Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

CATEGORY WEIGHT AND STEENROD OPERATIONS

BY EDWARD FADELL AND SUFIAN HUSSEINI

1. Introduction

If $u_1, \ldots, u_n$ are cohomology classes of positive dimension of a space X (over some coefficient system R) and their (cup) product $w = u_1 u_2 \ldots u_n \neq 0$, then the *length* $l(w)$ of w is said to be n. Then, $l(X)$ is defined to be $\max l(w)$ for all w of positive dimension. Then, an estimate of the Lusternik-Schnirelmann category of X, *cat* X, is classically given by *cat* $X \geq 1 + l(X)$. Notice that each cohomology class in the product $u_1 u_2 \ldots u_n$ is treated equally: namely, having "weight" 1. Our objective in this note is to observe that some indecomposable cohomology classes carry more weight than others, where category is concerned. We define category weight (*cwgt*) for $0 \neq u \in \tilde{H}^*(X; R)$ by saying that $cwgt(u) \geq k$ if for every closed subset $A \subset X$ with $cat_X A \leq k$, $u|A = 0$, i.e., u restricted to A is zero. Then, $cwgt\ u = k$ if k is maximal with $cwgt\ u \geq k$.

THEOREM (1.1). *If* $w = u_1 u_2 \ldots u_n \neq 0$ *in* $\tilde{H}^*(X; R)$, *then*

$$cwgt\ (w) \geq \sum_i cwgt\ (u_i), \tag{1}$$

and therefore

$$cat\ X \geq 1 + cwgt\ (w). \tag{2}$$

Our first general situation where $cwgt\ (u) > 1$ is given by the following.

THEOREM (1.2). *If* β *is the* mod p *Bockstein, and* $u \in H^1(X; \mathbb{Z}_p)$, *then* $\beta u \neq 0$ *implies* $cwgt\ (\beta u) \geq 2$.

These results yield the following generalization of a theorem of Krasnoselski ([8]) who proved it for the case of a single odd sphere S^{2n+1}.

THEOREM (1.3). *Let* M *denote the product of* k *copies,* $k \geq 1$, *of the odd sphere* S^{2n+1}, *and let* p *denote an odd prime. Suppose* $G = (\mathbb{Z}_p)^k$ *acts freely on* M *so that the induced action on the* $\mathbb{Z}_p$*-cohomology of* M *is trivial. Then, the category of* M/G *is the maximum possible, i.e.,* $cat\ M/G = 1 + \dim M/G$.

A variation of this result for free actions of $(\mathbb{Z}_p)^k$ on the complex Stiefel manifold $O_{n,k}(\mathbb{C})$ (see Theorem (3.8)) is also obtained using Theorems (1.1) and (1.2). If St_p^I is a mod p Steenrod operation, it is natural to inquire about the category weight of a non-zero cohomology class of the form $St_p^I u$.

THEOREM (1.4). *Let* $e(I)$ *denote the excess of the Steenrod operator* St_p^I. *If* $\dim u = e(I) \geq 2$ *and* $St_p^I u \neq 0$, *then* $cwgt\ (St_p^I u) \geq 2$.

Theorems (1.2) and (1.4) may be summarized briefly by saying that the Bockstein β is *universal* in dimension 1 for *cwgt* ≥ 2 and the Steenrod operation St_p^I is also universal in dimension $e(I) \geq 2$ for *cwgt* ≥ 2. Of course, it is possible for *cwgt* $(St_p^I u) \geq 2$ and $\dim u > e(I)$ for some particular class u. For example, if $X = Sp(2)$ and $p = 3$, we show that if $x_3 \in H^3(Sp(2); \mathbb{Z}_3)$ is a generator, *cwgt* $(P_3^1 x_3) \geq 2$, while the excess in this case is 2. This fact follows from a result of Schweitzer [10] that *cat* $Sp(2) = 4$.

