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OPTIMAL RATIONAL CURVES AND HOMOTOPY PLANES

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1. Introduction

This paper is concerned with the construction of acyclic affine surfaces from plane algebraic curves. Foundational material for this construction can be found in TOM DIECK – PETRIE [1989,1991]. Here we concentrate on the curves and explicit examples.

Surfaces will always be connected, non-singular, quasi-projective, algebraic surfaces over the complex numbers. Let R be a subring of the rational numbers. A surface V is called *R–homology plane* if $H_i(V;R) = 0$ for $i > 0$. In the case $R = \mathbb{Z}$ we simply refer to this as a *homology plane*.

We investigate homology planes via their compactifications. If X is a projective surface and $C \subset X$ a curve we call (X, C) or X a *compactification* of any surface V which is isomorphic to the complement $X \setminus C$. By the basic rationality theorem of GURJAR – SHASTRI [1989] we can and will assume that X is a rational projective surface and C a rational curve.

Suppose $X \setminus C = V$ is a homology plane and $p : X \to Y$ a contraction to a minimal rational surface Y. Then $D = p(C)$ will be called a *minimal divisor* of V. In case $Y = \mathbb{P}^2$ is the projective plane, we call D a *plane divisor* of V.

Only very special rational curves can be minimal divisors of $\mathbb{Q}$–homology planes. In qualitative terms: If $D = D_1 \cup \ldots \cup D_n$ is the decomposition into irreducible components, then the D_i should have very few intersection points or singularities with several branches. For instance, if $D \subset \mathbb{P}^2$ is irreducible, then D must be a *cuspidal* curve; this means, D is a topologically embedded 2–sphere $\mathbb{P}^1$.

The minimal divisors of a homology plane V are by no means uniquely determined. In fact, by applying an arbitrary Cremona transformation to a plane divisor of some V we obtain again a plane divisor of V. Therefore we are looking for some kind of normal form. The normal forms should be chosen so as to meet certain criteria: They should be as simple as possible (fewest number of components, lowest degree of components). They should be maximal in the following sense: It is shown in TOM DIECK – PETRIE [1991] that a minimal divisor of a homology plane gives raise to infinite families of surfaces, called towers of surfaces. (The members of a tower are in general $\mathbb{Q}$–homology planes.) The maximality condition askes for curves for which the corresponding towers are not contained in larger towers of other curves. For instance, the tower of a cuspidal curve $D \subset \mathbb{P}^2$ consists of the complement $\mathbb{P}^2 \setminus D$ alone. But all known cuspidal curves are members of infinite towers. We offer two conjectures which help to explain these heuristic conditions.

CONJECTURE (1.1). *Every homology plane has a plane divisor which consists of lines and quadrics.*

Unfortunately, it is very difficult to classify arrangements of lines and quadrics which can be plane divisors of homology planes. (For lines see TOM DIECK [1990]; for lines and quadrics NEUSEL [1992].) An affirmative answer to the corresponding conjecture for (rational) $\mathbb{Q}$–homology planes would imply that a cuspidal curve can be transformed into a line by a Cremona transformation (compare MATSUOKA – SAKAI [1989] for remarks concerning the latter problem). Actually, I believe that the methods of this paper lead to a construction of all cuspidal curves.

Let $D = D_1 \cup \ldots \cup D_n \subset Y$ be a rational curve in a minimal rational surface Y. There is a minimal expansion $p : X \to Y$ such that $C = p^{-1}(D)$ has normal crossings (embedded resolution of singularities). The intersection pattern of C is usually codified in its weighted dual graph ΓC. The curve D is called *optimal* if ΓC is connected and has $n - b_2(X)$ cycles. Here b_2 is the second Betti number; the number of cycles of a connected graph Γ is defined to be $1 - \chi(\Gamma)$, with $\chi(\Gamma)$ the Euler characteristic of (the geometric realization of) Γ. The term "optimal" refers to the algorithm for the construction of homology planes from minimal divisors which was described in TOM DIECK – PETRIE [1991]. This algorithm requires the choice of a selection function and the choice of a cutting set. For optimal curves there is only a trivial selection function with constant value one.

CONJECTURE (1.2). *Homology planes have optimal minimal divisors.*

Whereas (1.1) may well be true for $\mathbb{Q}$–homology planes, (1.2) seems to require integral homology planes. Also it is not enough to deal only with plane divisors. Optimal plane divisors will not, in general, consist of lines and quadrics.

One purpose of this paper is to describe the (maximal) optimal rational curves which have been found so far (up to Cremona transformations). Since homology planes of logarithmic Kodaira dimension $\bar{\kappa} \leq 1$ have been analyzed successfully by GURJAR – MIYANISHI [1987], we always concentrate on surfaces of general type $\bar{\kappa} = 2$. There probably exist no more than those coming from the curves described here. This belief is the main justification for this paper. But, loosely speaking, I conjecture that there exists a finite number of towers which comprise all homology planes of general type.

The results of this paper suggest a number of more technical conjectures which I will explain in due course.

In section 2 we construct the optimal curves and show that they are uniquely determined by their weighted dual graph. In section 3 we present some numerical results for the construction of homology planes. In section 4 we verify that the known homology planes have very special logarithmic Chern numbers.

2. Optimal curves

The curves are described by their intersection pattern (weighted dual graph, singularities). Existence and uniqueness up to automorphism of the minimal rational surface is shown. Recall that the minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces $\Sigma(n)$, $n = 0, 2, 3, \ldots$. We also show how our optimal curves are related to arrangements of lines and quadrics in order to check (1.1).

A. Six lines in $\Sigma(0)$.

Note that $\Sigma(0) = \mathbb{P}^1 \times \mathbb{P}^1$. The arrangement consists of

$$L_i = \mathbb{P}^1 \times i, \qquad M_i = i \times \mathbb{P}^1, \qquad i = 0, 1, \infty \in \mathbb{P}^1.$$

An equivalent optimal arrangement of Five Lines in $\mathbb{P}^2$ is the following: Take four lines in general position $L_1, \ldots, L_4$ and add a further line L which passes through two double points x and y of $L_1 \cup \ldots \cup L_4$. In order to obtain a normal crossing curve one has to blow up x and y. Then the proper transform of L becomes a (-1)–curve. Contracting this (-1)–curve leads to an arrangement of Six Lines in $\Sigma(0)$ as described above. The weighted dual graph has 4 cycles and is (the non-planar graph) given in Figure A. This is the only arrangement with 4 cycles.

