j} - a_{2i,2j+1}, \\ c_{2i,2j+1} = c_{2j+1,2i} = a_{2i+1,2j+1} + a_{2i,2j}, \end{cases}$$

where only $1 \leq j$ is significant.

Now, from the equality of the matrix $C$ with the explicit expression given by (3.3), it follows that

$$(3.5)\qquad b_1 = 0, \qquad b_{2j} = -a_{1,2j+1}, \qquad b_{2j+1} = a_{1,2j}.$$

Also we get that

$$(3.6)\qquad c_{ij} = 0 \quad \text{for} \quad i > 1 \quad \text{and} \quad j > 1.$$

Thus, (3.5) implies that

$$(3.7)\qquad B_3 = \begin{bmatrix} 0 \\ -a_{13} \\ a_{12} \\ -a_{15} \\ \cdot \\ \cdot \\ \cdot \\ a_{1,4h} \end{bmatrix},$$

and from (3.6) and (3.4), it follows that

$$\text{(3.8)} \qquad \begin{cases} a_{2i+1,2j} = a_{2i,2j+1}, \\ a_{2i+1,2j+1} = -a_{2i,2j}. \end{cases}$$

In particular, if $i = j$, from (3.8) and the alternate property of $A_3$, it follows that

$$\text{(3.9)} \qquad a_{2i,2i+1} = a_{2i+1,2i} = a_{jj} = 0.$$

Write

$$A_3 = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_p \end{bmatrix} = \begin{bmatrix} 0 & v_1 \\ a_{21} & v_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{p1} & v_p \end{bmatrix}$$

Where $v_j = (a_{j2}, a_{j3}, \cdots, a_{jp})$. The relations (3.9) imply that the following two consecutive components of $v_j$ are zero:

$$a_{2i,2i} = a_{2i,2i+1} \quad = 0 \quad \text{if } j = 2i,$$

$$a_{2i+1,2i} = a_{2i+1,2i+1} = 0 \quad \text{if } j = 2i+1.$$

Now, using the identities (3.8), the rows $v_j$ of odd index $j = 2i+1$ can be written as follows,

$$v_{2i+1} = (a_{2i,3}, -a_{2i,2}, \cdots, -a_{2i,2i-1}, 0, 0, a_{2i,2i+1}, \cdots, -a_{2i,p-1}),$$

and then it is easy to verify that

$$v_{2i+1} = -v_{2i}U,$$

where $U$ is the matrix of order $p-1$ defined in (2.3). Also, it follows from (3.7) that

$$\text{(3.10)} \qquad B_3 = \begin{bmatrix} 0 \\ (v_1U)^t \end{bmatrix}.$$

Then, since $a_{i1} = -a_{1i}$, the matrix $M_3$ can be written in the following form

$$\text{(3.11)} \qquad M_3 = [A_3, B_3] = \begin{bmatrix} 0 & v_1 & 0 \\ \begin{bmatrix} \\ -v_1{}^t \\ \\ \end{bmatrix} & \begin{bmatrix} v_2 \\ -v_2U \\ v_4 \\ -v_4U \\ \vdots \\ -v_{p-1U} \end{bmatrix} & \begin{bmatrix} \\ (v_1U)^t \\ \\ \end{bmatrix} \end{bmatrix}$$

Now, it will be established that the existence of a normal map $F^{4h+1} \times F^3 \to F^{4h+2}$ implies the existence of a Hurwitz-Radon map $K^{4h+2} \times K^3 \to K^{4h+2}$. This last statement is equivalent to the existence of two alternate square matrices $N_2$ and $N_3$ over $K$, each of order $4h+2$, such that [(see p. 34)]

$$N_2N_2{}^t = N_3N_3{}^t = I_{4h+2} \quad \text{and} \quad N_2N_3{}^t + N_3N_2{}^t = 0.$$

Set $N_2$ and $N_3$ in terms of $M_2$ and $M_3$, as follows

$$N_2 = \begin{bmatrix} M_2 \\ [-D_1{}^t, 0] \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} M_3 \\ [-B_3{}^t, 0] \end{bmatrix}.$$

Clearly, these matrices are alternate. To verify the above conditions proceed directly as follows. First, that

$$N_2N_2{}^t = \begin{bmatrix} M_2 \\ [-D_1{}^t, 0] \end{bmatrix} \begin{bmatrix} M_2{}^t \begin{bmatrix} -D_1 \\ 0 \end{bmatrix} \end{bmatrix} = I_{4h+2},$$

it is obtained from the equalities

$$M_2M_2{}^t = I_{4h+1},\ M_2 \begin{bmatrix} -D_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad [-D_1{}^t, 0]\begin{bmatrix} -D_1 \\ 0 \end{bmatrix} = D_1{}^tD_1 = 1.$$

