Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

FUNCTIONS WHOSE CRITICAL SET CONSISTS OF TWO CONNECTED MANIFOLDS

BY DUAN HAIBAO AND ELMER G. REES

In memory of José Adem

1. Introduction

A famous theorem due to G. Reeb [12] states that if $f : M \to \mathbf{R}$ is a smooth function on a closed manifold with only two (non-degenerate) critical points then M is homeomorphic to a sphere. J. Milnor [9] used this result to show that a certain seven dimensional manifold not diffeomorphic to a sphere was homeomorphic to S^7. Later, the requirement in Reeb's theorem that the critical points be non-degenerate was shown to be unnecessary (references are given in [10]).

In this paper we consider the more general situation where the critical set of the smooth map $f : M \to \mathbf{R}$ has only two components V_1 and V_2 both of which are smooth submanifolds of M. For example, Theorem 1 considers the case where V_2 is a point and $\dim V_1 > 0$, its conclusion is that the topology of the pair (M, V_1) is similar to that of the pair $(\mathbf{F}P^n, \mathbf{F}P^{n-1})$. We do not assume that the critical sets of f are non-degenerate in any sense. The ultimate aim would be to give a complete homotopy characterisation of the possible triples (M, V_1, V_2). The situation that we consider has been previously studied by L. Pontrjagin [11] and he obtained information on the relationship between the various Betti numbers. We exploit the considerable advances in algebraic topology that were started in the 1950's through the work of Jose Adem and others and we show that the topology of V_1 (say) imposes very severe restrictions on M and V_2. Similar methods have already been used by J. Eells and N.H. Kuiper [7] where they studied manifolds that admit a real valued function with three non-degenerate critical points.

Our Theorem 1 is closely related to the Bott-Samelson theorem concerning Blaschke manifolds (see [4, especially Chapter 7] for a thorough discussion); by applying Theorem 1 to the square of the distance from a point, one can recover the Bott-Samelson theorem. The other results in this paper are natural generalisations of Theorem 1 and the methods we use can surely be used to handle other cases.

2. Examples

We give various examples of functions satisfying our hypotheses. In all the cases we know, a function f can be chosen so that the maximum and minimum submanifolds are non-degenerate in the sense of R. Bott [5]. We will be particularly interested in cases where one of the critical manifolds is a real projective space.

1. Let M be the total space of a bundle over a sphere S^k with fibre the manifold V. Let $f : S^k \to \mathbf{R}$ be a function with only two critical points, then the composition of f with the projection map has two copies of V as its critical set.

2. Let M be a real projective space and f be defined by a quadratic form on the associated vector space. The critical set of f consists of a copy of a real projective space $\mathbf{R}P^{k-1}$ for each eigenvalue of f. The dimension k is determined by the fact that it is the dimension of the corresponding eigenspace. If the quadratic form defining f has only two distinct eigenvalues, one obtains an example of the required type. [There are similar examples for projective spaces over $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$.]

3. There is a function $f : S^4 \to \mathbf{R}$ whose critical set is two copies of $\mathbf{R}P^2$; this is described in more detail in [6]. The manifold S^4 is regarded as the space of real symmetric 3×3 matrices A satisfying $\operatorname{tr} A = 0$ and $\operatorname{tr} A^2 = 1$, the function f is defined by $f(A) = \operatorname{tr} A^3$.

4. Let M be $\mathbf{R}P^{k-1} \times \mathbf{R}P^{k-1}$, regarded as a quotient of $S^{k-1} \times S^{k-1}$ and $f(x,y) = (x.y)^2$. Then f attains its minimum (on $S^{k-1} \times S^{k-1}$) on the set $\{(x,y) \mid x.y = 0\}$ which is the Stiefel manifold $V_{k,2}$. This corresponds to a projective Stiefel manifold in M. On the other hand f attains its maximum value on the set where $x = \pm y$ which corresponds to a projective space $\mathbf{R}P^{k-1}$ in M. It can easily be checked, using Lagrange multipliers, that f has no other critical points.