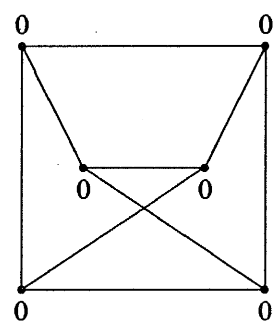


Figure A

We remark that Four Lines in general position in $\mathbb{P}^2$ are an optimal curve (three cycles). But the towers of this arrangement are contained in towers of the arrangement (A) above. In order to see this, ones has to cut one of the three cycles with multiplicities $(1, 1)$ (see TOM DIECK – PETRIE [1991] for terminology and background). Cutting a cycle with these multiplicities removes an edge in Figure A and replaces the weights of the adjacent vertices by -1. Contracting these (-1)–curves results in the graph of Four Lines (see TOM DIECK – PETRIE [1991], (5.2)).

B. Quadric with three tangents.

Up to automorphisms of $\mathbb{P}^2$ there exists a unique curve consisting of a regular quadric Q and three of its tangents T_1, T_2, T_3. The symmetric group S_3 still acts on any such arrangement by projective automorphisms. Blowing up the points of tangency (twice) produces a normal crossing curve. The weighted dual graph is given in Figure B.

As a matter of notation we display the negatives of the actual weights in order to avoid minus-signs.

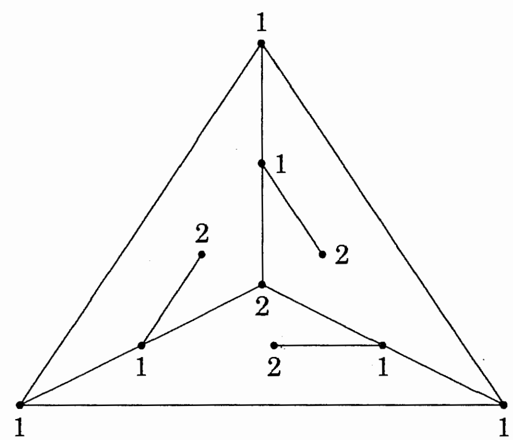


Figure B

C. Quadric, two tangents, line through double point.

Let Q be a quadric with two tangents T_1, T_2 and a further line L through $T_1 \cap T_2$. Any such arrangement has a projective symmetry group $\mathbb{Z}/2 \times \mathbb{Z}/2$: Reflection in L, interchanging T_1 and T_2; reflection in the line through $Q \cap T_i$. The weighted dual graph of a minimal resolution is given in Figure C.

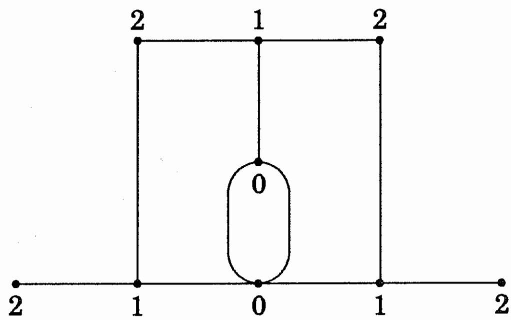


Figure C

PROPOSITION (2.1) *The surfaces in towers of* B *and* C *are obtainable from the following arrangement of Seven Lines* (Figure 1)*: Let $L_1, \dots, L_4$ be four lines in general position. Add the three possible lines M_1, M_2, M_3 which connect the double points of $L_1 \cup \dots \cup L_4$.*

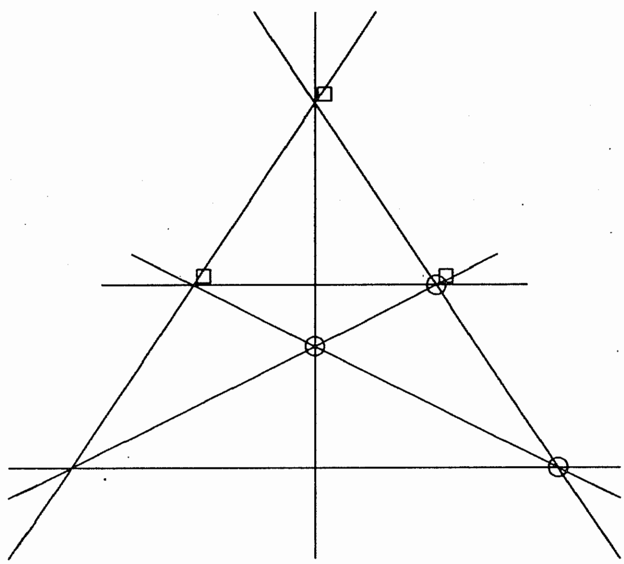


Figure 1

Proof. Note that the symmetric group S_4 acts on this arrangement by permuting the L_j. The construction of towers requires the choice of a selection function (see TOM DIECK – PETRIE [1991]). In the case of an arrangement of lines a selection function d assigns a value 0 or 1 to a point of multiplicity greater than two. The combinatorics requires in our case that d assumes the value 0 on three non-collinear points. (Value 0 means: When this point is blown up the corresponding exceptional divisor will not be included in the compactification divisor of the homology plane. If three collinear points have value zero, then the resulting graph is no longer connected.) Up to symmetry these are the cases marked □ and ○ in Figure 1. Case ○ leads to the graph B and case □ to the graph C.

In the case of graph B we can contract successively (-1)–curves. We end up with four curves Q, T_1, T_2, T_3 with self-intersections (respectively) $4, 1, 1, 1$ and intersection numbers $|Q \cdot T_j| = 2$. All this is read off from the graph. Thus any such graph arises from an arrangement B. A similar argument applies to case C. □

As a matter of notation we write $S \cdot T = \sum n_i x_i$ when the irreducible curves S and T intersect in the points x_i with multiplicity n_i. The intersection number $|S \cdot T|$ is then given by $\sum n_i$.

D. Cubic with two lines.

Let $C \subset \mathbb{P}^2$ be a cubic with a cusp s. There is a unique point $p \in C$ which has a tangent L of order 3, i.e. $C \cdot L = 3p$. Let M be an ordinary tangent to C in a regular point r. It intersects M in a further regular point q. The points q, r are different from p, since $C \setminus s$ carries a group structure (isomorphic to the additive group $\mathbb{C}$) such that the intersections of $C \setminus s$ with a line add up to zero in this group structure. A cubic with a cusp is projectively unique and still carries a $\mathbb{C}^*$–action with fixed points p, s and orbit $C \setminus \{p, s\}$. This shows that the configuration $C \cup L \cup M$ is projectively unique. The dual graph is specified in Figure D.