Recall that the conditions $M_3M_3{}^t = I_{4h+1}$ means that any two rows of $M_3$ are orthogonal and that the norm of any row of $M_3$ is equal to 1. The matrix $N_3$ is constructed by adding to $M_3$ the expression $[-B_3{}^t, 0] = [0, -v_1U, 0]$ as its $(4h + 2)$-row. Consequently, to prove that $N_3N_3{}^t = I_{4h+2}$ it is enough to show that the row $[0, -v_1U, 0]$ has norm 1 and that it is orthogonal to all other rows of $M_3$. These conditions are established using the form for $M_3$ given in (3.11) and the fact that $[0, v_1, 0]$ is its first row. Thus, because $U$ is orthogonal and $v_1$ has norm 1, it follows that $[0, -v_1U, 0]$ has also norm 1. Because $U$ is alternate, it follows that

$$(v_1U)v_1{}^t = v_1(v_1U)^t = (v_1U^t)v_1 = -(v_1U)v_1,$$

hence, $2(v_1U)v_1{}^t = 0$. Therefore, $[0, v_1, 0]$ and $[0, -v_1U, 0]$ are orthogonal.

To prove that the new row $[0, -v_1U, 0]$ is orthogonal to all the other rows, use the relations $v_{2i}v_1{}^t = 0$ and $(v_{2i}U)v_1{}^t = 0$ of $M_3$, as follows:

$$v_{2i}(v_1U)^t = (v_{2i}U^t)v_1{}^t = -(v_{2i}U)v_1{}^t = 0,$$
$$(v_{2i}U)(v_1U)^t = v_{2i}(UU^t)v_1{}^t = v_{2i}v_1{}^t = 0.$$

Consequently, the condition $N_3N_3{}^t = I_{4h+2}$ holds.

The verification of the condition $N_2N_3{}^t + N_3N_2{}^t = 0$ is automatic and is based on the fact that the same relation holds for $M_2$ and $M_3$. The details are omitted.

## 4. Nonexistence of normed maps for $p = 4h + 2$, $q = 3$ and $n = 4h + 3$

Now the following generalization of [(8.1)] will be proved.

THEOREM (4.1). *For any field $F$ of characteristic different from two, no normed map $F^{4h+2} \times F^3 \to F^{4h+3}$ can exist.*

*Proof.* As before, suppose that the map of (4.1) exists over some field $F \subset K$, where $K$ is algebraically closed. Let $M_1$, $M_2$ and $M_3$ be the $(4h+2)\times(4h+3)$ matrices of [(2.5)] associated with the map. Then, from [(5.1)] and (2.6), it follows that $M_1 = [I_{4h+2}, D]$, $M_2 = [U, D]$ and $M_3 = [A_3, B_3]$, can be regarded as matrices over $K$, where $U$ and $D$ are as in (2.3) and (2.4).

Let $U = (b_{ij})$ and $A_3 = (a_{ij})$ where $1 \le i, j \le p$ and $p = 4h+2$. The elements $b_{ij}$ of $U$ are given by

$$\text{(4.2)} \qquad \begin{cases} b_{2i-1,2i} = -b_{2i,2i-1} = 1 & \text{if} \quad 1 \le i \le 2h+1, \\ b_{ij} = 0 \text{ otherwise.} \end{cases}$$

Since $D$ is the zero $p$-column, condition [(3.8)] becomes

$$A_3U + UA_3 = 0.$$

In the same form that (3.8) was obtained, here, using (4.2) in the above relation, it follows that

$$\text{(4.3)} \qquad \begin{cases} a_{2i-1,2j} = a_{2i,2j-1}, \\ a_{2i-1,2j-1} = -a_{2i,2j}. \end{cases}$$

In particular, if $i = j$, from (4.3) and the fact that $A_3$ is an alternate matrix, it follows that

$$\text{(4.4)} \qquad a_{2i-1,2i} = a_{2i,2i-1} = a_{jj} = 0.$$

Let

$$A_3 = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix},$$

where $w_j = (a_{j1}, \cdots, a_{jp})$ and $b_j \in K$. As before, the relations (4.4) imply that the following two consecutive components of each $w_j$ are zero:

$$\begin{cases} a_{2i,2i-1} = a_{2i,2i} = 0 & \text{if} \quad j = 2i, \\ a_{2i-1,2i-1} = a_{2i-1,2i} & \text{if} \quad j = 2i-1. \end{cases}$$

Then, using the identities (4.3) the $w_{2i}$ can be written as follows,

$$w_{2i} = [a_{2i-1,2}, \cdots, -a_{2i-1,2i-3}, 0, 0, a_{2i-1,2i}, \cdots, -a_{2i-1,p-1}].$$

Then, it is easy to check that

$$\text{(4.5)} \qquad w_{2i} = -w_{2i-1}U.$$

Using this relation (4.5) and the fact that the alternate matrix $U$ is orthogonal, the $p$-column $B_3$ will be shown to be the zero column.