2. Preliminaries

We let X denote an ANR (sep. metric) and take $A \subset X$. Then, $cat_X A \leq k$ if A can be covered by k sets $U_1, \ldots, U_k$, open in X such that U_i is contractible in X. Then $cat_X A = k$ if k is minimial with this property. There are numerous reformulations of this concept due to (e.g.) A. Svarc [12], G. Whitehead [13], T. Ganea [5]. We review those given by A. Svarc [12] for later use. The first is based on the fiberwise join of two (Hurewicz) fibrations over X. If $p_1 : E_1 \to X, p_2 : E_2 \to X$ are fibrations over X, then "the" fiberwise join (called the "sum" of two fiberings in [10]) will be denoted by $p : E_1 *_X E_2 \to X$. If $p_1^{-1}(x) = F_1$ and $p_2^{-1}(x) = F_2$, then the fiber $p^{-1}(x) = F_1 * F_2$, the usual join of the spaces F_1 and F_2. When $E = E_1 = E_2$, we denote the fiberwise join by $E^{*_X 2}$ and the usual join $F * F$ by F^{*2}. Then, we have a sequence of fibrations over X

$$(S) \qquad \begin{array}{ccccccccc}
F & \longrightarrow & F^{*2} & \longrightarrow & \cdots & \longrightarrow & F^{*k} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
E & \longrightarrow & E^{*_X 2} & \longrightarrow & \cdots & \longrightarrow & E^{*_X k} & \longrightarrow & \cdots \\
{\scriptstyle p}\downarrow & & {\scriptstyle p_2}\downarrow & & & & {\scriptstyle p_k}\downarrow & & \\
X & \xrightarrow{id} & X & \xrightarrow{id} & \cdots & \xrightarrow{id} & X & \xrightarrow{id} & \cdots
\end{array}$$

where, e.g., $E^{*_X k} = E^{*_X (k-1)} *_X E$, and the horizontal maps are injections. The (Serre) fibration

$$(S_\infty) \qquad F^{*\infty} \longrightarrow E^{*_X \infty} \xrightarrow{p_\infty} X$$

is obtained by setting $F^{*\infty} = \bigcup_k F^{*k}$ and $E^{*_X \infty} = \bigcup_k E^{*_X k}$, both with the inductive limit (weak) topology. The basic relation of (S) to category is the following.

THEOREM (2.1) [12, Theorems 3 and 18]. *If* $A \subset X$, *let* $E_A = \mathcal{P}(X, A, x_0)$ *denote the space of paths* α *in* X *such that* $\alpha(0) = x_0 \in A, \alpha(1) \in A$. *Then, the sequence* (S) *for* $\Omega X \to E_A \xrightarrow{p} A$, *where* $p(\alpha) = \alpha(1)$, *has the property that* $cat_X A \leq k$ *if, and only if,* $p_k : (E_A)^{*_X k} \to A$ *has a section.*

Remark (2.2). Since $F^{*\infty}$ in (S_∞) is contractible, p_∞ is a homotopy equivalence.

A useful proposition in [12,Theorem 21] is the following.

PROPOSITION (2.3). $E^{*_X 2} = E *_X E \xrightarrow{p_2} X$ *may be identified (up to homotopy) with the map* $q : S\Omega X \to X$ *where* $q(\alpha, t) = \alpha(t), \alpha \in \Omega(X)$. *More precisely, there is a homotopy equivalence* ν

$$\begin{array}{ccc} E *_X E & \xrightarrow{\nu} & S\Omega X \\ p_2 \downarrow & & \downarrow q \\ X & \xrightarrow{\text{id}} & X \end{array}$$

with $q\nu = p_2$.

The next useful sequence is the Milnor sequence [7] of principal G-bundles

$$(M) \qquad \begin{array}{ccccccccc} G & \longrightarrow & E_2(G) & \longrightarrow & \cdots & \longrightarrow & E_k(G) & \longrightarrow & \cdots \\ q_1 \downarrow & & q_2 \downarrow & & & & q_k \downarrow & & \\ * & \longrightarrow & B_2(G) & \longrightarrow & \cdots & \longrightarrow & B_k(G) & \longrightarrow & \cdots \end{array}$$

where $E_k(G) = G^{*k}$ and the limit is the universal G-bundle $G \to E_\infty(G) \to B_\infty(G)$. The connection with category is the following. First, there is the Svarc concept of "genus" of a fibration $\mathcal{B} : F \to E \to B$. Namely, genus $\mathcal{B} \leq k$, if B admits an open covering $\{U_1, \ldots, U_k\}$ and maps $\sigma_j : U_j \to E$ such that $p\sigma_j(x) = x$ for $x \in U_j, j = 1, \ldots, k$. Then, genus $\mathcal{B} = k$ if genus $\mathcal{B} \leq k$ and k is minimal with this property.