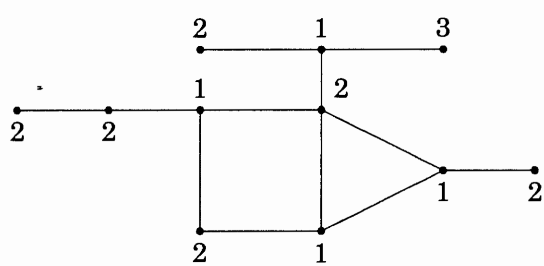


Figure D

E. Four sections in $\Sigma(2)$.

Let $\pi : \Sigma(2) \to \mathbb{P}^1$ be the ruling, F a general fibre of π and E the section with self-intersection $|E \cdot E| = -2$. The arrangement consists of curves Q_2, Q_1, L, E linearly equivalent to

$$Q_2 \sim 3F + E, \quad Q_1 \sim 2F + E, \quad L \sim 2F + E.$$

The intersection pattern is given as follows

$$Q_2 \cdot E = x, \quad Q_2 \cdot Q_1 = 3y, \quad Q_1 \cdot L = 2v, \quad Q_2 \cdot L = 2z + u.$$

Here x, y, z, u, v denote 5 different points. From this intersection pattern one computes the dual graph of a minimal resolution to be as in Figure E.

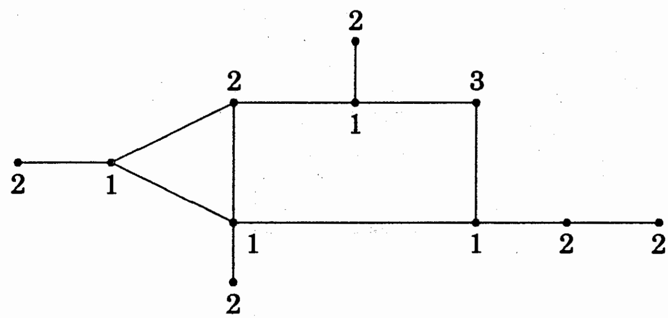


Figure E

PROPOSITION (2.2) *An arrangement with the intersection pattern as above exists and is unique up to automorphism of* $\Sigma(2)$.

Proof. We start with the following arrangement of lines L, M and quadrics Q_1, Q_2 in $\mathbb{P}^2$. The intersection pattern is specified as follows:

$$Q_1 \cdot Q_2 = a + 3c, M \cdot Q_1 = 2a, M \cdot Q_2 = a + d, L \cdot Q_1 = 2b, L \cdot Q_2 = d + e, L \cdot M = d.$$

The construction is done in the following order:

(1) Choose Q_1, Q_2 with the given intersection pattern. This is projectively unique.

(2) M is the tangent to Q_1 in a. Then M is not tangential to Q_2 in a and produces the other intersection d with Q_2. We must have $c \neq d$.

(3) From d draw the second tangent L to Q_1. This determines $L \cdot Q_1 = 2b$. We have $b \neq c$ since otherwise L would be tangent to Q_2. We find e from $L \cdot Q_2$ and e is different from b and c. Having chosen Q_1, Q_2, the construction is uniquely determined.

We apply the following expansions and contractions to the arrangement.

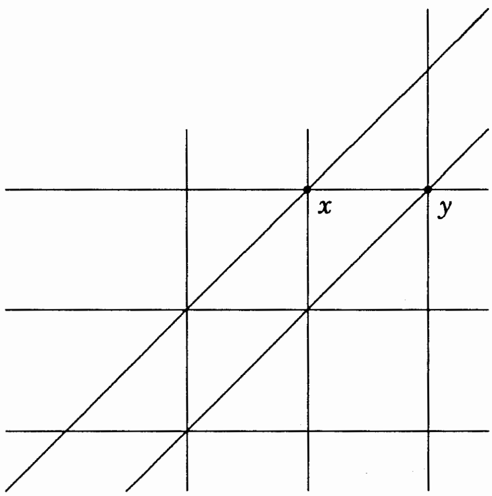
(α) Blow up a. This yields a (-1)–curve E and the proper transforms L', M', Q_1', Q_2'.

(β) E, M' and Q'_1 have a common point. We blow up this point. Then M' becomes a (-1)–curve M''.

(γ) Contract M''.

The proper transforms of L, E, Q_1, Q_2 are the curves we are looking for.

Given an arrangement L, E, Q_1, Q_2 as in the beginning we can certainly reverse $(\alpha) - (\gamma)$ and arrive at a uniquely determined arrangement of lines and quadrics. □



with ∞ – line

Figure 2

PROPOSITION (2.3). *The surfaces in towers* D *and* E *are obtainable from the following arrangement of Nine Lines in* $\mathbb{P}^2$ (Figure 2)*: The lines*

$$z_1 = 0, 1, 2; \quad z_2 = 0, 1, 2; \quad z_1 = z_2; \quad z_1 = z_2 + 1;$$

and the infinite line (affine coordinates z_1, z_2*).*

Proof. A selection function d must have value one on two points of multiplicity 3. In order to obtain a connected graph one of these points has to be a point of multiplicity 3 on the infinite line. By symmetry of the figure the other point must be x or y. The case $d(x) = 1$ leads to the graph D and the case $d(y) = 1$ leads to the graph E. By contracting successively (-1)–curves in such graphs one arrives at a system of curves with intersection pattern as described under D and E. We have already seen that the intersection pattern determines the arrangement involved uniquely. □

F. Four sections in $\Sigma(2)$.

As in case (E) we find curves with linear equivalence classes

$$C \sim 3F + E, \quad L_1 \sim L_2 \sim 2F + E, \quad E.$$

But this time the intersection pattern has the following structure:

$$C \cdot E = x, \quad C \cdot L_1 = 3y, \quad C \cdot L_2 = 3z, \quad L_1 \cdot L_2 = u + v.$$

This intersection pattern leads to the following dual graph (Figure F).

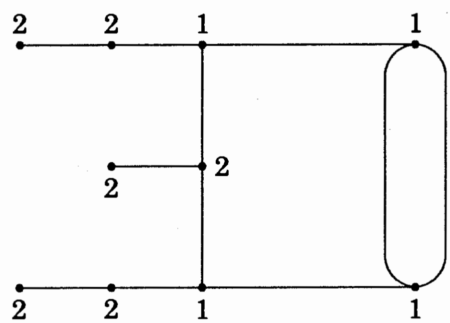


Figure F

PROPOSITION (2.4). *An arrangement with the intersection pattern as above exists and is unique up to automorphism of* $\Sigma(2)$.

