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THE COHOMOLOGY OF BLOW UPS

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0. Introduction

The blowing up of a manifold along a submanifold is a construction that has been used extensively in algebraic geometry. In this expository note we give a description of the cohomology of both complex and real blow ups. We use Thom spaces and the Thom–Pontryagin mappings to define isomorphisms which allows more of the algebra of the cohomology of the blow ups and the module over the Steenrod algebra structure to become apparent. We have tried to make this note accessible to topologists.

Given a complex manifold X of complex dimension n and a submanifold Y of complex of codimension d, the blow up of X along Y is a complex manifold $\widetilde{X}$ of the same dimension as X, which comes with a holomorphic map $f : \widetilde{X} \to X$ which is an isomorphism outside $f^{-1}(Y)$ and where $f^{-1}(Y)$ can be identified with $\widetilde{Y}$ the projective normal bundle of the embedding of Y in X. The mapping f is of degree one for closed manifolds and this produces a splitting using the Gysin map $f! : H^*(\widetilde{X}) \to H^*(X)$

$$H^*(\widetilde{X}) = f^*H^*(X) \oplus \operatorname{Ker} f! \tag{0.1}$$

This splitting is not a splitting of algebras. We have a natural commutative diagram:

$$\begin{array}{ccccc} \widetilde{Y} & \xrightarrow{\widetilde{k}} & \widetilde{X} & \xrightarrow{\widetilde{t}} & \widetilde{Y}^{\widetilde{N}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{k} & X & \xrightarrow{t} & Y^N \end{array} \tag{0.2}$$

where $\widetilde{Y}^{\widetilde{N}}$ and Y^N are the Thom spaces of the normal bundles of the embeddings of $\widetilde{Y}$ in $\widetilde{X}$ and of Y in X, respectively, and $k, \widetilde{k}$ are the inclusions, $t, \widetilde{t}$ the Thom–Pontryagin maps; g is the restriction of f to $\widetilde{Y}$ and h is the induced map in Thom spaces. This diagram induces a short exact sequence

$$0 \to H^*(Y^N) \xrightarrow{\alpha} H^*(X) \oplus H^*(\widetilde{Y}^{\widetilde{N}}) \xrightarrow{\beta} H^*(\widetilde{X}) \to 0 \tag{0.3}$$

where $\alpha = (t^*, h^*), \beta = f^* - \widetilde{t}^*$ which enables us to identify $\operatorname{Ker} f!$ with $H^*(\widetilde{Y})/g^*H^*(Y)$ and to determine the ring structure even though β is not a ring map. In particular, when $k^* : H^*(X) \to H^*(Y)$ is onto, the description of $H^*(\widetilde{X})$ as an algebra is particularly simple, see Theorem (3.10).

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A similar exact sequence appears in [5]. However Manin does not consider Thom spaces.

In section 4, we describe the Chern classes of the tangent bundle to $\widetilde{X}$.

One may also make the blow up construction for real manifolds. The methods in this note, may equally apply to this situation and yield the mod 2 cohomology of the real blow up, see section 5.

1. The blow up construction

Let X be a (real or) complex manifold of dimension n and Y a submanifold of codimension d. Let F be the real or complex numbers and F^{m+d} the corresponding $(m+d)$-space. Let D^{m+d} be the disk with coordinates $(z,z') = (z_1,\ldots,z_m,z'_1,\ldots,z'_d)$. Let D_0^m be the subspace of D^{m+d} given by $D_0^m = (z,z')|z'=0$, and let FP^{d-1} denote the projective space of dimension $d-1$ with homogeneous coordinate $t=[t_1,\ldots,t_d]$. We now recall the blow up construction as given in [2, p. 603]. The blow up of D^{m+d} along D_0^m is the submanifold $B(D^{m+d},D_0^m)$ of $D^{m+d}\times FP^{d-1}$ defined by:

$$B(D^{m+d},D_0^m) = ((z,z'),[t])|z'_it_j = z'_jt_i \text{ for } 1\geq i,j\geq d.$$

The projection onto the first factor induces $f: B(D^{m+d},D_0^m)\to D^{m+d}$ and it is clear that f induces an isomorphism (diffeomorphism or biholomorphism)

$$B(D^{m+d},D_0^m) - f^{-1}(D_0^m) \cong D^{m+d} - D_0^m. \tag{1.1}$$

The general construction is now obtained by patching. Namely choose a family of charts $\{U_\alpha\}$ for X so that $Y=\bigcup(U_\alpha\cap T)$ and so that it

$$\varphi_\alpha : U_\alpha \to D^{m+d} \quad \text{is a diffeomorphism,}$$

then $\varphi_\alpha(U_\alpha\cap Y) = D^m$.

