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WHITEHEAD SQUARE IN EX-HOMOTOPY THEORY

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We deal here with the order of the Whitehead square in ex-homotopy theory. Also, we prove that under what conditions the fibre ex-space has a Hopf ex-space structure *.

Introduction

Let P be a principal $SO(m)$-bundle over $S^n (n \geq 2)$. The unreduced suspension $\hat{a}$ of the antipodal self-map a of S^{m-1} is a pointed $SO(m)$-map of S^m into itself. Since $P_\sharp$ constitutes a functor from the category of equivariant spaces to the category of ex-spaces, $P_\sharp \hat{a}$ is an ex-map of $E = P_\sharp S^m$ into itself. Let $w(\Sigma E)$ denote the Whitehead square $[\iota_{\Sigma E}, \iota_{\Sigma E}]$ of the ex-homotopy class of the identity on ΣE.

In [4, §8], I.M.James has shown that $4w(\Sigma E) = 0$ for m even, and raised the question when $w(\Sigma E) = 0$, i.e, when ΣE is a Hopf ex-space (see [4, p. 236]). In this note we give a partial answer to this question.

1. The case $w(\Sigma E) = 0$

Let $m = 2$ and the subgroup $SO(2)$ of $SO(4)$ act on S^3 which leaves the first and last basis vectors fixed. If $n > 2$, then the principal $SO(2)$-bundle P over S^n is a trivial bundle. Then P reduces to a principal bundle P' with a group $SO(1)$. The multiplication of quaternion gives S^3 a Hopf $SO(2)$-space structure. Hence $P \times_{SO(2)} S^3 = P_\sharp S^3$ has a Hopf ex-space structure. Since $P'_\sharp S^2 = P' \times_{SO(1)} S^2 \cong P \times_{SO(2)} S^2 = P_\sharp S^2 = E$, we have $P \times_{SO(2)} S^3 \cong P' \times_{SO(1)} S^3 \cong P' \times_{SO(1)} \Sigma S^2 = \Sigma P'_\sharp S^2 = \Sigma E$ (cf. [5, p. 53]). Thus ΣE is a Hopf ex-space; $w(\Sigma E) = 0$.

Let $m = 6$ and the subgroup $SO(6)$ of $SO(8)$ act on S^7, with base point the point $(1, 0, ..., 0)$, which leaves the first and last basis vectors fixed. Suppose that the structure group $SO(6)$ of the principal bundle P reduces to the subgroup $SU(2) \times 1 \times 1$ with a reduced principal bundle P'. Then, we have the isomorphism of the associated bundles;

$$P' \times_{SU(2)} S^7 \cong P \times_{SO(6)} S^7.$$

Since the multiplication of the Cayley numbers gives S^7 a Hopf $SU(2)$-structure, $P' \times_{SU(2)} S^7 \cong P'_\sharp S^7$ is a Hopf ex-space. Since $P'_\sharp S^6 = P' \times_{SU(2)} S^6 \cong P \times_{SO(6)} S^6 = E$, we have

$$P' \times_{SU(2)} S^7 \cong P' \times_{SU(2)} \Sigma S^6 = P'_\sharp \Sigma S^6 = \Sigma P'_\sharp S^6 \cong \Sigma E$$

(cf. [5, p. 53]). Hence ΣE is a Hopf ex-space; $w(\Sigma E) = 0$. Therefore we have

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THEOREM (1.1). *If* $m = 2$ *and* $n > 2$ *or* $m = 6$ *and the structure group* $SO(6)$ *of the principal bundle* P *reduces to the subgroup* $SU(2) \times 1 \times 1$*, then* $w(\Sigma E) = 0$.

2. The case $2w(\Sigma E) = 0$

Consider the sequence

$$\pi_{n-1}(SO(m)) \xrightarrow{F} \pi_n(SO(m)) \xrightarrow{J} \pi_{n+m}(S^m) \xrightarrow{\Sigma_*^{m+1}} \pi_{n+2m+1}(S^{2m+1})$$

where F denotes the Bott suspension. In [4, §6], James introduced the operator $D : \pi_{2m+2}(S^{m+1}) \longrightarrow \pi_{n+2m+1}(S^{m+1})$ which is given by $D\alpha = \alpha \circ \Sigma_*^{m+2}J\theta - 2\Sigma_* J\theta \circ \Sigma_*^{n-1}\alpha$ for the classifying element $\theta \in \pi_{n-1}(SO(m))$ of P. The following theorem was proved