Proof. We start with the following arrangement in $\mathbb{P}^2$: Let C be a cubic with node c. Let T be one of the tangents to C in c. Then there is no further intersection of C and T. There are exactly 3 regular points of C with tangents of order 3 (flexes). Let L_1 and L_2 be two of these tangents in d and e. We obtain intersections $T \cdot L_1 = a$, $T \cdot L_2 = b$, $L_1 \cdot L_2 = c$. Up to automorphism of $\mathbb{P}^2$ there is exactly one such arrangement.

We now apply the following expansions and contractions to this arrangement:

(α) Blow up the point c. Let E be the corresponding (-1)–curve.

(β) Blow up the intersection of E with the proper transforms C' and T'.

(γ) Contract the resulting (-1)–curve T''.

The proper transforms of C, L_1, L_2 and E constitute the arrangement we are looking for. □

G. Nodal cubic with two tangents.

This is the subsystem $C \cup L_1 \cup L_2$ which was described in the proof of (2.4). It leads to the following dual graph (Figure G).

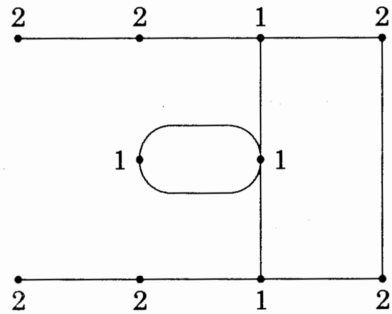


Figure G

PROPOSITION (2.5). *The surfaces in towers* F *and* G *are obtainable from the following arrangement of Nine Lines in* $\mathbb{P}^2$:

$$z_1 = 1, \omega, \omega^2; \quad z_2 = 1, \omega, \omega^2; \quad z_1 = z_2; \quad \omega z_1 = z_2;$$

and the infinite line ($\omega \neq 1, \omega^3 = 1$). *(This arrangement is not realizable over the real numbers.)*

Proof. A selection function must have value one on three points of multiplicity 3. The points can be x, y or x, z (up to symmetry) with $x = \{z_1 = 1\} \cap \{z_2 = 1\}, y = \{z_1 = 1\} \cap \{z_2 = \omega\}, z = \{z_1 = \omega\} \cap \{z_2 = \omega\}$. The proof is finished as for (2.3). □

H. Cubic, quadric, line.

PROPOSITION (2.6). *There exists an arrangement in* $\mathbb{P}^2$ *consisting of a cuspidal cubic* C, *a regular quadric* Q, *and a line* L *with the following intersection pattern:*

$$C \cdot L = 3z, \quad C \cdot Q = 3x + 2z + y, \quad Q \cdot L = 2z.$$

The points $x, y, z \in C$ *are regular.*

Proof. There certainly exist C and L with $C \cdot L = 3z$. We use again the group structure on the regular part of C. Six regular intersection points x_j of C and Q (counted with multiplicities) must satisfy $x_1 + \ldots + x_6 = 0$ in this additive group structure. Since $C \cdot L = 3z$, the point z corresponds to the zero element of this group. The points x and y therefore have to satisfy $3x + y = 0$. The quadric which is determined by these conditions is regular: Otherwise there would exist a line intersecting C in a subset of $\{x, y, z\}$ and this is impossible. Again the configuration is unique up to automorphism of $\mathbb{P}^2$. □

The dual graph of the minimal resolution is given in Figure H.

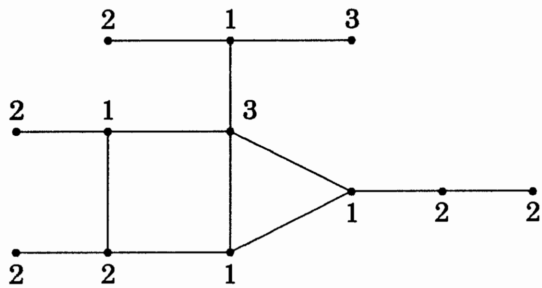


Figure H

Remark (2.7). The surfaces in towers of H are obtainable from the following arrangement: A quadric Q with three tangents T_1, T_2, T_3 together with two lines L_1 through $T_1 \cap T_2$ and $Q \cap T_3 = y$ and L_2 through $Q \cap L_1 \setminus \{y\}$ and $T_1 \cap T_3$.

In order to prove this one has to find a selection function which produces the graph H. Any configuration with this graph can be contracted to the arrangement of (2.6): Contract successively (-1)–curves which show up in the curves of the graph. This last remark applies to all other graphs.

I. Quartic with bitangent.

A quartic with three cusps $S \subset \mathbb{P}^2$ is projectively unique (Steiner quartic) and has a bitangent T. The graph of the minimal resolution of $S \cup T$ is given in Figure I.

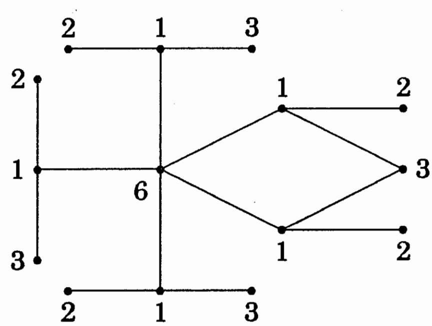
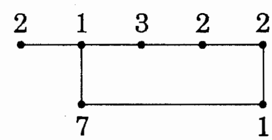


Figure I

This arrangement was used in TOM DIECK [1990] to produce homology planes with $\mathbb{Z}/3$–action. There it is also shown that the towers of (I) can be obtained from an arrangement of two quadrics and three lines.

There is another arrangement which yields the same result. There exists a cuspidal quintic D with four cusps x_1, x_2, x_3, and y (compare NAMBA [1984], p. 182, curve number 6). The points x_j are ordinary cusps of multiplicity two. There exists a tangent L in y which intersects D in a single further point z which is a regular point of D. In order to see that this arrangment $D \cup L$ exists and yields the same result as $S \cup T$ above, one blows up one of the points $S \cap T$. Then the graph of (I) is modified so as to contain the piece



and the vertical string can be contracted to the singular point y. This construction also shows a $\mathbb{Z}/3$–symmetry on the quintic which permutes the cusps.

J. A 2–section and two sections in $\Sigma(2)$.

There exists an arrangement S, L, E in $\Sigma(2)$ with the following intersection properties

$$S \sim 5F + 2E, \quad L \sim 2F + E, \quad E$$

$$S \cdot E = a, \quad S \cdot L = 4b + c.$$

The curve S has two cuspidal singularities and a, b, c are regular points of S. The intersection pattern yields the graph of Figure J.