Use the diffeomorphisms φ_α to constant $B_\alpha(U_\alpha, U_\alpha\cap Y)$ over U_α.

Now $B(X,Y) = \coprod_\alpha B_\alpha(U_\alpha, U_\alpha\cap Y)$ modulo identifications, where the identifications are as follows. Consider U_β another chart, then we can form $B_\alpha(U_\alpha\cap U_\beta, U_\alpha\cap U_\beta\cap Y)\subset B_\alpha(U_\alpha, U_\alpha\cap Y)$ and $B_\beta(U_\alpha\cap U_\beta, U_\alpha\cap U_\beta\cap Y)\subset B_\beta(U_\beta, U_\beta\cap Y)$. We identify a point $x\in B_\alpha(U_\alpha\cap U_\beta, U_\alpha\cap U_\beta\cap Y)$ with $\varphi_\beta^{-1}\varphi_\alpha(x)\in B_\beta(U_\alpha\cap U_\beta, U_\alpha\cap U_\beta\cap Y)$. It is then not hard to see that $B(X,Y)$ is a manifold of the same dimension as X, closed and compact if X is. Moreover, if X and Y are orientable manifolds (the problem of orientability only arises for real X and Y), then so is $B(X,Y)$. By general position, because Y has codimension at least 1, the map $f: B(X,Y)\to X$ is of degree 1. Also $f^{-1}(Y) = \widetilde{Y}$ is the projective normal bundle of the embedding of Y in X. This projective normal bundle $\widetilde{Y}$ has a canonical line bundle L consisting of

(y, l, v), where $y \in Y$, l is a line in the normal space to Y at y, and v is a vector in that line. We also denote by $\widetilde{X} = B(X, Y)$. We now have a diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "k"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[r, "j"'] & X \end{tikzcd} \tag{1.2}$$

the blow up diagram, k,j the embeddings with normal bundles $\widetilde{N} = L$ and N respectively, where $f : \widetilde{X} - \widetilde{Y} \to X - Y$ is an isomorphism.

2. Degree one mappings

In this section, we recall some well known results like the Gysin map, Poincaré duality and degree one maps, which can be found in [1].

Let $f : \widetilde{X} \to X$ be a mapping of manifolds, which are orientable (otherwise take coefficients in Z_2) and closed. Denote by $[\widetilde{X}]$, $[X]$ their fundamental homology classes. Recall that using Poincare duality, we can define the Gysin map, $f! : H^q(\widetilde{X}) \to H^{q+e}(X)$ where $e = \dim X - \dim \widetilde{X}$, as follows:

$$[X] \cap f! \, \widetilde{x} = f_*([\widetilde{X}] \cap \widetilde{x}) \tag{2.1}$$

where $\widetilde{x} \in H^q(\widetilde{X})$, f_* is the map induced in homology and $\cap$ is the cap product.

If $e = 0$, and f is of degree 1, namely $f_*[\widetilde{X}] = [X]$, then for $x \in H^q(X)$,

$$[X] \cap f!f^*x = f_*([\widetilde{X}] \cap f^*x) = f_*[\widetilde{X}] \cap x = [X] \cap x$$

thus $f!f^*$ is the identity and $f!$ is a left inverse of f^*. We thus obtain:

$$H^q(\widetilde{X}) \cong f^*H^q(X) \oplus (\mathrm{Ker} f!)^q. \tag{2.2}$$

Suppose now that $x \in H^p(X)$, $y \in (\mathrm{Ker} f!)^q$, then

$$y \cup f^*x \in (\mathrm{Ker} f!)^{p+q} \tag{2.3}$$

for

$$\begin{aligned} [X] \cap (f!(y \cup f^*x)) &= f_*(([\widetilde{X}] \cap y) \cap f^*x) \\ &= (f_*([\widetilde{X}] \cap y)) \cap x \\ &= ([X]) \cap f!y) \cap x = 0. \end{aligned}$$