5. Let M be $G_{k,r}$ the Grassmanian of all r dimensional linear subspaces of $\mathbf{R}^k$ and let e be a fixed non-zero vector in $\mathbf{R}^k$. Define f by: $f(\gamma)$ is the maximum value of $(x.e)^2$ as x varies over all unit vectors in γ. By applying the method of Lagrange multipliers to the induced function on the Stiefel manifold $V_{k,r}$ of all orthonormal r-frames in $\mathbf{R}^k$, one can show that the critical set of f consists of

$$G_{k-1,r} = \{\gamma \mid e \perp \gamma\} \text{ and } G_{k-1,r-1} = \{\gamma \mid e \in \gamma\}.$$

In the case $r = 2$, the critical set consists of $G_{k-1,2}$ and $\mathbf{R}P^{k-2}$.

6. Let V_0, V_1 be connected closed manifolds and ξ_0, ξ_1 be vector bundles over them whose sphere bundles are diffeomorphic. Let M be a manifold obtained by using a diffeomorphism between the sphere bundles to glue the disc bundles of ξ_0 and ξ_1 together. Then M clearly admits a function of the required type. All examples where the critical manifolds are non-degenerate can be described in this way and so, in this case the question we are studying can be reduced to understanding which sphere bundles are diffeomorphic. However, it is not clear how to carry this out methodically.

Section 3

THEOREM (1). *Let $f : M \to \mathbf{R}$ be a smooth function defined on a compact manifold such that its critical set consists of a point p and of a connected smooth submanifold V of positive dimension. Then, M has the cohomology ring structure of a projective space and the cohomology of V corresponds to a codimension one projective subspace. More precisely one has one of the following*

(i) *M is homotopically equivalent to the projective space* $\mathbf{F}\mathrm{P}^n$ *and V to* $\mathbf{F}\mathrm{P}^{n-1}$ *with either $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, or*

(ii) *M has the same cohomology ring as $\mathbf{F}P^n$ and V as* $\mathbf{F}\mathrm{P}^{n-1}$ *with either $\mathbf{F} = \mathbf{H}$ or $\mathbf{F} = \mathbf{O}$ (the Cayley numbers) and $n = 2$.*

The simplest case of this theorem is when $V = S^1$ and the conclusion is that M is $\mathbf{R}\mathrm{P}^2$.

Remark. As in [4, p.186], one could say slightly more in the case $\mathbf{F} = \mathbf{R}$, namely, that the Browder-Livesay invariant of M vanishes (but this is essentially the same as the existence of a codimension one subspace V that is homotopy equivalent to a real projective space).

Remark. One cannot deduce that M is diffeomorphic to $\mathbf{F}\mathrm{P}^n$ (or even homotopy equivalent in the cases $\mathbf{F} = \mathbf{H}$ or $\mathbf{O}$) since there are 'exotic' such examples which admit functions of the given type.

First, we prove the following

PROPOSITION (1). *Let W be a compact, connected, smooth submanifold of the manifold M and let $f : M \to \mathbf{R}$ be a proper smooth function whose minimum set is W with $f(W) = 0$. Let $c > 0$ be such that $f^{-1}[0, c]$ contains no critical points of f other than those in W. Then, the inclusion map*

$$W \to f^{-1}[0, c]$$

is a homotopy equivalence.

Proof. Choose a tubular neighbourhood N of W in M such that

$$D = f^{-1}[0, c] \supset N.$$

Since ∂N is compact, one can choose $c' > 0$ such that $c' < f(x)$ for every $x \in \partial N$. Let $D' = f^{-1}[0, c']$. Now choose a smaller tubular neighbourhood N' of W inside D'. The inclusion maps $D' \to D$, and $N' \to N$ are both homotopy equivalences; hence, by Lemma 1 below, so is $D' \to N$. The inclusion map $W \to N$ is also a homotopy equivalence; hence so is $W \to D$ as required.

Clearly, Proposition 1 can also be applied to maxima.