THEOREM (2.4) [12,Theorem 9]. *Let* $\mathcal{B} : G \to E \to X$ *denote a principal G-bundle. Then, genus* $\mathcal{B} \leq k$ *if, and only if, there is a G-bundle map*

$$\begin{array}{ccc} E & \longrightarrow & E_k(G) \\ \downarrow & & \downarrow q_k \\ X & \longrightarrow & B_k(G) \end{array}$$

Now, suppose X is a locally finite simplicial complex. Let x_0 denote a vertex of X and let $\mathcal{P}_\sigma(X,x_0) = \mathcal{P}_\sigma(X)$ and $\Omega_\sigma(X,x_0) = \Omega_\sigma(X)$ denote the Milnor simplicial analogues [6] of $\mathcal{P}(X,x_0)$ and $\Omega(X,x_0)$. Then, $\Omega_\sigma(X) \to \mathcal{P}_\sigma(X) \xrightarrow{p'} X$, where $p'(\alpha) = \alpha(1)$, is a principal G-bundle, with $G = \Omega_\sigma(X)$, that is fiber homotopy equivalent to $\Omega(X) \to \mathcal{P}(X) \xrightarrow{p} X$. Let $\mathcal{B}_A : \mathcal{P}_\sigma(X,A) \to A$ denote the $\Omega_\sigma(X)$-bundle over X induced by the inclusion $A \to X$. Then, we have the following proposition.

PROPOSITON (2.5) [10, Theorem 19]. *Genus $\mathcal{B}_A = cat_X A$ and hence $cat_X A \leq k$ if, and only if, there is a G-bundle map*

$$\begin{array}{ccc} \mathcal{P}_\sigma(X,A) & \longrightarrow & E_k(G) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B_k(G) \end{array}$$

where $G = \Omega_\sigma(X)$.

3. Category weight of cohomology classes

Let X denote a 0-connected space, which is ANR (sep. metric) throughout, R a coefficient ring (e.g. a P.I.D.), and $H^*(X;R)$ the cohomology of X over R. If $u \in H^q(X,R)$ and $j : A \to X$ is an inclusion map, then we denote $j^*(u)$ by $u|A$, the restriction of u to A. $C_k(X)$ will denote the family of closed subsets of $A \subset X$ such that $cat_X A \leq k$.

Definition (3.1). Let $u \in H^q(X,R)$ denote a non-zero cohomology class with $q \geq 1$. Define $cwgt\ (u) \geq k$ to mean that $u|A = 0$ for every $A \in C_k(X)$. Then, set

$$cwgt\ (u) = \max\{k : cwgt\ (u) \geq k\}.$$

Remark (3.2). $cwgt\ (u)$ may be infinite if $cat\ X = \infty$. Also, $cwgt\ (u) \geq k$ implies $cwgt\ (u) \geq k - 1$.

Example (3.3). Let M denote a compact triangulate manifold of dimension m with $cat\ M = m + 1$ (i.e. M has maximal category). Let μ denote the fundamental class of M (over $\mathbb{Z}$ if M is orientable, otherwise over $\mathbb{Z}_2$). Then $cwgt\ (\mu) = m$ as follows. Let $cat_X A \leq m$. Then $A \neq M$ and hence $A \subset M - x_0$. But $M - x_0$ is deformable into the $(m-1)$-skeleton of M and $\mu|A = 0$. Thus, we see that there are cohomology classes of arbitrarily high category weight. Of course, the class μ may be decomposable. However, there are cases where it is indecomposable. For example, let $M = S^3/G$, where G is a perfect group acting freely on S^3. The fundamental class μ of M is indecomposable because M is a homology sphere. On the other hand, one can show (using Theorem (3.7)) that $cat\ M = 4$, thus obtaining an indecomposable class μ of category weight equal to 3. Theorems (3.6) and (3.12) below will show, more generally, how to find indecomposable classes of category weight greater than 1, by means of cohomology operations.

Observation (3.4). If u is any non-zero cohomology class of dimension ≥ 1, then

$$1 \leq cwgt\ (u) \leq cat\ X - 1.$$

The main property of category weight is the following analogue of the classical result relating cup length and category.