Hence if $p + q = 2n$,

$$y \cup f^*x = 0. \tag{2.4}$$

3. The Mayer–Vietoris sequences

Consider the following diagram:

$$
\begin{tikzcd}
\widetilde{Y} \arrow[r, "\widetilde{k}"] \arrow[d, "g"] & \widetilde{N} \arrow[r, "\widetilde{j}"] \arrow[d, "\widehat{g}"] & \widetilde{X} \arrow[r, "\widetilde{t}"] \arrow[d, "f"] & \widetilde{Y}^{\widetilde{N}} \arrow[d, "\widehat{f}"] \\
Y \arrow[r, "k"'] & N \arrow[r, "j"'] & X \arrow[r, "t"'] & Y^{N}
\end{tikzcd}
\tag{3.1}
$$

where we have extended the diagram (1.2) to include the inclusion of $\widetilde{Y}$ in $\widetilde{N}$ and Y in N where $\widetilde{N}$ and N are the normal bundles of the embeddings. Then

$$\widetilde{X} = (\widetilde{X} - \widetilde{Y}) \cup \widetilde{N} \qquad (\widetilde{X} - \widetilde{Y}) \cap \widetilde{N} = \widetilde{N} - \widetilde{Y} \tag{3.2}$$

$$X = (X - Y) \cup N \qquad (X - Y) \cap N = N - Y. \tag{3.3}$$

Recall from the construction, of $\widetilde{X}$, that f induces isomorphisms $\widetilde{X} - \widetilde{Y} \to X - Y$, $\widetilde{N} - \widetilde{Y} \to N - Y$. Now (3.1), (3.2) and (3.3) induce a map of Mayer–Vietoris sequences:

$$
\begin{tikzcd}
\cdots \arrow[r] & H^{q-1}(\widetilde{N} - \widetilde{Y}) \arrow[r] & H^{q}(\widetilde{X}) \arrow[r] & {} \\
\arrow[r] & H^{q-1}(N - Y) \arrow[r] \arrow[u, "f_2^*"] & H^{q}(X) \arrow[r] \arrow[u, "f^*"] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\arrow[r] & H^{q}(\widetilde{N}) \oplus H^{q}(\widetilde{X} - \widetilde{Y}) \arrow[r] & H^{q}(\widetilde{N} - \widetilde{Y}) \arrow[r] & \cdots \\
\arrow[r] & H^{q}(N) \oplus H^{q}(X - Y) \arrow[r] \arrow[u, "\overline{f}^* \; f_1^*"] & H^{q}(N - Y) \arrow[r] \arrow[u, "f_2^*"] & \cdots
\end{tikzcd}
\tag{3.4}
$$

where f_1^* and f_2^* are isomorphisms and f^* and $\overline{f}^*$ are monomorphisms. The fact that $\overline{f}^*$ is a monomorphism follows from the homotopy equivalences $\widetilde{N} \simeq \widetilde{Y}$, $N \simeq Y$ and the well known monomorphism $H^*(Y) \xrightarrow{g^*} H^*(\widetilde{Y})$. A little bit of diagram chasing now produces that $\widetilde{j}^*$ induces

$$\operatorname{Ker} f! \cong H^*(\widetilde{Y})/g^*H^*(Y). \tag{3.5}$$

Now using (2.2) we obtain

$$H^*(\widetilde{X}) \cong f^*H^*(X) \oplus (H^*(\widetilde{Y})/g^*H^*(Y)) \tag{3.6}$$

and we need to describe the ring structure. This will follow from

THEOREM (3.7). *We have short exact sequences*

$$0 \to H^q(Y^N) \xrightarrow{\alpha} H^q(X) \oplus H^q(\widetilde{Y}^{\widetilde{N}}) \xrightarrow{\beta} H^q(\widetilde{X}) \to 0$$

where

$$\alpha(y) = (t^*y, \widehat{f}^*y), \quad \beta(x,z) = f^*x - \widetilde{t}^*z.$$