We have used the following well known result.

LEMMA (1). *Let*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$$

be maps between spaces such that $\beta\alpha$ and $\gamma\beta$ are homotopy equivalences. Then, α, β, γ are homotopy equivalences.

Proof. Let $\varphi : C \to A$ and $\psi : D \to B$ be homotopy inverses to $\beta\alpha$ and $\gamma\beta$ respectively. Then β has a homotopy inverse because $\psi\gamma\beta \simeq 1_B$ and $\beta\alpha\varphi \simeq 1_C$ and hence one has that $\psi\gamma \simeq \psi\gamma\beta\alpha\varphi \simeq \alpha\varphi$ is a homotopy inverse for β. Now one can show that $\varphi\beta$ and $\beta\psi$ are homotopy inverses for α and γ respectively. For example $\varphi\beta\alpha \simeq 1_A$ and $\alpha\varphi\beta \simeq \psi\gamma\beta \simeq 1_B$.

With the notation of the proof of Proposition 1 one has

PROPOSITION (2). *$D\backslash\operatorname{int} N'$ is an h-cobordism between ∂D and ∂N.*

Proof. Consider the inclusion maps

$$\partial N' \to D'\backslash\operatorname{int} N' \to N\backslash\operatorname{int} N' \to D\backslash\operatorname{int} N'.$$

Since $D\backslash\operatorname{int} D'$ and $N\backslash\operatorname{int} N'$ are diffeomorphic to $\partial D' \times I$ and $\partial N' \times I$ respectively, one can apply Lemma 1 to conclude that $\partial N' \to D\backslash\operatorname{int} N'$ is a homotopy equivalence. Similarly, by considering

$$\partial D \to D\backslash\operatorname{int} N \to D\backslash\operatorname{int} D' \to D\backslash\operatorname{int} N'$$

one concludes that $\partial D \to D\backslash\operatorname{int} N'$ is also a homotopy equivalence.

We will also need

PROPOSITION (3). *Suppose the sphere S^N is homotopy equivalent to the total space E of a sphere bundle over a manifold B of positive dimension:*

$$S^{d-1} \to E \to B$$

then $d = 1, 2, 4$ or 8 and $N + 1 = d(s + 1)$ for some $s \geq 1$. Moreover, if $d = 8$ then $s = 1$ or 2. The various cases are

$d = 1$ $B \simeq \mathbf{RP}^s$

$d = 2$ $B \simeq \mathbf{CP}^s$

$d = 4$ *B has the same cohomology ring as* $\mathbf{HP}^s$

$d = 8$ *B has the same cohomology ring as* $\mathbf{OP}^s$, $s = 1, 2$.

Proof. If B is not simply connected, then the homotopy exact sequence shows that $d = 1$ and so $E \to B$ is a double cover.

If B is simply connected, then the Gysin sequence of the fibration shows that

$$H^*(B; \mathbf{Z}) \cong \mathbf{Z}[e]/e^{s+1},$$

the truncated polynomial algebra on the Euler class of the bundle. The solution of the Hopf invariant one problem [1] now yields the restrictions on d. The restriction on s when $d = 8$ is proved in [3].

When $d = 1$ or 2 there will be a map $B \to \mathbf{F}P^n$ classifying the given bundle and inducing an isomorphism of cohomology rings. It is easy to check, using Whitehead's theorem, that this map is a homotopy equivalence.

Proof of Theorem 1. Applying Proposition 1 with $W = p$ we see that each intermediate level set of f is a sphere, hence the sphere bundle of the normal bundle of V is also a sphere. Proposition 3 then implies that V is a homology projective space and since M is obtained by taking a cone on the boundary of the neighbourhood of V one sees that M is a homology projective space of one higher dimension. Since a space with the cohomology of a "Cayley projective space of dimension three" cannot exist [3], one sees that the case where V is like $\mathbf{O}P^2$ cannot occur.