THEOREM (3.5). *If u_i, $1 \leq i \leq n$, are non-zero cohomology classes of positive dimension and the cup product $u_1u_2\ldots u_n \neq 0$, then*

$$cwgt\ (u_1u_2\ldots u_n) \geq \sum_i cwgt\ (u_i), \tag{1}$$

and hence

$$cat\ X \geq cwgt\ (u_1u_2\ldots u_n) + 1 \geq 1 + \sum_i cwgt\ (u_i). \tag{2}$$

Proof. We need only prove (1). Let $u = u_1u_2\ldots u_n, k_i = cwgt\ (u_i)$ and $k = \sum_i k_i$. Assume first that k is finite. Let $A \in C_k(X)$ and let $\Omega = \{F_1, \ldots, F_k\}$ denote a categorical closed cover of A, with $F_i \subset A$, i.e., each F_i is contractible to a point in X. Partition Ω into n families $\Omega_1, \ldots, \Omega_n$ such that Ω_i has k_i closed sets and let

$$A_i = \bigcup_{F \in \Omega_i} F.$$

Then, $cat_X A_i \leq k_i$ and $u_i|A_i = 0$. Then, by a standard argument, $u|A = 0$ on $A = \cup A_i$ and $cwgt\ (u) \geq k$. In case some k_i, say k_1, is infinite, let A be a set such that $cat_X A \leq k$ for k arbitrarily large. Let $\Omega = \{F_1, \ldots, F_k\}$ as above and partition Ω into n families $\Omega_1, \ldots, \Omega_n$, where each $\Omega_i, i \geq 2$ has one of the F_i's and Ω_1 has all the rest, and let A_i be defined as above. Then, $u_i|A_i = 0$ for all i and hence $u|A = 0$. Thus, $cwgt\ (u) \geq k$ and hence $cwgt\ (u) = \infty$.

Our next result illustrates that there is a large class of cohomology classes of category weight greater than 1 derived from a cohomology operation.

THEOREM (3.6). *Let X denote a space, p a prime, and $\beta : H^q(X; \mathbb{Z}_p) \to H^{q+1}(X; \mathbb{Z}_p)$ the Bockstein homomorphism. Then, if $u \in H^1(X; \mathbb{Z}_p)$ and $\beta u \neq 0$, then $cwgt\ (\beta u) \geq 2$.*

Proof. Let $f : X \to K(\mathbb{Z}_p, 1)$ be such that $f^*(\iota_1) = u$. If we set $Y = K(\mathbb{Z}_p, 1)$, then ΩY consists of p components $C_1, \ldots, C_p$, each of which is contractible and hence the suspension $S\Omega Y$ has the same homotopy type as a wedge of circles. Thus, $H^q(S\Omega Y; \mathbb{Z}_p) = 0$ for $q \geq 2$. Now let A denote a closed subset of X such that $cat_X A \leq 2$. We may also assume that X is 0-connected. Let (for a base point in $A \subset X$)

$$E = \mathcal{P}X, E_A = \mathcal{P}(X,A), E_Y = \mathcal{P}Y.$$

Then, we have a diagram of fibrations (from the Svarc sequence (S) in §2)

$$\begin{array}{ccccccc} E_A & \longrightarrow & E_A *_A E_A & \xrightarrow{\bar{i}*\bar{i}} & E *_X E & \xrightarrow{\bar{f}*\bar{f}} & E_Y *_Y E_Y \\ \downarrow & & \downarrow p & & \downarrow & & \downarrow q \\ A & \xrightarrow{id} & A & \xrightarrow{i} & X & \xrightarrow{f} & Y \end{array}$$

Using the diagram,

$$p^*(\beta u|A) = p^* i^* f^* \beta(i_1) = (\bar{i} \circ \bar{i})^* (\bar{f} \circ \bar{f})^* q^*(\beta i_1) = 0$$

because $E_Y *_Y E_Y \sim S\Omega Y$. On the other hand $cat_X A \leq 2$ implies that p admits a section. Hence $\beta u|A = 0$ and $cwgt\ (\beta u) \geq 2$.