Proof. The fact that α is $1-1$ follows from the fact that $\widehat{f}^*$ is $1-1$. Namely if $\mu \in H^{2d}(Y^N)$ is the Thom class, then

$$\widehat{f}^*\mu = U \cup \chi \tag{3.8}$$

where U is the Thom class of $\widetilde{Y}^{\widetilde{N}}$ and χ is described below in (3.10). Then $\widehat{f}^*(\mu \cup y) = U \cup \chi \cup y$, for any class $y \in H^*(Y)$. But $\chi = (-1)^{d-1}x^{d-1} + \cdots$ and $H^*(\widetilde{Y}) \cong (H^*(Y) \otimes P(\chi))/I_N$ where I_N is the ideal generated by $x^d - x^{d-1}C_1(N) + \cdots + (-1)^d C_d(N)$. Thus $U \cup \chi \cup y = \pm U \cup x^{d-1} \cup y + \cdots$ and $U \cup \chi \cup y = 0$ if and only if $U \cup x^{d-1} \cup y = 0$. Now β is an epimorphism, for by (3.6) β is onto $f^*H^*(X)$. Now the classes $[x^k y]$ generate $H^*(\widetilde{Y})/g^*H^*(Y)$ with $k > 0$, and if we look at $x^k y$ in $H^*(\widetilde{Y})$, $k^*\widetilde{t}^*(U \cdot x^{k-1}y) = x^k y$ so $\overline{k}^*\widetilde{t}^*(U \cdot x^{k-1}y) = [x^k y]$ and β is onto. Now $\beta\alpha = 0$, for $\beta\alpha(y) = \beta(t^*y, \widetilde{f}^*y) = f^*t^*y - \widetilde{t}^*\widetilde{f}^*y = 0$ because of the commutativity of (3.1).

Finally, suppose $\beta(u,z) = 0$, i.e. $f^*u = \widetilde{t}^*z$. Now z is of the form $z = U(y_0 + y_1 x + \ldots + y_{d-1}x^{d-1})$, then $k^*\widetilde{t}^*z = y_0 x + y_1 x^2 + \cdots + y_{d-1}x^d = k^*f^*u$. Thus $y_0 x + y_1 x^2 + \cdots + y_{d-2}x^{d-1} + y_{d-1}(C_1 x^{d-1} - \cdots - (-1)^{d+1}C_d) = k^*f^*u$. Then we must have $(-1)^{d+1}y_{d-1}C_d = k^*f^*u$ and all the coefficients of $x, \ldots, x^{d-1}$ zero. Thus,

$$\begin{aligned} y_0 + (-1)^d y_{d-1}C_{d-1} &= 0 \\ y_1 + (-1)^{d-1} y_{d-1}C_{d-2} &= 0 \\ &\vdots \\ y_{d-2} - y_{d-1}C_2 &= 0. \end{aligned}$$

Thus

$$y_0 x + \cdots + y_{d-1}x^{d-1} = (-1)^{d-1} y_{d-1}\chi$$

and

$$z = (-1)^{d-1} u y_{d-1}\chi = (-1)^{d-1}(-1)^{(d-1)(d-1)} u\chi y_{d-1} = u\chi y_{d-1}.$$

Thus

$$z = \widehat{g}^*(\mu \cdot y_{d-1}) \text{ and } t^*(\mu \cdot y_{d-1}) = f^*u.$$

Suppose E and F are vector bundles over X, then the embedding $E \longrightarrow E \oplus F$ given by $e \to (e, 0)$ induces a mapping of Thom spaces $\widehat{i} : X^E \to X^{E\oplus F}$.

PROPOSITION (3.9). *We have* $\widehat{i}^* U_{E\oplus F} = U_E \cup \chi(F)$, *where* $U_E | U_{E\oplus F}$ *are the Thom classes and* $\chi(F)$ *is the Euler class of* F.

Proof. Consider the product bundles

$$\begin{array}{ccc} E\times X & \xrightarrow{i} & F\times F \\ \big\downarrow & & \big\downarrow \\ X\times X & \xrightarrow{id} & X\times X \end{array}$$

where $i(e,x) = (e, s_F(x))$ and s_F is the 0-section of F. It induces a map of Thom spaces

$$X^E \wedge X \overset{k\wedge s}{\to} X^E \wedge X^F$$

and $(k \wedge s_F)^*(U_E \otimes U_F) = U_E \otimes \chi(F)$.