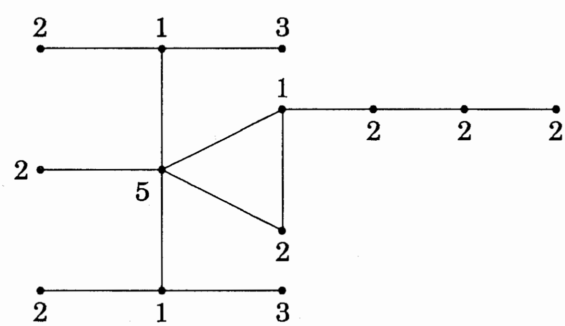


Figure J

PROPOSITION (2.8). *An arrangement as above exists and is unique up to automorphism of* $\Sigma(2)$.

Proof. Start with two quadrics Q_1, Q_2 in $\mathbb{P}^2$ such that $Q_1 \cdot Q_2 = 4x$ and choose a further point $y \in Q_1$. The triple (Q_1, Q_2, y) is unique up to automorphism of $\mathbb{P}^2$. Let T_1, T_2 be the tangents of Q_2 which pass through y. Let L be the line through $Q \cap T_1$ and $Q_1 \cap T_2$ and let M denote the line through y and one of the intersections $Q_2 \cap L$. By symmetry it does not matter which of the points in $Q_2 \cap L$ is chosen.

Now apply a plane Cremona transformation with centers $y, Q_1 \cap L$ to Q_1, Q_2, T_1, T_2, L, M. Then Q_2 becomes a quartic Q_2' with two cusps and a node z; and M and Q_1 become lines M', Q_1'. The next step is to contract the curves L, T_1, T_2. Moreover M' is one of the tangents of Q_1' in y. Now one blows up y twice and contracts the proper transform of M'. The proper transforms S of Q_2, L of Q_1, and E of the exceptional divisor of y are the arrangement we are looking for. By reversing this procedure we obtain uniqueness, since the arrangement Q_1, Q_2, T_2, T_2, L, M was unique up to automorphism. □

K. Quartic with tangent.

In the previous section J we obtained a quartic S with two cusps, a node and a tangent L of order 4. The curve $S \cup L$ is optimal and has the graph:

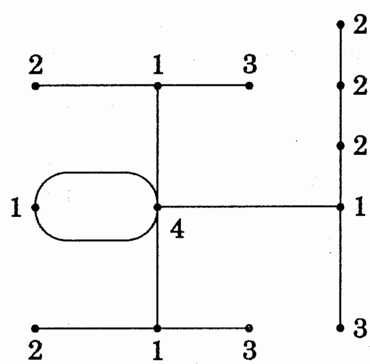


Figure K

The corresponding tower can be obtained from the arrangement Q_1, Q_2, T_1, T_2, L of the previous section. There exists an involution on $\mathbb{P}^2$ which interchanges the cusps of S and fixes $S \cap L$ and the node of S.

L. A quintic with cuspidal tangent.

PROPOSITION (2.9) *There exists a quintic* $Q \in \mathbb{P}^2$ *with three cusps. The multiplicity sequence of each cusp is* $(2,2)$. *The tangent* T *at a cusp intersects* Q

in a single further point which is regular. A quintic with these types of cusps is projectively unique. There exists a $\mathbb{Z}/3$–action on $\mathbb{P}^2$ which permutes the cusps of Q.

Proof. We give a geometric contruction which exhibits the $\mathbb{Z}/3$–symmetry. The starting point is an arrangement of a quadric with six or seven lines.

Let S be a quadric with three tangents T_1, T_2, T_3. Set $x_{ij} = T_i \cap T_j$. There exist lines L_k through x_{ij}, $k \notin \{i,j\}$ such that $L_k \cap L_l \in S$ for $k \neq l$. Existence is shown as follows by working in the real plane: Let S be a circle about the origin in the plane. Suppose the T_i are permuted under a $2\pi/3$–rotation about the origin. Add lines L_k such that $L_k \cap L_l \in S$ and such that the L_k are permuted under a $2\pi/3$–rotation. Now rotate the configuration of the L_k until L_k passes through X_{ij}, $k \notin \{i,j\}$. Starting from this construction it is also possible to show uniqueness of the arrangement.

The quintic Q is now obtained by applying the following Cremona transformation: Blow up the points x_{ij} and $L_k \cap L_l$. Then T_i becomes a (-1)–curve and L_i a (-2)–curve. Contract these (-1)–curves and then the (-2)–curves. The proper transform of S is Q. It is clear that this process is $\mathbb{Z}/3$–equivariant. Reverse this procedure to show uniqueness of Q.

If we add a line through x_{13} and $S \cap L_2$ in the primary configuration, then T is the proper transform of this line under the birational map of the previous paragraph. □

The curve $Q \cup T$ is optimal. The graph of the minimal resolution contains a (-1)–curve. We contract this curve and arrive at the graph of Figure L.

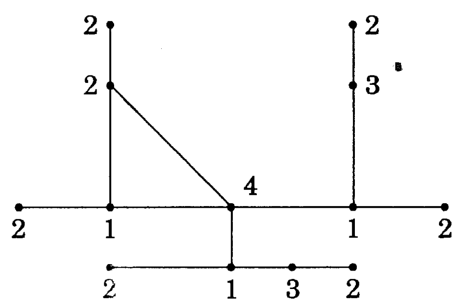


Figure L

Remark (2.10). The graph of figure L can be contracted to a curve $S' \cup T'$ in $\mathbb{P}^2$. In this case S' is again a quintic, but this time with two cusps and a tacnode t, and T is a line passing through t. By applying a Cremona transformation with centers x_{ij} to the primary configuration one obtains another interesting one: A Steiner quartic together with three lines through its cusps; the lines intersect on the quartic. The Cremona transformation with centers $L_k \cap L_l$ yields another interesting configuration of three quadrics with a common tangent.

M. Nodal cubic, quadric, line.

PROPOSITION (2.11). *Let $C \subset \mathbb{P}^2$ be a nodal cubic. There exists a quadric Q and a line L such that we have the following intersections:*

$$C \cdot Q = 4x + 2y, \quad C \cdot L = 3y, \quad Q \cdot L = 2y.$$

Proof. We use the group structure $\mathbb{C}^*$ on the complement of the double point of C. Then we take $y = \exp(2\pi i/3)$ and $x = \exp(2\pi i/12)$. There exists a regular quadric Q which intersects C in $x_1, \dots, x_6$ if and only if $\prod x_i = 1$ in this group structure. Since $x^4y^2 = 1$ we can realize the intersection pattern $C \cdot Q$ as stated. A similar argument applies to L. □

The dual weighted graph is displayed in Figure M.