Alternate Proof. We use some of the notation above and assume (without loss of generality; see [6]) that X and Y are locally finite simplicial complexes and $f : X \to Y$ is simplicial. Then, using Milnor paths and loops (§2), we have Milnor sequences with $G = \Omega_\sigma(X), G' = \Omega_\sigma(Y)$ and a diagram of principal fibrations

$$\begin{array}{ccccccccccc} E_2G & \longrightarrow \cdots \longrightarrow & E_\infty G & \longrightarrow & \mathcal{P}_\sigma(X) & \longrightarrow & \mathcal{P}_\sigma(Y) & \longleftarrow & E_\infty G' & \longleftarrow \cdots \longleftarrow & E_2G' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_2G & \longrightarrow \cdots \longrightarrow & B_\infty G & \xrightarrow{\alpha} & X & \xrightarrow{f} & Y & \xleftarrow{\beta} & B_\infty G' & \longleftarrow \cdots \longleftarrow & B_2G' \end{array}$$

where α and β are homotopy equivalences and $\beta^{-1} f \alpha \sim f_\infty : B_\infty G \to B_\infty G'$, the map induced from $\Omega f : G \to G'$. Let $\mathcal{P}_\sigma(X,A) \to A$ denote the G-bundle induced from $\mathcal{P}_\sigma X \to X$ by the inclusion map i. Then, there is a fiber homotopy equivalence

$$\begin{array}{ccc} \mathcal{P}_\sigma(X,A) & \longrightarrow & E_A \\ p_2 \downarrow & & \downarrow q \\ A & \xrightarrow{\text{id}} & A \end{array}$$

and $cat_X A \leq 2$ implies $E_A \to A$ has genus ≤ 2 and, hence, the same is true for

$\mathcal{P}_\sigma(X,A) \to A$. It then follows that there is a G-bundle map

$$\begin{array}{ccc} \mathcal{P}_\sigma(X,A) & \longrightarrow & E_2(G) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\lambda} & B_2(G) \end{array}$$

and we have a diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\lambda} & B_2(G) & \xrightarrow{j} & B_\infty(G) & \xrightarrow{\alpha} & X \\ & & {\scriptstyle f_2}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ & & B_2(G') & \xrightarrow{j'} & B_\infty(G') & \xrightarrow{\beta} & Y \end{array}$$

where f_2 is induced by $\Omega f : G \to G'$ and j and j' are inclusions. Then, if (as in the first proof) $\beta u \in H^2(X,\mathbb{Z}_p)$, $\beta u|A$ factors through $B_2(G')$; but G' is up to homotopy the suspension of $\mathbb{Z}_p$ and hence $\beta u|A = 0$. Thus, *cwgt* $(u) \geq 2$. A simple application of Theorem (3.6) is the following new result on free actions on products of spheres.

THEOREM (3.7). *Let M denote the product of k copies of the sphere S^{2n+1} ($k \geq 1$) and let p denote a prime. Suppose that the group $G = (\mathbb{Z}_p)^k$ acts freely on M (not necessarily linearly or coordinate-wise) and the induced action of G on $H^*(M;\mathbb{Z}_p)$ is trivial. Then, cat$(M/G) = 1 + \dim M$, i.e. the category of M/G is maximum possible.*

Proof. We restrict our proof to the case where p is an odd prime. The case $p = 2$ is considerably simpler (and the sphere may be even dimensional) and is left to the reader. Using a result of Carlsson [2, Corollary 7, page 399], there is a surjection $\alpha : H^*(BG;\mathbb{Z}_p) \to H^*(M/G;\mathbb{Z}_p)$. Recall that $H^*(BG;\mathbb{Z}_p) = E(x_1,\ldots,x_k) \otimes P(y_1,\ldots,y_k)$, where $\dim x_i = 1$ and $\dim y_i = 2$. Then, since M/G is orientable, the fundamental class μ is a linear combination of terms of the form

$$\alpha\left(x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} y_1^{\epsilon'_1} y_2^{\epsilon'_2} \cdots y_k^{\epsilon'_k}\right),$$

at least one of which is non-zero. Thus, letting $\alpha(u) = \bar{u}$, there is an element in $H^*(M/G;\mathbb{Z}_p)$ of the form

$$0 \neq v = \bar{x}_1^{\epsilon_1} \cdots \bar{x}_k^{\epsilon_k} \bar{y}_1^{\epsilon'_1} \cdots \bar{y}_k^{\epsilon'_k},$$

where $\sum \epsilon_i + 2\sum \epsilon'_j = k(2n+1)$. Since *cwgt* $(x_i) \geq 1$ and *cwgt* $(\bar{y}_j) =$ *cwgt* $(\beta\bar{x}_j) \geq 2$, we have *cwgt* $(v) \geq k(2n+1)$ and, hence, *cat* $M/G \geq$

$k(2n+1)+1$. Since, in general, $cat \leq \dim + 1$, the result follows.