Now using $\Delta : X \to X \times X$ to pass to Whitney sums, (3.9) follows.

Now the mapping $\widehat{f} : \widetilde{Y}^{\widetilde{N}} \to Y^N$ is really the composite

$$\widetilde{Y}^{\widetilde{N}} \to \widetilde{Y}^{g^*N} \to Y^N$$

where

$$0 \to \widetilde{N} \to g^*N \to W \to 0.$$

Then by (3.9) $\widehat{f}^* U_N = U_{\widetilde{N}} \cup \chi(W)$

Now $\chi(W)$ is computed in:

PROPOSITION (3.10). *The Euler class* $\chi(W)$ *is given by*

$$\chi(W) = C_{d-1}(N) - xC_{d-2}(N) + \cdots + (-1)^{d-1}x^{d-1}.$$

Proof. We have

$$C(W) \cup C(N) = g^*C(N).$$

Now $C(\widetilde{N}) = 1 + x$, hence $C(W) = g^*C(N)(1+x)^{-1}$ or $C(W) = g^*C(N)(1 - x + x^2 - \cdots + (-1)^k x^k + \cdots)$ and since $\chi(W) = C_{d-1}(W)$ the result follows.

The Euler class $C_d(N) \in H^{2d}(Y)$ is in the image of $H^*(X)$ since we have $H^*(Y^N) \xrightarrow{t^*} H^*(X) \xrightarrow{k^*} H^*(Y)$ and $U_N \to C_d(N)$. Let $\overline{C_d} = t^*U_N$.

When $k^* : H^*(X) \to H^*(Y)$ is onto, as for example where $X = Y \times Y$ and $Y \hookrightarrow Y \times Y$ is the diagonal embedding having the tangent bundle as normal bundle, then we may *choose* classes $\overline{C_i} \in H^{2i}(X)$ with $k^*\overline{C_i} = C_i, i = 1, 2, \ldots, d-1$. Let $P(V)$ denote the polynomial algebra over Z on a single generator V of dimension 2. Then we have:

THEOREM (3.11). *When* $k^* : H^*(X) \to H^*(Y)$ *is onto the mapping* $\phi : H^*(X) \otimes P(V) \to H^*(\widetilde{X})$ *given by*

$$\phi(x \otimes v^r) = \begin{cases} f^*x & \text{if } r = 0 \\ [k^*x \cup u^r] & \text{if } r > 0 \end{cases}$$

is an epimorphism of algebras. The kernel is the ideal generated by $(\mathrm{Ker}\, k^*) \otimes \overline{P}(V)$ *and the class*

$$z = V^d - V^{d-1}\overline{C}_1 + \cdots + (-1)^d \overline{C}_d$$

where $k^*\overline{C}_i = C_i$.

This theorem follows easily from the above results.

4. The Chern classes of $\widetilde{X}$

In this section, we compute the Chern classes of (the tangent bundle of) $\widetilde{X}$. Formulas similar to these appear in [3], [4] and [6]. Our formula looks a little bit different from that of A.T. Lascu and D.B. Scott [4, p.227], because we did not use reverse formulas, but more importantly, we split the image of $j^*f^*(TX)$ more than they do. Their formula can be obtained by the same argument I use below.

We begin by recalling that if E is a complex d vector bundle over a complex manifold Y and

$$q : P(E) \to Y$$

is the projective bundle, then q^*E splits

$$0 \to L \to q^*E \to W \to 0$$

where L is the canonical line bundle. Also

$$T(P(E)) = q^*T(Y) \oplus B(T\mathbb{C}P^{d-1})$$

where $B(T\mathbb{C}P^{d-1})$ is the bundle along the fibers. In this case

$$B(T\mathbb{C}P^{d-1}) \cong \text{ Hom } (L, W) \cong L^* \otimes W$$

hence

$$T(P(E)) = q^*T(Y) \oplus L^* \otimes W \tag{4.1}$$

From this formula, one may compute $C(P(E))$, or $C(\widetilde{Y})$ namely $C(\widetilde{Y}) = g^*C(Y)C(L^* \otimes W)$.