Section 4

We now consider a more general situation. Let $f : M \to \mathbf{R}$ be a smooth function defined on a closed manifold M and whose critical set consists of two smooth submanifolds V_0 and V_1. We can arrange that $f(V_0) = 0$ and $f(V_1) = 1$ and choose tubular neighbourhoods N_0, N_1 of V_0 and V_1 respectively, such that $N_0 \subset f^{-1}[0, 1/2[$, $N_1 \subset f^{-1}]1/2, 1]$. Using Proposition 2 we see that $M \backslash \{\mathrm{int}(N_0) \cup \mathrm{int}(N_1)\}$ is an h-cobordism between ∂N_0 and ∂N_1.

PROPOSITION (4). *The inclusion map $V_0 \to M$ is a* $(\dim M - \dim V_1 - 1)$ *equivalence.*

Proof. By a general position argument, one shows that every element in $\pi_k(M)$ has a representative that misses V_1 if $k < \dim M - \dim V_1$ and so lies in some $f^{-1}[0, c]$ for $0 < c < 1$. Using Proposition 1 one sees that it is homotopic to a map whose image lies in V_0. Hence $\pi_k(V_0) \to \pi_k(M)$ is surjective for $k < \dim M - \dim V_1$. Similarly it is injective for $k < \dim M - \dim V_1 - 1$.

Section 5

In this section we consider a particular case of the situation of §4 namely, that in which the critical set consists of $\mathbf{R}P^k$ and a connected manifold V of dimension t with $0 < t < k$. We will show that there are only three types of example, which we now describe.

Type I. This type is modelled on Example 2 of §1. M is homotopy equivalent to $\mathbf{R}P^{k+t+1}$ and V to $\mathbf{R}P^t$; the inclusions of V and $\mathbf{R}P^k$ in M are homotopic to the standard inclusions.

For the other two types, it is convenient to consider the general level set L of $f : M \to \mathbf{R}$; that is, the inverse image of a regular value of f. The manifold L is a sphere bundle over both $\mathbf{R}P^k$ and V. For example, in the standard case of Type I, L is diffeomorphic to both $S(t\lambda_k)$ and $S(k\lambda_t)$ where λ_r denotes the canonical real line bundle over $\mathbf{R}P^r$.

Type II. An example of this type is one that has the same cohomology as one of the following examples. M is the connected sum of $\mathbf{R}P^{k+1}$ and $\mathbf{F}P^r$ where

$k+1 = dr$ ($d = \dim_{\mathbf{R}} \mathbf{F}$) ; $r \geq 2$ for $d = 2, 4$ and $r = 2$ for $d = 8$; V is $\mathbf{F}\mathrm{P}^{r-1}$ and L is S^k. The function on M is obtained by considering quadratic functions on $\mathbf{R}\mathrm{P}^{k+1}$ and $\mathbf{F}\mathrm{P}^r$ whose critical sets consist of a point and a co-dimension one subspace. Choose ball neighbourhoods of the single critical points whose boundaries are level sets, remove them from both manifolds and take M to be the connected sum; the function on M is then constructed in the obvious way.

Type III. First we describe some examples. Let M_1 denote the total space of a smooth bundle with base $\mathbf{F}\mathrm{P}^r$ and fibre $\mathbf{R}\mathrm{P}^s$. Consider a function $f_1 : M_1 \to \mathbf{R}$ obtained by composing the bundle projection with a quadratic function on $\mathbf{F}\mathrm{P}^r$ whose critical set consists of a point and $\mathbf{F}\mathrm{P}^{r-1}$. Let $f_2 : M_2 = \mathbf{R}\mathrm{P}^{s+dr} \to \mathbf{R}$ be a quadratic function whose critical set consists of $\mathbf{R}\mathrm{P}^s$ and $\mathbf{R}\mathrm{P}^{dr-1}$. The integers s and dr are chosen so that the normal bundle of $\mathbf{R}\mathrm{P}^s \subset \mathbf{R}\mathrm{P}^{s+dr}$ is trivial. Then M is formed by glueing M_1 to M_2 along level sets of f_1 and f_2 after removing suitable neighbourhoods of a fibre from M_1 and of $\mathbf{R}\mathrm{P}^s$ from M_2. The function f can now be easily constructed by modifying f_1 and f_2 suitably, its critical set consists of $\mathbf{R}\mathrm{P}^{dr-1}$ and the total space V of an $\mathbf{R}\mathrm{P}^s$ bundle over $\mathbf{F}\mathrm{P}^{r-1}$ and the level set L is diffeomorphic to $\mathbf{R}\mathrm{P}^s \times S^{dr-1}$. The restrictions on the integers s, d, r that ensure that $0 < \dim V < k = dr - 1$ and that $\mathbf{R}\mathrm{P}^s$ has trivial normal bundle in $\mathbf{R}\mathrm{P}^{s+dr}$ are:

