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FREE CIRCLE ACTIONS ON RATIONAL SUSPENSIONS

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The aim of this text is to show that, given a suspension Y, there exists another one of the same rational homotopy type admitting a free circle action if and only if some condition depending on the Betti numbers is satisfied. It is clear that the existence of a free circle action is not a homotopy invariant. For instance a wedge of at least two spheres does not admit a free circle action. However S^3 trivially admits a free circle action and therefore also the space Y union of two spheres S^3 along a S^1 orbit. We now remark that Y is a suspension of the same homotopy type as $S^2 \vee S^3 \vee S^3$. Here we prove :

THEOREM. *Let $Y = \Sigma X$ be the suspension of a connected finite CW complex. The space Y has the rational homotopy type of a finite CW complex admitting a free circle action if and only if the two following conditions are satisfied :*

(1) $\chi(Y) = 0$

(2) *The degrees $n_1, \ldots, n_{2r+1}$ of some homogeneous basis of $H_+(Y;\mathbb{Q})$ are such that $n_{2i} - n_{2i-1}$ is odd for $i = 1, \ldots r$ with absolute value less than n_{2r+1}.*

Hence there exists in the rational homotopy type of $S^3 \vee S^3 \vee S^4$ a finite CW complex admitting a free circle action (an explicit model is given by the union of $(S^1 \times S^3)$ and S^3 along a particular S^1 orbit), but there does not exist any in the rational homotopy type of $S^3 \vee S^3 \vee S^6$.

Another family of examples of suspensions admitting free circle actions is obtained by the following process : we begin with the wedge of S^4 with a wedge of spheres S. We denote by $f : S \vee S^4 \to S^4$ the projection on the second factor. The pullback of the Hopf fibration $S^3 \to S^7 \to S^4$ along f gives a fibration

$$S^3 \to E \to S^4 \vee S .$$

The space E admits , via S^3, a free circle action. On the other hand, a simple computation shows that E has the rational homotopy type of a suspension.

The basic tool in the proof of our theorem is the rational minimal model of the space. In the first section we compute the minimal model of the Borel fibration associated to the S^1 action. The proof of the theorem is contained in section 2. For the definitions and basic properties concerning minimal models we refer to [6], [2], [7].

1. Minimal model of the Borel fibration

Let Y be a suspension which is a finite CW complex admitting a free circle action. The K.S. minimal model ([2]) associated to the Borel fibration $Y \to EY = ES^1 \times_{S^1} Y \to BS^1$ ([5]) is given by

$$(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \wedge Z, D) \to (\wedge Z, d) .$$

Here, the variable t is of degree 2, $(\wedge Z, d)$ is the minimal model of Y, and the differential D is a perturbation of d which means that

$$(D-d)(Z) \subset \mathbb{Q}^+[t] \otimes \wedge Z\,.$$

A classical theorem of Borel ([5]) asserts that the dimension of the cohomology of the total space EY is finite. Therefore $H^*(\mathbb{Q}[t] \otimes \wedge Z, D))$ is finite dimensional. Now, in ([3]), S. Halperin gives the following converse to this theorem :

THEOREM (A). *Let X be a simply connected CW complex with minimal model $(\wedge Z, d)$ and satisfying dim $H^*(X;\mathbb{Q}) < \infty$. The following properties are equivalent :*

(a) *There exists a finite simply connected CW complex Z with the same rational homotopy type as X admitting a free circle action.*

(b) *There exists on $\mathbb{Q}[t] \otimes \wedge Z$ a differential D such that $D(t) = 0$, $(D-d)(Z) \subset \mathbb{Q}^+[t] \otimes \wedge Z$ and $H^*(\mathbb{Q}[t] \otimes \wedge Z, D) < \infty$.*

Each suspension Y has the rational homotopy type of a wedge of spheres and is formal which means ([4]) that its minimal model $(\wedge Z, d)$ admits a gradation by a lower degree $Z = \oplus_{p\geq 0} Z_p$ such that $d(Z_p) \subset (\wedge Z)_{p-1}$, and $H(\wedge Z, d) = H_0(\wedge Z, d)$. This implies that we have an isomorphism $H^+(\wedge Z, d) \cong Z_0$.