Consider now

$$\begin{CD} \widetilde{X}-\widetilde{Y} @>\widetilde{j}>> \widetilde{X} \\ @VgVV @VVfV \\ X-Y @>j>> X \end{CD}$$

then since g is an isomorphism and $\widetilde{X}-\widetilde{Y}$ and $X-Y$ are open submanifolds of $\widetilde{X}$ and X respectively, we have

$$g^*j^*T(X) = \widetilde{j}^*f^*T(X) = \widetilde{j}^*(T\widetilde{X})$$

and hence $\widetilde{j}^*(C(\widetilde{X}) - f^*C(X)) = 0$.

It follows that $C(\widetilde{X}) - f^*C(X) = \widetilde{t}^*(u \cdot a)$, since $H^*(\widetilde{X}, \widetilde{X} - \widetilde{Y}) \cong H^*(\widetilde{Y}^{\widetilde{N}})$ by excision.

We now describe the class $a \in H^*(\widetilde{Y})$.

First we define a certain "Chern class" $B(L,E)$ for any bundle E and line bundle L.

PROPOSITION (4.2). *For E and L as above, one has:*

$$C(E \otimes L) = C(E) + C_1(L)B(L,E).$$

Proof. By the splitting principle, one has

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_k.$$

Suppose $C_1(L_i) = x_i, C_1(L) = x$, then

$$\begin{aligned} C(E \otimes L) &= \prod_{i=1}^{k}(1+x+x_i) \\ C(E) &= \prod_{i=1}^{k}(1+x_i) \end{aligned}$$

and

$$C(E \otimes L) - C(E) = xB(L,E) = C_1(L)B(L,E).$$

Now we are ready to state the result on the Chern classes of $\widetilde{X}$.

THEOREM (4.3) *Suppose $Y \hookrightarrow X$ is an embedding with normal bundle N, $\widetilde{Y}$ and $\widetilde{X}$ the divisor and the blow up of X along Y with $0 \to \widetilde{N} \to g^*N \to W \to 0$ the splitting of N when lifted to $\widetilde{Y}$. Then*

$$C(\widetilde{X}) = f^*C(X) + \widetilde{t}^*[u(g^*C(Y)C(\widetilde{N})B(\widetilde{N}^*, W))].$$

Proof. Consider the blow up diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{k}"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[r, "k"] & X \end{tikzcd}$$

We need to compute $\widetilde{k}^*(C(\widetilde{X}) - f^*C(X))$.

Since $\widetilde{k}^*T(\widetilde{X}) = T(\widetilde{Y}) \oplus \widetilde{N}$,

$$\widetilde{k}^*(C(\widetilde{X})) = C(\widetilde{Y})C(\widetilde{N})$$

Again $C(\widetilde{Y}) = g^*C(Y)C(L^* \otimes W)$ so $\widetilde{k}^*(C(\widetilde{X}) - f^*C(X)) = g^*C(Y)C(\widetilde{N}^* \otimes W)C(\widetilde{N}) - \widetilde{k}^*f^*C(X)$ But $\widetilde{k}^*f^*(C(X)) = g^*k^*C(X)$ and since $k^*T(X) = T(Y) \oplus N$ and $g^*N = \widetilde{L} \oplus W$ we have $g^*k^*C(X) = g^*C(Y)C(\widetilde{N})C(W)$ and $\widetilde{k}^*(C(\widetilde{X}) - f^*C(X)) = (g^*C(Y))C(\widetilde{N})[C(\widetilde{N}^* \otimes W) - C(W)]$ thus $\widetilde{k}^*(C(\widetilde{X}) - f^*C(X)) = g^*(C(Y))C(\widetilde{N}) \cdot C_1(\widetilde{N}^*)B(\widetilde{N}^*, W))$ and thus the result follows.

5. Real blow ups

The definition is that given in § 1 where instead of $\mathbb{C}$, now use $\mathbb{R}$. Again, § 2 is already made to apply to $\widetilde{X} \xrightarrow{f} X$ the real blow up. In section 3, the Mayer-Vietoris sequence (3.4) holds with Z_2 coefficients, and (3.5), (3.6) and (3.7) hold. Proposition (3.10) holds replacing C by W where x now is 1 dimensional.

Theorem (3.11) holds also, where dimension of V is 1 and $\overline{C}_i$ are replaced by $\overline{W}_i$. Theorem 4.3 holds replacing Chern classes by Stiefel Whitney classes.

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