either $d = 4$, $r \geq 2$ and $s = 1$ or 2

or $d = 8$, $r = 2$ and $1 \leq s \leq 6$.

A function $f : M^n \to \mathbf{R}$ is of Type III if it has certain features in common with the examples just described. Namely, if the critical set consists of $\mathbf{R}\mathrm{P}^k$ and V^t, that there are integers d, r, s satisfying

(i) either $d = 4$ and $r \geq 2$ or $d = 8$ and $r = 2$;

(ii) $n = dr + s$, $k = dr - 1$ and $t = n - d$;

(iii) $\pi_1(\mathbf{R}\mathrm{P}^k) \to \pi_1(M)$ is an isomorphism and either $\pi_1(V) \to \pi_1(M)$ is an isomorphism (when $s > 1$) or it is the epimorphism $\mathbf{Z} \to \mathbf{Z}/2$(when s $= 1$)} ;

(iv) the normal bundle ν of $\mathbf{R}\mathrm{P}^k$ in M has non-zero mod 2 Euler class;

(v) the additive cohomology of the double cover $\tilde{L}$ of L is isomorphic to $H^*(S^k \times S^s)$.

THEOREM (2). *Let $f : M \to \mathbf{R}$ be a smooth function defined on the compact smooth manifold M and whose critical set consists of* $\mathbf{R}\mathrm{P}^k$ *and a connected manifold V^t with $0 < t < k$. Every other level set of M is diffeomorphic to a manifold denoted by L. Then the manifolds M, V, L are described by one of the above three types.*

COROLLARY. *Let $f : M \to \mathbf{R}$ be a smooth function whose critical set consists of* $\mathbf{R}\mathrm{P}^k$ *and* $\mathbf{R}\mathrm{P}^s$ *with $k \neq s$. Then M is homotopy equivalent to* $\mathbf{R}\mathrm{P}^{k+s+1}$.

Of course, if $k = s$ there are several other possibilities, for example M could be any bundle over a sphere with fibre $\mathbf{R}\mathrm{P}^k$ or it could be example 3 of §2.

6. Proof of Theorem 2

Let h and s denote the codimensions of $\mathbf{RP}^k$ and V^t respectively; therefore $h+k=s+t=\dim M$, and $s>h$ since $t<k$.

First, we consider the case where $h=1$. If the normal bundle of $\mathbf{RP}^k$ is trivial, then L is diffeomorphic to $\mathbf{RP}^k\times S^0$. However, L is also an S^{s-1} bundle over V and $s-1>0$ since $s>h=1$; this implies that L is connected — a contradiction. Therefore the normal bundle must be the non-trivial line bundle over $\mathbf{RP}^k$; hence L is homotopy equivalent to S^k. By Proposition 3, V is a cohomology projective space and the situation is of Type II (we are assuming $\dim V>0$).

Now we consider the case $h>1$.