The condition $M_3M_3^t = I_p$ is equivalent in terms of $w_i$ and $b_j$, with the set of conditions given in (2.25). Then, from

$$w_{2i}w_{2i}^t = (w_{2i-1}U)(w_{2i-1}U)^t = w_{2i-1}w_{2i-1}^t,$$

it follows that $b_{2i}^2 = b_{2i-1}^2$. On the other hand, since $U$ is alternate, it quickly follows that

$$w_{2i-1}w_{2i}^t = -w_{2i-1}(w_{2i-1}U) = 0.$$

Hence, $b_{2i-1}b_{2i} = 0$. Therefore $b_1 = \cdots = b_p = 0$, and $B_3$ is the zero $p$-column.
Thus, in the case, we have

$$B_1 = B_2 = B_3 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now, under the assumption that the matrices $M_1$, $M_2$ and $M_3$ over $K$, satisfy conditions [(2.5)], by taking out the part $B_i$ in each $M_i$, it follows that the three square matrices $A_1 = I_{4h+2}$, $A_2 = U$ and $A_3$ also fulfill conditions [(2.5)]. Consequently, the existence of these matrices imply the existence of a normed map $K^{4h+2} \times K^3 \to K^{4h+2}$ and its restriction $K^{4h+1} \times K^3 \to K^{4h+2}$ is also a normed map. But this implies a contradiction with (3.1). Hence, this ends proof of (4.1).

## 5. Minimal values of $n$ for $1 \leq q \leq 3$

The results of the last two sections will be used to prove the following generalizations of the first three cases of [(9.1)].

THEOREM (5.1). *Let $F$ be a field of characteristic different from two and let $1 \leq q \leq 3$. Then, for any $p$, the minimal $n$ for the existence of a normed map $F^p \times F^q \to F^n$ is independent of $F$ and its value is given by the following table*:

| q\p | $4h+1$ | $4h+2$ | $4h+3$ | $4h+4$ |
|---|---|---|---|---|
| 1 | $4h+1$ | $4h+2$ | $4h+3$ | $4h+4$ |
| 2 | $4h+2$ | $4h+2$ | $4h+4$ | $4h+4$ |
| 3 | $4h+3$ | $4h+4$ | $4h+4$ | $4h+4$ |

*Proof.* The same arguments already given in [(9.1)] are used here to construct the maps and to prove that the values of $n$ are minimal.

The main difficulty is to settle the cases $(p, q) = (4h + 1, 3)$ and $(p, q) = (4h + 2, 3)$. For these cases, the results (3.1) and (4.1) imply, respectively, that $n \geq 4h + 3$ and that $n \geq 4h + 4$. On the other hand, the direct sum of $F^{4h} \times F^3 \to F^{4h}$ and $F^1 \times F^3 \to F^3$, constructs the map for the first case and the restriction of $F^{4h+4} \times F^4 \to F^{4h+4}$ to $F^{4h+4} \times F^2 \subset F^{4h+4} \times F^4$, gives the map for the second case. Hence, these two cases are decided. All other details are omitted and this ends the proof.

## 6. Concluding remarks

Independent of the field, the results already established imply that $n = 4h + 4$ is the minimal value for the existence of a normed map, in the following three cases: $(p, q) = (4h + i, 4)$, for $i = 2, 3, 4$.

The case $(p, q) = (4h + 1, 4)$ with $h > 1$ has not been settled and we have the following question:

*Does there exists for some field $F$ a normed map $F^{4h+1} \times F^4 \to F^{4h+3}$?*

If $h = 1$ this was proved not to exist in [(7.1)]. If $h > 1$ the answer is not known even if $F = C$ is the complex field and $h = 2$.

If $F$ is a formally real field the Hopf theorem [(1.6)] gives a negative answer to the existence of such map. For other fields, the possibility of having isotropic vectors seems to complicate the problem. A difficult general question appears to be the following:

*Does the Hopf theorem hold for any field of characteristic not two?*

*Added remark.* After this paper was written, and due to some delay in the publication of the Boletín, I can report that in July 1982 K. Y. Lam and T. Y. Lam informed me that they had settled the two above problems for any field of characteristic zero. The problems remain open for fields of characteristic $p > 0$. Also, quite recently, D. B. Shapiro has informed me of some extensions and substantial simplifications he has obtained regarding the results of this paper.

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