LEMMA. *With the preceeding notations, every differential graded algebra of the form $(\mathbb{Q}[t] \otimes \wedge Z, D)$ satisfying $(D-d)(Z) \subset \mathbb{Q}^+[t] \otimes \wedge Z$ is isomorphic, as a $\mathbb{Q}[t]$-differential graded algebra, to the graded differential algebra $(\mathbb{Q}[t] \otimes \wedge Z, D')$ which satisfies a stronger condition :*

$$(D'-d)(Z_p) \subset \mathbb{Q}^+[t] \otimes (\mathbb{Q} \oplus Z_{<p})\,.$$

Proof. We give by induction on n a vector space Z' in $(\mathbb{Q}[t] \otimes \wedge Z)^n$ such that

(1) There is an isomorphism of graded differential algebras

$$(\mathbb{Q}[t] \otimes \wedge Z', D') \xrightarrow{\cong} (\mathbb{Q}[t] \otimes \wedge Z, D)\,.$$

(2) $Z' \subset (\mathbb{Q} \otimes Z) \oplus (\mathbb{Q}^+[t] \otimes \wedge Z)$. This gives to Z' a lower gradation, and finally we have

(3) $(D'-d)(Z'_p) \subset \mathbb{Q}^+[t] \otimes (\mathbb{Q} \oplus Z'_{\leq p})$. As t has degree two, this is trivially true for $n \leq 3$. We suppose this is true for $n < m$ and by a change of generators, we can suppose $Z^{<m} = (Z')^{<m}$ and $Z^m_{<p} = (Z')^m_{<p}$. Let then x be an element of Z^m_p. We write

$$D(x) = d(x) + \Sigma_{i=0}^r \alpha_i\,,$$

with α_i in $\mathbb{Q}^+[t] \otimes \wedge^i(Z)$. Suppose first that $r > 1$. In this case $d(\alpha_r) = 0$. But in $H^*(\mathbb{Q}[t] \otimes \wedge Z, d)$ each cocycle in $\mathbb{Q}[t] \otimes \wedge^{\geq 2} Z$ is a d-coboundary. Therefore

α_r is a d-coboundary : $\alpha_r = d(\gamma_r)$. Replacing x by $x - D(\gamma_r)$, we may suppose $\alpha_r = 0$. It follows that $D(x) = d(x) + \alpha_0 + \alpha_1$. We can now decompose $\alpha_1 = \Sigma_{i=1}^s \alpha_{1,i}$, with $\alpha_{1,i}$ in Z_i. In this decomposition, the integer s has to be smaller than or equal to p because otherwise $\alpha_{1,s}$ is a nonzero d-cocycle belonging to Z_+ which is impossible. □

2. Proof of the theorem

We suppose first that there exists a finite simply connected CW complex S in the rational homotopy type of Y admitting a free circle action and we prove that the rational cohomology of S satisfies the prescribed conditions. The KS extension associated to the Borel fibration $S \to ES^1 \times_{S^1} S \to BS^1$ has the form

$$(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \wedge Z, D) \to (\wedge Z, d),$$

where $(\wedge Z, d)$ is the bigraded minimal model of Y. By the lemma, the differential D can be written

$$D = D_0 + D_1 + \dots ,$$

where $D_i : Z_r \to \mathbb{Q}[t] \otimes (\mathbb{Q} \oplus Z_{r-i})$. It results that $\mathbb{Q}[t] \otimes (\mathbb{Q} \oplus Z_0)$ is stable under the differential.

Necessarily, there must exist some element x in Z_0 such that $D(x) = t^n + \Sigma \alpha_i t^{r_i} x_i$, with $x_i \in Z_0$ and $\alpha_i \in \mathbb{Q}$. Otherwise, in the cohomology Serre spectral sequence associated to the Borel fibration, Image $(d_r) \subset E_r^{*,+}$, for all $r \geq 2$, so that $E_\infty^{*,0} = E_2^{*,0} = \mathbb{Q}[t]$ and E_∞ has infinite dimension in opposition with Borel theorem.

We choose some element x with a minimal n satisfying the above property. Denote $R = \mathbb{Q}[t]$. By changes of generators in the R-module $R \otimes Z_0$, we can suppose that the R-module generated by x admits a summand T stable for the differential D. We have only to avoid the elements y of Z_0 such that $D(y) = \alpha t^m + \gamma$, $\gamma \in R \otimes Z_0, \alpha \in \mathbb{Q}$.