PROPOSITION (5). *Under the given hypotheses, the (mod 2) Euler class of the normal bundle ν of $\mathbf{RP}^k$ in M is non-zero, and hence, $h\le k$.*

Proof. By Proposition 4, the inclusion map $\mathbf{RP}^k\to M$ is an $(s-1)$-equivalence and we have that $s\ge 3$; hence the generator $x\in H^1(\mathbf{RP}^k)$ is the restriction of a class $y\in H^1(M)$ and so $y^k\ne 0$. If the Euler class of ν vanished, then one would have an injection $\pi^*:H^*(\mathbf{RP}^k)\to H^*(L)$ induced by the projection $\pi:L\to\mathbf{RP}^k$ and so $z=\pi^*x$ would satisfy $z^k\ne 0$. Consider the diagram

$$\begin{tikzcd}
L \arrow[r, "\pi"] \arrow[d, "\pi_V"'] & \mathrm{RP}^{\mathrm{k}} \arrow[d, "j"] \\
V \arrow[r, "i"] & M
\end{tikzcd}$$

It is commutative up to homotopy because both compositions are homotopic to the inclusion of L in M. Hence $z=\pi^*j^*y=\pi_V^*i^*y$ and $z^k\ne 0$ so $(i^*y)^k\ne 0$; this contradicts the fact that $\dim V<k$ and so proves the result.

PROPOSITION (6). *Under the above hypotheses one has $h\le t+1$.*

Proof. By our hypotheses we have $2\le t+1\le k$; assume also that $h>t+1$. Since $t+1\le k$ there is a linear inclusion $g:\mathbf{RP}^{t+1}\to\mathbf{RP}^k$. Let ν denote the normal bundle of $\mathbf{RP}^k$ in M, its dimension is $h>t+1$, so its pullback $g^*\nu$ has a nowhere vanishing section. Hence, the composition

$$\mathbf{RP}^{t+1}\to\mathbf{RP}^k\to M$$

is homotopic to a map g' which factors through L and hence through a neighbourhood $N(V)$ of V. Since we are assuming that $h-1>t$ and so $h-1\ge 2$ we can use Proposition 4 to deduce that the inclusion $V\to M$ induces an isomorphism on fundamental groups; similarly for $\mathbf{RP}^k\to M$, since $s-1\ge 3$.

Now consider the diagram

$$\begin{tikzcd} \mathbf{R}P^{t+1} \arrow[r, "g'"] \arrow[d] & V \arrow[d, "i"] \\ \mathbf{R}P^{k} \arrow[r, "j"] & M \end{tikzcd}$$

which commutes up to homotopy. Since the composition $\mathbf{R}P^{t+1} \to M$ induces an isomorphism

$$\mathbf{Z}/2 = H^1(M) \to H^1(\mathbf{R}P^{t+1});$$

then, $(i^*y)^{t+1} \neq 0$ where $y \in H^1(M)$ is the generator. This contradicts the fact that $\dim V = t$ and hence the result is proved.

Now we consider the case $h = t + 1$. We will conclude that we are in the situation of Type I.

By Proposition 4, the inclusions $\mathbf{R}P^k \to M$ and $V^t \to M$ are k and t equivalences respectively. Hence V is a manifold of dimension t having its homotopy groups isomorphic to those of $\mathbf{R}P^k$ in dimensions less than t. So V must be homotopy equivalent to $\mathbf{R}P^t$.

The inclusion $\mathbf{R}P^k \to M^{k+h}$ is a k-equivalence; so if $k > h$, M^{k+h} is homotopy equivalent to $\mathbf{R}P^{k+h}$ by Poincaré duality.

Now assume also that $k = h$. Consider the double cover $\tilde{M}$ of M. There is a k-equivalence $j : S^k \to \tilde{M}$. If j is non-trivial on π_k, then $\pi_k(\tilde{M}) = \mathbf{Z}$ and $\tilde{M}$ has non-zero homology in dimensions 0, k and 2k only. Hence $\tilde{M}$ has the fixed point property but it admits a free involution (whose quotient space is M). Hence $\pi_k\tilde{M} = 0$ and $\tilde{M}$ is a homotopy sphere, so M is a homotopy projective space, as required.

The cases that remain have the following properties.