Remark (3.8). The case $k = 1$ in Corollary (1.6) is due to Krasnoselski [8]. Several others (e.g., T. Bartsch, W. Marzantowicz, Z. Wang, J.C. Gómez-Larrañaga, F. González-Acuña) have provided alternative proofs of this case. However, their methods do not seem strong enough to obtain our more general result. It is interesting to note that, in this special case ($k = 1$), the fundamental class has the form $x\beta x\beta x \ldots \beta x$, where the number of letters that appear in the notation for this class is $2n+1$. Finally, the case $p = 2$ in the proof of Theorem (3.6) requires only classical cup length.

The methods used to prove Theorem (3.7) may be employed to compute the category of the orbit space of a free action on the complex Stiefel manifold $M = O_{n,k}(\mathbb{C})$ of k-frames in $\mathbb{C}^n$ as follows. M is homomorphic to the space of isometric linear imbeddings of $\mathbb{C}^k$ in $\mathbb{C}^n$. Then, the unitary group $U(k)$ acts freely on M by post multiplication. Recall that the p-rank r of a subgroup G of $U(k)$ is the maximal r for which G contains $(\mathbb{Z}_p)^r$ as a subgroup.

THEOREM (3.9). *Suppose G is a finite subgroup of $U(k)$ of rank k. Then, cat $M/G = \dim M + 1$, i.e., cat M/G, is maximal.*

Proof. Since, for a covering space $\tilde{X} \to X$, we have $cat\ X \geq cat\ \tilde{X}$, it suffices to assume $G = (\mathbb{Z}_p)^k$. Also, after conjugation if necessary, we may assume that $G \subset T$, where T is the standard maximal torus of $U(k)$. $M/T = \mathbb{F}_k(\mathbb{C}^n)$ is the space of k-flags in $\mathbb{C}^n$. The inclusion $G \subset T$ induces a fibration

$$T/G = (S^1)^k \longrightarrow M/G \longrightarrow M/T = \mathbb{F}_k(\mathbb{C}^n).$$

We also have a commutative diagram

$$\begin{array}{ccc} T/G & \xrightarrow{id} & T/G \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{f} & BG \\ \alpha\downarrow & & \downarrow\gamma \\ M/T & \xrightarrow{g} & BT \end{array}$$

where, as usual, BG and BT are classifying spaces for G and T, respectively, and the horizontal maps are classifying maps of $M \to M/G$ and $T \to T/G$, respectively. Observe that the Leray-Hirsch Theorem applies to α and γ and g induces a surjection $H^*(BT; \mathbb{Z}_p) \to H^*(M/T; \mathbb{Z}_p)$. This forces f^* : $H^*(BG; \mathbb{Z}_p) \to H^*(M/G; \mathbb{Z}_p)$ to be surjective. Thus, we may consider the fundamental class μ of the compact orientable manifold M/G and proceed as in the proof of Theorem (3.7) to obtain a cohomology class of *cwgt* $=$ $\dim M/G = \dim M$. Thus, *cat* $M/G = \dim M + 1$.

Theorem (3.6) may be interpreted as follows. Let θ denote a cohomology operation of type (n, q), $\theta : H^n(\cdot, \mathbb{Z}_p) \to H^q(\cdot, \mathbb{Z}_p)$, p a prime.

Definition (3.10). θ is called universal in dimension n for category weight c if for any space X and $u \in H^n(X; \mathbb{Z}_p)$, such that $\theta u \neq 0$, then *cwgt* $(u) \geq c$. Then, Theorem (3.6) says that the Bockstein homomorphism β is universal in dimension 1 for category weight 2.