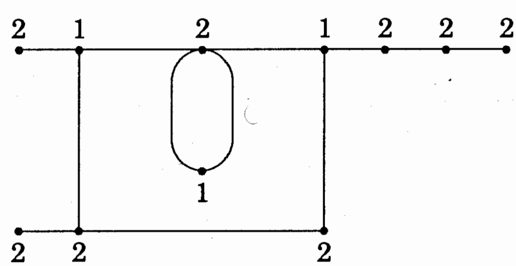


Figure M

The towers of the resulting $\mathbb{Q}$–homology planes can be obtained by the following arrangement of lines $L_1, \dots, L_5$ and a quadric Q (see NEUSEL [1992]): L_1, L_2, L_3 are tangents to Q and L_4 passes through $Q \cap L_1$ and $Q \cap L_2$. Finally L_5 passes through $L_1 \cap L_2$ and $L_3 \cap L_4$.

N. Four sections in $\Sigma(0)$.

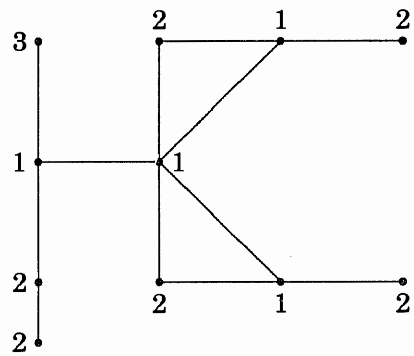


Figure N

PROPOSITION (2.12). *There exist four sections $S_1, \dots, S_4$ of* $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ *with the following properties:* $S_1 = \mathbb{P}^1 \times \{y_i\}$, $i = 1, 2, 3$ *and* $S_4 \cdot S_1 = 3x_1$, $S_4 \cdot S_2 = 2y_1 + y_2$, $S_4 \cdot S_3 = 2z_1 + z_2$.

Proof. We start with configuration D and add the line which passes through the transvers intersection of C and M. We then blow up the cusp s and $L \cap M = t$ and contract the proper transform of the line which connects s and t. The proper transforms of L, M, and Q become S_1, S_2, and S_3 and the exceptional divisor of s becomes S_4. □

The towers of $\mathbb{Q}$–homology planes are obtainable from the following arrangement of a quadric Q and lines $L_1, \dots, L_6$: The lines L_1, L_2, L_3 are tangents of Q, L_4 passes through $L_1 \cap L_2$ and $Q \cap L_3$ and L_5 (L_6) passes through $Q \cap L_4$ and $L_1 \cap L_3$ $(L_2 \cap L_3)$.

Remark (2.13). A surface V is said to admit a $\mathbb{C}^{n*}$–fibration if there is a morphism $p : V \to U$ onto a curve U such that the general fibre of p is the n–punctured affine line $\mathbb{C}$. From the construction of our optimal curves it is easy to see, that their complement admits a $\mathbb{C}^{2*}$–fibration (cases A,B,C) or a $\mathbb{C}^{3*}$–fibration (the other cases). Since the cases A, B, C also admit $\mathbb{C}^{3*}$–fibrations we state:

CONJECTURE (2.14). *Every homology plane of general type admits a $\mathbb{C}^{3*}$–fibration.*

The classification of $\mathbb{C}^{3*}$–fibrations will be the subject of another essay.

3. Discriminants

This section explains the explicit construction of homology planes from the previously describrd curves. This construction is based on a blow-up algorithm. The details of the algorithm are justified in the paper TOM DIECK – PETRIE [1991]. We recall from that paper that the construction of towers from a minimal divisor requires:

(1) The choice of a selection function.

(2) The choice of a cutting set.

Since we are dealing with optimal curves there is no choice of a selection function. A cutting set is a set of edges in the graph such that its omission produces a tree. The surfaces in a tower belonging to a given cutting set are parametrized by multiplicities: To each edge in a cutting set is assigned a pair (u, v) of positive coprime integers u and v, called multiplicities. There is a multilinear function in the multiplicities, called discriminant of the tower, such that its absolute value gives the order of the first homology group of the associated surface in the tower (provided this value is non zero). The surface is a homology plane if and only if the discriminant has absolute value one.

All that matters for our present purpose is a description of the discriminant algorithm. We first describe this algorithm abstractly in words. Since this is slightly involved and its theory is not helpful at this point, the reader should carefully follow the steps in the example after the formal description of the algorithm.

The discriminant is calculated via the following algorithm: Start with a curve $C = C_1 \cup \ldots \cup C_n$ in a minimal rational surface Y and let $C_i \in H^2(Y;\mathbb{Z})$ also denote the Poincaré dual of the cycle C_i. Let $R_1, \ldots, R_k$ denote a basis of the relations among the C_i; the R_j are certain integral linear combinations of the indeterminates C_i. Now blow up the surface to resolve the singularities of C. Rewrite R_j in terms of the proper transforms of the C_i and the exceptional divisors of the expansion. From the resulting linear combinations R'_j the discriminant is produced in the following manner:

Let x_e be an edge in the cutting set with vertices (= curves) A and B. Replace A and B in the R'_j by $a(A)x_e, a(B)x_e$, where $a(A), a(B) \in \mathbb{Z}$ are the multiplicities. If A occurs in several edges of the cutting set, replace A by the sum of the corresponding terms $a(A)x_e$. Set $A = 0$, if A is not a vertex of an edge in a cutting set. It is a fact that k is also the cardinality of a cutting set. The coefficients of the x_e in the R'_j after these replacements yield a (k, k)–matrix whose determinant is the value of the discriminant at the given multiplicities.

Since some of our graphs have a large number of cutting sets (e.g. B has 11, up to symmetry), we give only the linear polynomials R'_j in those variables which can appear in cutting sets. As for notation, we use the symbols for the curves which were introduced in the definition of the arrangement (A) to (N) in section 2. Moreover we need notation for some of the exceptional divisors: We draw again the relevant graphs and specify the symbol for the divisor at the corresponding vertex.