(i) $2 \leq h \leq t < k$;
(ii) $h < s \leq k$;
(iii) the normal bundle of $\mathbf{R}P^k$ in M has non-trivial mod 2 Euler class;
(iv) $\mathbf{R}P^k \to M$ is an $s - 1$ equivalence; so $\pi_1 M \cong \mathbf{Z}/2$; and
(v) $V \to M$ is an $h - 1$ equivalence.

We will need the following result.

PROPOSITION (7). *Let X be a space whose cohomology is isomorphic to*

$$\mathbf{Z}[b]/(b^{r+1}) \otimes \Lambda(a)$$

with $\dim b = s \geq 3$, $\dim a \leq s - 2$ *and* $r \geq 2$. *Then either* $s = 4$ *or* $s = 8$ *and* $r = 2$.

Proof. We first consider cohomology with mod 2 coefficients. The Adem relations [13, p.7] show that Sq^s is decomposable in terms of Steenrod operations if s is not a power of 2. J.F. Adams [1] showed that Sq^s is decomposable in terms of secondary operations if s is a power of 2 and $s \geq 16$. So, when $s \geq 5$ and $s \neq 8$ one has $\mathrm{Sq}^s = \sum_i \alpha_i \Phi_i$ with α_i an element of the Steenrod algebra and Φ_i a secondary operation. Hence, since $r \geq 2$

$$0 \neq b^2 = \mathrm{Sq}^s(b) = \sum_i \alpha_i \Phi_i(b)$$

one has that $\Phi_i(b) = ab$ and $\alpha_i(ab) = b^2$ for some i. Since $H^j(X)$ vanishes for $\dim(ab) < j < \dim(b^2)$, α_i is indecomposable and so α_i must be Sq^k for one of $k = 2, 4$ or 8. Using the Cartan formula one obtains

$$0 = \mathrm{Sq}^{2k}(ab)^2 = \mathrm{Sq}^k(ab)\mathrm{Sq}^k(ab) = b^2$$

which is a contradiction. Hence $s = 4$ or 8.

It remains to show that $r = 2$ in the case $s = 8$. We consider $H^*(X; \mathbf{Z}/3)$ and use [13, pp.72, 73] as a reference for the properties of the Steenrod operations P^i. Since $s = \dim b = 8$ one has $P^4(b) = b^3$ and using the Adem relation $P^4 = P^1P^3$ one sees that $P^3(b) = \pm ab^2$ and $P^1(ab^2) = \pm b^3$; hence $\dim a = 4$. By the Cartan formula,

$$P^1(ab^2) = P^1(a)b^2 + 2aP^1(b)^2$$

But $P^1(b) = \lambda ab$ so the second term vanishes; hence $P^1(a) = \pm b$. Using the Cartan formula and $a^2 = 0$, one obtains

$$0 = P^2(a^2) = 2aP^2(a) + P^1(a)^2 = 2a^4 + b^2 = b^2\,;$$

this is a contradiction, hence $r = 2$.

We now return to the main proof.

First consider the case $h = 2$ and (so) $k \geq 3$, then, by Proposition 4, the map $\pi_1(V) \to \pi_1(M)$ induced by inclusion is onto. The normal bundle ν of $\mathbf{R}\mathrm{P}^k$ in M is 2-dimensional and, by (iii) above, $w_2(\nu) \neq 0$. Such bundles have been classified (by J. Levine [8] for $k = 3$ and 4, and by J.F. Adams [2, Prop.4] for $k \geq 5$), the only possibility being that ν is isomorphic to 2λ, where λ is the canonical line bundle over $\mathbf{R}\mathrm{P}^k$. Hence L is diffeomorphic to $(S^1 \times S^k)/\mathbf{Z}/2$, where the action is antipodal on both factors. Also L fibres over V with fibre S^{s-1} where $s > 2$, so $\pi_1(V) \cong \pi_1(L) \cong \mathbf{Z}$.