Our next result investigates universality for Steenrod operations in dimension $n \geq 2$ for category weight 2. First we recall some standard notation [3, Exposé 15]. A sequence of non-negative integers $I = (a_1, \ldots, a_k)$ is called admissible if $a_j = 2\lambda_i(p-1) + \epsilon_i$, where $\epsilon_i = 0$ or 1, and $a_i \geq pa_{i+1}$. Then, if P_p^λ is a Steenrod reduced power operation and $a = 2\lambda(p-1) + \epsilon$,

$$St_p^a = P_p^\lambda \quad \text{if} \quad \epsilon = 0$$

and

$$St_p^a = \beta P_p^\lambda \quad \text{if} \quad \epsilon = 1.$$

Then, set

$$St_p^I = St_p^{a_1} \circ St_p^{a_2} \circ \cdots \circ St_p^{a_k}.$$

Furthermore, set $|I| = \sum_{i=1}^{k} a_i$ and the so-called excess

$$e(I) = 2\lambda_1 p + 2\epsilon_1 - |I|.$$

In [3, Exposé 15], $e(I)$ is denoted by $n(I)$. We will also make use of the following classical result (see H. Cartan [3]).

PROPOSITION (3.11). *If I is an admissible sequence and $u \in H^n(X; \mathbb{Z}_p)$ where $n < e(I)$, then $St_p^I(u) = 0$.*

THEOREM (3.12) *St_p^I is universal in dimension $e(I)$ for category weight 2.*

Proof. Suppose X is a space and $A \subset X$ with $cat_X A \leq 2$. Then, if $u \in H^n(X; \mathbb{Z}_p)$ and $n = e(I)$, we need to show that $St_p^I(u)|A = 0$. Consider the Svarc sequence as in the first proof of Theorem (3.6), where $E = \mathcal{P}X$, $E_A = \mathcal{P}(X, A)$,

$$\begin{array}{ccccccc}
E_A *_A E_A & \xrightarrow{\bar{i} \circ \bar{i}} & E *_X E & \longrightarrow & \cdots & \longrightarrow & E^{*_X \infty} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_\infty} \\
A & \xrightarrow{i} & X & \xrightarrow{id} & \cdots & \longrightarrow & X
\end{array}$$

where the fiber of p_∞ is $(\Omega X)^{*\infty}$, which is contractible, forcing p_∞ to be a homotopy equivalence. Also $cat_X A \leq 2$ implies that p admits a section. Let $j : E *_X E \to E^{*X\infty}$, and recall (Prop. 2.3) that there is a homotopy equivalence ν:

$$\begin{array}{ccc} E *_X E & \xrightarrow{\nu} & S\Omega X \\ p_2 \downarrow & & \downarrow q \\ X & \xrightarrow{id} & X \end{array}$$

with $q\nu = p_2$. If μ is a homotopy inverse for ν, then we assert that $\mu^* j^* p_\infty^*(St_p^I u) = 0$. Consider the suspension isomorphism, with, $q \geq 2$,

$$H^{q-1}(\Omega X; \mathbb{Z}_p) \xrightarrow{\Delta^*} H^q(S\Omega X; \mathbb{Z}_p).$$

Then, if $\Delta^*(v) = \mu^* j^* p_\infty^*(u), St_p^I(v) = 0$ because $\dim v < e(I)$ and $0 = \Delta^*(St_p^I v) = St_p^I \Delta^*(v) = \mu^* j^* p_\infty^*(St_p^I u)$. Of course, this implies $j^* p_\infty^*(St_p^I u) = 0$. To complete the proof, observe that now

$$p^*(St_p^I u|A) = (\bar{i} \circ \bar{i})^* p_2^*(St_p^I u)$$

and

$$p_2^*(St_p^I u) = j^* p_\infty^*(St_p^I u) = 0.$$

Hence $p^*(St_p^I u|A) = 0$ and since p^* injects, $St_p^I u|A = 0$.

Remark (3.13). Let $\iota_n \in H^n(\mathbb{Z}_p, n; \mathbb{Z}_p)$ denote the fundamental class of the Eilenberg-MacLane space $K(\pi, n)$, and p is an odd prime. Then, $St_p^I(\iota_n)$, as I ranges over admissible sequences such that $e(I) \leq n$, generate the cohomology algebra $H^*(\mathbb{Z}_p, n; \mathbb{Z}_p)$. Those with $e(I) = n$ are precisely the generators that do not arise as transgressions in $H^*(\mathbb{Z}_p, n-1; \mathbb{Z}_p)$ (see Postnikov [9]). However, when $p = 2$, the algebra $H^*(\mathbb{Z}_2, n; \mathbb{Z}_2)$ is generated by the elements defined by those I with $e(I) < n$ (see Serre [11]).