Example for the discriminant algorithm.
We refer to arrangement H in the following list. We start with the curve $C \cup Q \cup L$ in $Y = \mathbb{P}^2$. Then L generates $H^2(\mathbb{P}^2;\mathbb{Z}) = \mathbb{Z}$ and $C = 3L$, $Q = 2L$. A relation basis is a basis of the kernel of

$$\mathbb{Z}^3 \to H^2(\mathbb{P};\mathbb{Z}), \quad (a,b,c) \mapsto aC + bQ + cL.$$

In this case we can take

$$R_1 = C - 3L, \qquad R_2 = Q - 2L.$$

Now one has to resolve the singularities in x, z given by the intersection $C \cdot Q = 3x + 2z + y$. This requires to blow up x three times and z twice. Moreover, C has a cuspidal singularity c which resolves after two blow up processes. The images of the (Cartier) divisors C, Q, L after these blow up processes are then (using e.g. HARTSHORNE [1977], V.3) respectively

$$C \mapsto C + 3A + 2B + B' + 3E + 2E' + E'' + F' + 2f,$$

$$Q \mapsto Q + 2A + 2B + B' + 3E + 2E' + E'',$$

$$L \mapsto L + 3A + 2B + B'$$

where $A, B, B', E, E', E'', F, F'$ are names for exceptional divisors and L, B', E', E'', F, F' are not involved in cycles of the graph. The dual graph of the total transform is shown under arrangement H. It has two cycles. We have only named those vertices which appear in cycles. We substitute the expressions for C, Q, L into the relation basis $R_1 = C - 3L$, $R_2 = Q - 2L$. The resulting linear combinations are R_1' and R_2'. If we omit the vertices which are not involved in cycles there remain the two linear combinations which are written next to the figure in arrangement H.

Since the dual graph has two cycles a cutting set consists of two edges which, when removed, produce a tree. Let us take e. g. the edges AC and CE.

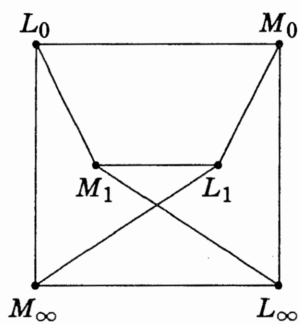
The next step in the algorithm requires to replace A by ax, C by $bx + cy$, E by dy with unknowns x and y and integers a, b, c. Moreover we have to put B and Q equal to zero since they are not involved in cutting edges. The resulting linear forms are

$$(b - 6a)x + (c + 3d)y$$

$$-4ax + 3dy.$$

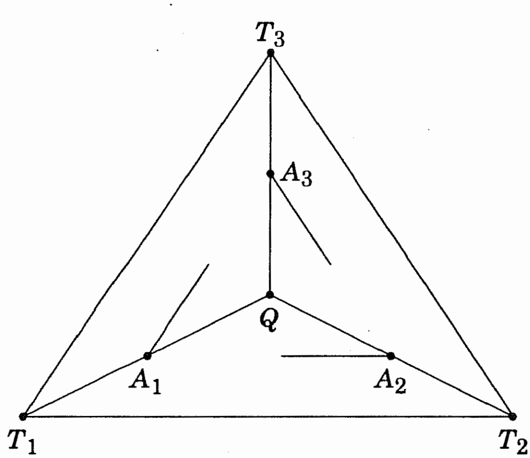
The discrimant is the resulting determinant $3d(b - 2a) + 4ac$ of the coefficients. This is the discriminant of the corresponding tower. The multiplicities $(a, b) = (c, d) = (1, 1)$ e.g. yield a homology plane.

In order to understand what is going on, one has to recall that an edge in the graph is an intersection point of two curves. The choice of multiplicities means a standard iterated blow up process of this point with linear dual graph of exceptional divisors. It is well known that such expansions are parametrized by a pair of multiplicities via a continued fraction algorithm.



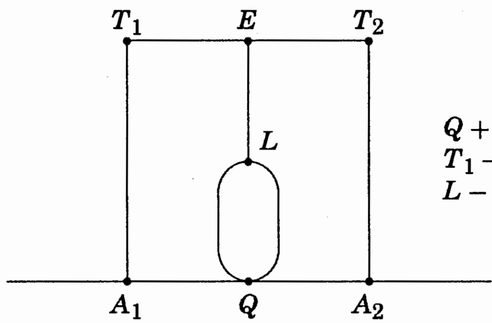
$L_0 - L_1,\ L_0 - L_\infty$
$M_o - M_1,\ M_0 - M_\infty$

Arrangement A



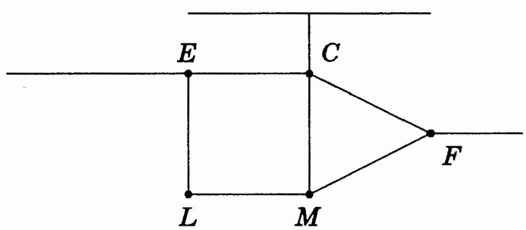
$T_1 - T_2 + 2A_1 - 2A_2$
$T_2 - T_3 + 2A_2 - 2A_3$
$Q - 2T_3 + 2A_1 + 2A_2 - 2A_3$

Arrangement B



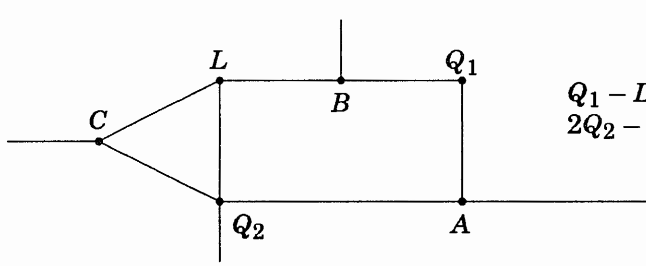
$$Q + 2A_1 + 2A_2 - 2L - 2E$$
$$T_1 - T_2 + 2A_1 - 2A_2$$
$$L - T_1 - 2A_1$$

Arrangement C



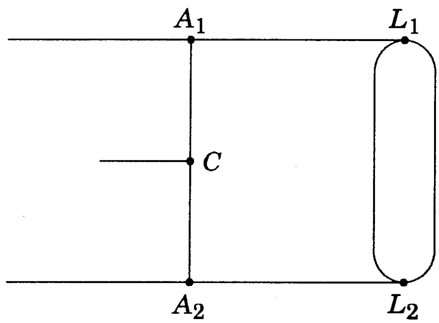
$$C - 3L + 2F - 6E$$
$$L - M + 3E - 2F$$

Arrangement D



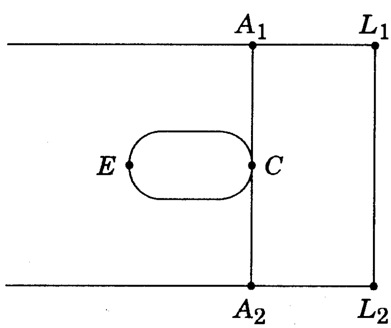
$$Q_1 - L + 3A - 2C$$
$$2Q_2 - 3L + 6A - 6B - 2C$$