Let $\tilde{L}$ and $\tilde{V}$ be the connected double covers of L and V respectively. One has the fibration

$$S^{s-1} \to \tilde{L} \to \tilde{V}.$$

Let $b \in H^s(\tilde{V})$ denote the transgression class of this fibration; then, using the Serre spectral sequence one obtains an isomorphism

$$H^*(\tilde{V}) \cong \mathbf{Z}[b]/b^{r+1} \otimes \Lambda(a) \text{ where } \dim a = 1.$$

By Proposition 7 one has that, either $s = 4$ and $r \geq 2$, or $s = 8$ and $r = 2$. Hence we have the situation of Type III.

The case $h \geq 3$ is easier because both inclusions $\mathbf{R}P^k \to M$ and $V \to M$ induce isomorphisms on fundamental groups. Let $\tilde{M}$ denote the (universal) double covering of M and $\tilde{L}$ the corresponding covering of the level set L. There is a fibration

$$S^{h-1} \to \tilde{L} \to S^k$$

and, since $h < k$, there is an isomorphism

$$H^*(\tilde{L}) \cong H^*(S^k \times S^{h-1}).$$

Now, we consider the bundle

$$S^{s-1} \to \tilde{L} \to \tilde{V}$$

where $\tilde{V}$ denotes the (universal) double cover of V. If the transgression for this bundle is zero, then the Serre spectral sequence collapses and one has that

$$H^*(\tilde{V}) \otimes H^*(S^{s-1}) \cong H^*(S^k) \otimes H^*(S^{h-1})$$

and so either $s - 1 = k$ or $s - 1 = h - 1$. Both these possibilities are, however, ruled out by our hypotheses. So we have that the transgression is non-zero; if $b \in H^s(\tilde{V})$ denotes the image of the fundamental class, then

$$H^*(\tilde{V}) \cong \mathbf{Z}[b]/b^{r+1} \otimes \Lambda(a)$$

for some r and $\dim a = h - 1$. As before we are in the situation described by Type III.

DEPARTMENT OF MATHEMATICS
PEKING UNIVERSITY
BEIJING 100871
PEOPLE'S REPUBLIC OF CHINA.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF EDINBURGH
THE KING'S BUILDINGS
EDINBURGH EH9 3JZ
SCOTLAND.

REFERENCES

[1] J.F. ADAMS *On the non-existence of elements of Hopf invariant one*, Ann. of Math (2) **72** (1960), 20–104.

[2] J.F. ADAMS *Geometric dimension of bundles over RP^n*, Pub. Res. Inst. Math. Sci. Kyoto **9** (1974), 1-17.

[3] J. ADEM *Relations on iterated reduced powers* Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 636-638.

[4] A. BESSE Manifolds all of whose geodesics are closed, Springer Verlag (1978).

[5] R. BOTT *Nondegenerate critical manifolds* Ann. of Math. (2) **60** (1954) 248-261.

[6] H. DUAN AND E. REES *Bitangent spheres and critical points* LMS Lecture Notes No.139, Advances in homotopy theory, C.U.P.(1989).

[7] J. EELLS AND N.H. KUIPER *Manifolds that are like projective planes* Inst. Hautes Etudes Sci. Publ. Math. No. 14 (1962), 5-46.

[8] J. LEVINE *Spaces with involution and bundles over P^n*, Amer. J. Math **85** (1963), 516-540.

[9] J. MILNOR *On manifolds homeomorphic to the 7-sphere* Ann of Math. (2) **64** (1956), 399-405.

[10] J. MILNOR *Morse theory* Ann. of Math. Studies No. 51, Princeton Univ. Press, (1963).

[11] L. PONTRJAGIN *A method of calculation of homology groups* Rec. Math. [Mat. Sbornik] N.S. **11** (53) (1942), 3-14.

[12] G. REEB *Sur certaines propriétés topologiques des variétés feuilletées* Actualités Scientifiques et Industrielles 1183, Hermann, Paris (1952).

[13] N.E. STEENROD AND D.B.A. EPSTEIN *Cohomology operations* Ann. of Math. Studies No. 50, Princeton Univ. Press, (1962).