As R is a principal ideal domain, it is now easy to show that T is a free R-module with basis $(a_i, b_i), i = 1, \dots n$ satisfying

$$D(a_i) = 0, \qquad D(b_i) = t^{n_i} a_i .$$

We consider once again the Serre spectral sequence associated to the Borel fibration. This spectral sequence is isomorphic up to the term E_2 to the spectral sequence obtained by filtering the KS model $(R \otimes \wedge Z, D)$ by the degrees in R ([1]). Therefore the spectral sequence is also isomorphic up to the term E_2 to the spectral sequence obtained by filtering the complex $R \otimes (\mathbb{Q} \oplus Z_0)$ by the degrees in R. At each stage of the spectral sequence the R-module E_r has the form $E_r = \oplus_i v_i \mathbb{Q}[t]/t^{m_i}$ with the elements v_i belonging to $E_r^{0,*}$. There exists an integer r, such that some power t^m of t in $E_{*,0}$ has to be a boundary. As E_r is generated by $E_r^{*,0}$, this implies that $r = 2m$ and that there exists an element v in $E_r^{0,r-1}$ with $d_r(v) = t^m$.

The Serre spectral sequence is a spectral sequence of algebras. As the products in $E_2^{0,*}$ are zero, the same is true for $E_r^{0,*}$. Therefore we have

$$0 = d_r(0) = d_r(v \cdot a_i) = t^m \cdot a_i .$$

This implies that for each $i, 1 \leq i \leq n$ there exists an element c_i with $d(c_i) = t^{p_i} a_i$, for some $p_i < m$. The sequence consisting of the degrees of $a_1, c_1, a_2, c_2, \ldots, a_n, c_n$ and v satisfies the hypothesis.

We now prove the other implication. Let Y be a suspension whose rational reduced cohomology admits a basis $x_1, \ldots, x_n, y_1, \ldots, y_n, x$ such that

$$| y_i | > | x_i | , \text{ and}$$

$$2\rho_i - 1 = | y_i | - | x_i | < | x | = 2m - 1 , \qquad i = 1, \ldots n .$$

We define the algebra

$$E = \frac{\wedge x \otimes \wedge(a_{i,j}, b_{i,j})}{\wedge^{\geq 2}(a_{i,j}, b_{i,j})}_{1 \leq i \leq n, j \geq 0} .$$

The degrees are defined by the rules

$$| a_{i,j} | = | x_i | + 2(m-1)j, \text{ and } | b_{i,j} | = | y_i | + 2(m-1)j .$$

We put on the graded algebra $R \otimes E$ a differential D defined by

$$D(x) = t^m \quad D(a_{i,0}) = 0 \quad D(b_{i,0}) = -t^{\rho_i} a_{i,0} ,$$

and for $j \geq 1$,

$$D(a_{i,j}) = x a_{i,j-1} + t^{m-\rho_i} b_{i,j-1} \qquad D(b_{i,j}) = x b_{i,j-1} - t^{\rho_i} a_{i,j} .$$

The quotient $\overline{E} = \mathbb{Q} \otimes_R (R \otimes E)$ inherits a differential $\overline{D}$ satisfying

$$\overline{D}(a_{i,0}) = \overline{D}(b_{i,0}) = \overline{D}(x) = 0 ,$$

$$\overline{D}(a_{i,j}) = x a_{i,j-1} , \quad \overline{D}(b_{i,j}) = x b_{i,j-1} \quad j \geq 1 .$$

The projection $q : E \to \mathbb{Q} \oplus \mathbb{Q}(a_{i,0}, b_{i,0})$ is then a quasi-isomorphism. This implies that E is formal and that its minimal model is isomorphic to the minimal model of Y.

The dimension of the cohomology of $R \otimes E$ is finite : indeed the filtration of $R \otimes E$ by the degrees in R generates a spectral sequence whose E_2-term

$$E_2 = R \otimes H^*(E)$$

is a finitely generated R-module. The algebra R is noetherian, so that each E_r, $r \geq 2$, is also a finitely generated R-module. In E_{2m+1}, $t^m = 0$, so that

E_{2m+1} is a finitely generated R/t^m-module. This implies that E_∞ and therefore $H^*(R \otimes E, D)$ is finite dimensional.

We now consider the K.S. minimal model of the extension

$$(R,0) \to (R \otimes E, D) \to (E, \overline{D}).$$

This gives the following diagram where the vertical arrows are all quasi-isomorphisms.

$$\begin{array}{ccccc} (R,0) & \to & (R \otimes E, D) & \to & (E, \overline{D}) \\ \uparrow = & & \uparrow & & \uparrow \\ (R,0) & \to & (R \otimes \wedge Z, D) & \to & (\wedge Z, d). \end{array}$$

As $H^*(R \otimes \wedge Z, D) < \infty$, Theorem A shows the existence of a free action. □

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