Arrangement E



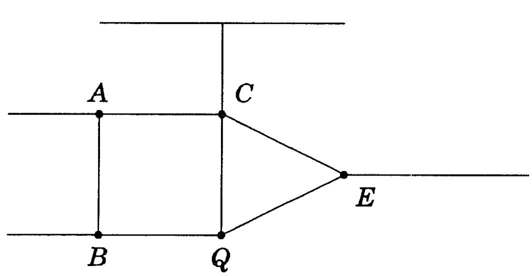
$$L_1 - L_2 + 3A_1 - 3A_2$$
$$2C - 3L_1 - 6A_1 + 3A_2$$

Arrangement F



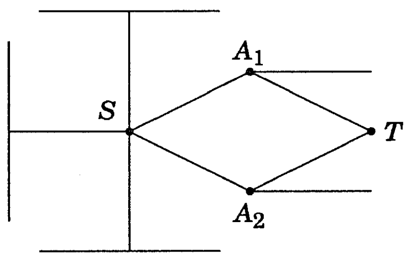
$$L_1 - L_2 + 3A_1 - 3A_2$$
$$C - 3L_1 - 6A_1 + 3A_2 + 2E$$

Arrangement G



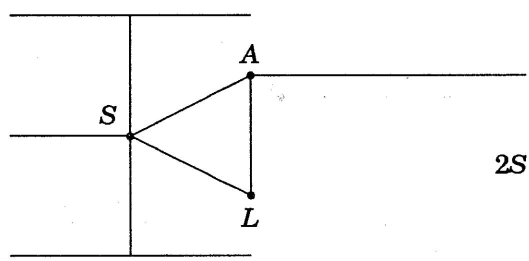
$$C + 3E - 6A - 4B$$
$$Q + 3E - 4A - 2B$$

Arrangement H



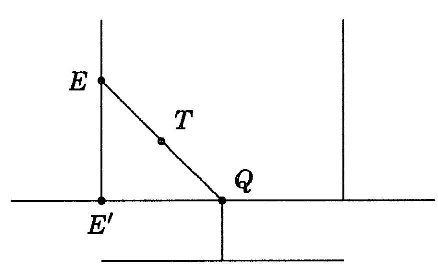
$S - 4T - 6A - 6B$

Arrangement I



$2S - 5L - 8A$

Arrangement J



$Q - 10E' - 6E - 5T$

Arrangement L

4. Chern numbers

In this section we compute the logarithmic Chern numbers of the homology planes which were constructed in section 2. Let X be a non-singular rational projective variety and $D \subset X$ a normal crossing divisor with rational curves $D_1, \ldots, D_u$ as irreducible components. Let $\Gamma(D)$ denote the dual graph of D. We use the following notation:

a_0	number of path-components of $\Gamma(D)$,
a_1	number of cycles of $\Gamma(D)$,
a_2	number of vertices of $\Gamma(D)$.

Let K_X be a canonical divisor of X. The logarithmic Chern numbers of $V = X \setminus D$ are (defined as)

$$\begin{aligned} \bar{c}_2(V) &= e(X) - e(D) && \text{difference of Euler numbers,} \\ \bar{c}_1(V)^2 &= (K_X + D)^2 && \text{self-intersection number.} \end{aligned}$$

Let $b_2 = b_2(X)$ denote the second Betti-number of X. From the homological and combinatorial definition of the Euler characteristic we obtain:

$$\text{(4.1)} \qquad \begin{array}{ll} (1) & e(X) = 2 + b_2(X). \\ (2) & e(D) = a_0 - a_1 - a_2, \end{array}$$

Next we compute $(K_X + D)^2 = K_X^2 + (K_X + D) \cdot D + K_X \cdot D$.

LEMMA (4.2).
(1) $K_X^2 = 10 - b_2(X)$.
(2) $(K_X + D) \cdot D = 2(a_1 - a_0)$.
(3) $K_X \cdot D = -2a_2 - \sum_i D_i^2$.

Proof.
(1) is true for $X = \mathbb{P}^2$ and follows in general by using the change of K_X^2 and $b_2(X)$ under an expansion (HARTSHORNE [1977], V.3.3).
(2). For a smooth rational curve C the adjunction formula says $(K_X + D) \cdot C = -2$ (HARTSHORNE [1977], V.1.5). Using this, (2) is proved by induction on the number of irreducible curves.
(3) is a rewriting of the adjunction formula. □

From (4.1) and (4.2) we obtain (compare GURJAR – SHASTRI [1989]):

$$\text{(4.3)} \qquad \begin{array}{ll} (1) & \bar{c}_2(V) = 2 + b_2 - a_0 + a_1 - a_2, \\ (2) & \bar{c}_1(V)^2 = 10 - b_2 - 2a_0 + 2a_1 - 2a_2 - \sum D_i^2, \\ (3) & 3\bar{c}_2(V) - \bar{c}_1(V)^2 = 4b_2 - 4 - a_0 + a_1 - a_2 + \sum D_i^2. \end{array}$$

We abbreviate the value (4.3.3) by $p(V)$.

PROPOSITION (4.4). *The varieties V which appear in towers of arrangements* (A) – (N) *have* $p(V) = 5$.

Proof. One calculates $p(V)$ by using (4.3.3) for the weighted graphs Figure (A) – (N) and by noting that cutting a cycle increases the number $p(V)$ by one. □

Remark (4.5). For a homology plane V one always has $\bar{c}_2(V) = 3$. Therefore (4.4) is equivalent to $\bar{c}_1(V)^2 = -2$ in the case of homology planes.

Note that (4.4) is a sharpening of the Miyaoka inequality (MIYAOKA [1984], Theorem 1.1) for homology planes. This fact was also observed by FLENNER and ZAIDENBERG. They relate it to the deformation theory of the surfaces. The result seems surprising. So one is tempted to state the following conjecture.

CONJECTURE (4.6). *For every homology plane V of general type the relation* $p(V) = 5$ *holds.*

If a homology plane of general type is obtained from an arrangement of lines and quadrics one can translate $p(V) = 5$ into a combinatorial assertion about the arrangement. E.g., if V is obtained from an arrangement of n lines which has t_r points of multiplicity r, then the relation $p(V) = 5$ amounts to

$$\sum_{r \geq 2} (r-2) t_r = 2n - 8. \tag{4.7}$$

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