THEOREM (2.1). (see [4, Theorem 8.5]). *Let* m *be even. Then* $2w(\Sigma E) = 0$ *if and only if* $w_{m+1} \circ \Sigma_*^{m+1}JF\theta$ *lies in the image of* $D : \pi_{2m+2}(S^{m+1}) \to \pi_{n+2m+1}(S^{m+1})$*, where* $w_{m+1} \in \pi_{2m+1}(S^{m+1})$ *denotes the ordinary Whitehead square of the generator of* $\pi_{m+1}(S^{m+1})$.

First,we consider the stable range ($n < m$). Since $F\theta = \theta\eta_{n-1}$(cf. [3, (17.2)]) we have $JF\theta = J(\theta \circ \eta) = J\theta \circ \eta_{n+m-1}$. If $m \equiv 0 \bmod 4$ then $\pi_{2m+2}(S^{m+1}) = G_{m+1} + Z_2[\eta_{m+1}, \iota_{m+1}]$, where G_{m+1} denote the $(m+1)$-stem stable homotopy groups of spheres (cf. [6, 3.7(b)]). Then

$$w_{m+1} \circ \Sigma_*^{m+1}JF\theta = [\iota_{m+1}, \iota_{m+1}] \circ \Sigma_*^{m+1}JF\theta = [\iota_{m+1}, \iota_{m+1}] \circ \Sigma_*^{m+1}J\theta \circ \eta_{n+2m}.$$

This lies in the image of D, since

$$\begin{aligned} &D([\eta_{m+1}, \iota_{m+1}]) \\ &= (-1)^{n+m-1}D([\eta_{m+1}, \iota_{m+1}]) \\ &= (-1)^{n+m-1}([\eta_{m+1}, \iota_{m+1}] \circ \Sigma_*^{m+2}J\theta - 2\Sigma_* J\theta \circ \Sigma_*^{n-1}[\eta_{m+1}, \iota_{m+1}]) \\ &= (-1)^{n+m-1}[\iota_{m+1}, \iota_{m+1}]\eta_{2m+1} \circ \Sigma_*^{m+2}J\theta \\ &= [\iota_{m+1}, \iota_{m+1}] \circ \Sigma_*^{m+1}J\theta \circ \Sigma^{m+n-1}\eta_{m+1} \end{aligned}$$

(cf. [7, p. 481(8.12) and p. 484(8.18)]). So, by theorem 2.1 we have

COROLLARY (2.2). *If* $n < m$ *and* $m \equiv 0 \bmod 4$*, then* $2w(\Sigma E) = 0$.

However, in the case $m \equiv 2 \bmod 4$ the homotopy group $\pi_{2m+2}(S^{m+1})$ is isomorphic to the stable group (cf. [6, 3.2(b)]).

Here is an example outside the stable range. Let $m = 2^{k+1} - 2$ and recall the strong form of the Kervaire invariant problem: does there exist an element $\theta_k \in \pi_m^S$ with $2\theta_k = 0$ and Kervaire invariant 1?. If there is such an element, then there is an element $\gamma \in \pi_{2m+1}(S^{m+1})$ with $2\gamma = w_{m+1} = [\iota_{m+1}, \iota_{m+1}]$ (cf. [1]). By [7, p. 479 (8.1) and (8.2)] we have the following relations

$$\begin{aligned} &w_{m+1} \circ \Sigma_*^{m+1}JF\theta = 2\gamma \circ \Sigma_*^{m+1}JF\theta \\ &= (\gamma + \gamma)\Sigma_*^{m+1}JF\theta = \gamma \circ \Sigma_*^{m+1}JF\theta + \gamma \circ \Sigma_*^{m+1}JF\theta \\ &= (\gamma \circ \Sigma_*^{m+1})(JF\theta + JF\theta) = (\gamma \circ \Sigma_*^{m+1})2JF\theta \\ &= 0 \end{aligned}$$

since $2F = 0$ (cf. [3, p. 110]). Hence

COROLLARY (2.3). *Let $m = 2^{k+1} - 2$. If there exists an element $\theta_k \in \pi_m^S$ with $2\theta_k = 0$ and Kervaire invariant one, then $2w(\Sigma E) = 0$.*

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