Remark (3.14). While $St_p^I(u) \neq 0$ always has $cwgt \geq 2$, whenever $\dim u = e(I)$, it is possible for some elements u to have $cwgt(St_p^I u) \geq 2$, even though $\dim u > e(I)$. Let X denote the symplectic group $Sp(2)$, considered by Schweitzer in [10]. Then, according to [1], $H^*(Sp(2); \mathbb{Z}_3) = E(x_3, x_7)$ and $x_7 = P_3^1 x_3$. Note that $P_3^1 = St_3^I$, where $I = (2)$ and $e(I) = 2$. Thus, $\dim x_3 > e(I)$ and Theorem (3.11) does not apply. Nevertheless, we can show that $cwgt(P_3^1 x_3) \geq 2)$ as follows, using Schweitzer's result [10] that *cat* $Sp(2) = 4$.

Proof. We use the cellular structure of $Sp(2)$ which is of the form $S^3 \cup e^7 \cup e^{10}$. Let $X = S^3 \cup e^7$, the 7-skeleton of $Sp(2)$. Since *cat* $Sp(2) = 4$, it follows that *cat* $X \geq 3$. We assert that *cwgt* $(x_7) \geq 2$ in $Z = S^3 \cup e^7 \cup e^{10}$. To prove this, let A denote a subspace of Z, with $cat_Z A \leq 2$. Then, since $cat\ Z = 4$, some interior point of e^{10} is not in A and thus A is deformable relative to X to a set B in X. Then, since $cat\ X \geq 3$, B is deformable into S^3 relative to S^3 and $x_7|B = 0$. Thus $x_7|A = 0$ and $cwgt\ (x_7) \geq 2$.

Remark (3.15). We wish to thank the referee for the careful reading of our paper and for pointing out an error in the use of a result of Ganea [5] in our original proof that $cat\ X \geq 3$, which did not use the result that $cat\ Sp(2) = 4$. Another alternative argument for the result that $cat\ X \geq 3$, independent of the Schweitzer result that $cat\ Sp(2) = 4$, can be provided as follows. Let $\alpha = [f] \in \pi_6(S^3) = \mathbb{Z}_{12}$ denote the generator with $f : S^6 \to S^3$ the characteristic map of the bundle $S^3 \to Sp(2) \to S^7$. Then, as shown in Borel-Serre [1,p.442], the generalized Hopf-invariant $H(f)$ is non-zero and $H(f)$ can in turn be identified with the obstruction to finding a cross section in the second term of the Milnor sequence (M) in section 2. Thus, $cat\ X \geq 3$. This sketch will be pursued more generally in a future work.

Remark (3.16). Fary [4] gave a definition of the category of a cohomology class that is unrelated to our *cwgt* and also not useful in considering product lengths.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 53706, U.S.A.

REFERENCES

[1] A. BOREL AND J.P. SERRE, *Groupes de Lie et puissances de Steenrod*, Amer. J. Math. **75** (1953), 409-448.
[2] G. CARLSSON, *On the rank of abelian groups acting freely on* $(S^n)^k$, Invent. Math. **69** (1982), 393-400.
[3] H. CARTAN, Séminaire H. Cartan, 1954-1955, Paris.
[4] J. FARY, *Sur la catégorie des classes de homologie d'un espace*, Proc. Internat. Congress Math. **2** (1954), 215.
[5] T. GANEA, *Lusternik-Schirnelmann category and strong category*, Illinois J. Math. **11** (1967), 417-427.
[6] J. MILNOR, *Construction of universal bundles I*, Annals of Math. **63** (1956), 272-284.
[7] J. MILNOR, *Construction of universal bundles II*, Annals of Math. **63** (1956), 430-436.
[8] M.A. KRASNOSELSKI, *On special coverings of a finite dimensional sphere*, (Russian). Dokl. Akad. Nauk. SSSR **103** (1955), 961-1064.
[9] M.M. POSTNIKOV, *On Cartan's theorem*, Russian Math. Surveys. **21** (1966), 25-36.
[10] P. SCHWEITZER, *Secondary operations induced by the diagonal mapping*, Topology. **3** (1965), 337-355.
[11] J.P. SERRE, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comm. Math. Helv. **27** (1953), 198-232.
[12] A. SVARC, *The genus of a fiber space*, Amer. Math. Soc. Translations. **55** (1966), 49-140.
[13] G.W. WHITEHEAD, *Elements of Homotopy Theory*, GTM vol. **61**, Springer-Verlag (1978).