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DEFORMATIONS OF FOLIATED QUATERNIONIC STRUCTURES AND THE TWISTOR CORRESPONDENCE

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In memory of José Adem

0. Introduction

The deformation theory of complex structures on a complex manifold evolved from a deep inter-relationship between the analysis of elliptic operators and sheaf-theoretic/cohomological methods (see [14] for a survey).

For a connected Lie group G, the deformation theory of G-structures on an arbitrary manifold M, is generally not so exact and the analytic setting is difficult to achieve.

When M is a quaternionic manifold ($\dim_{\mathbb{R}} M = 4m$) and $G = GL(m, \mathbb{H})$ $GL(1, \mathbb{H})$ (see §3), one may consider quaternionic structures in the sense of [12]. There is then a complex manifold $Z(\dim_{\mathbb{C}} Z = 2m + 1)$ which is the total space of an S^2 — fibration $Z \longrightarrow M$. The geometry of this fibration has been well thought out and goes by the name of the *Penrose-Salamon correspondence* or the *(generalized) twistor correspondence*.

Using some general notions pertaining to families of G-structures and quaternionic geometry, we describe how deformations of such an M can, in a certain way, be regulated by deformations of the corresponding (almost) complex structures on Z where a well-developed holomorphic theory prevails. Furthermore, the aspects of deformation theory discussed here also extend to that of holomorphic vector bundles on Z. In principle, the comparison of the deformation theories via the twistor correspondence should be applicable to the study of associated moduli spaces and their deformations with regards to *'special'* bundles on Z(*cf*. [10]).

We show that such an approach applies equally well when M carries a foliation locally spanned by quaternionic vector fields (§4). Lifting such a foliation to Z gives a holomorphic distribution which may very often induce a holomorphic foliation on Z (which will then be automatically *'transversely holomorphic'* [5]). Such a lifting was one of the principal ideas studied in a recent work of the first named author et al. [6] in which several important examples were presented. Bringing into play the generalized notion of self-dual connections on vector bundles on M, we can consider the corresponding holomorphic vectorbundles induced via pullback on Z. This is an application of the Atiyah-Ward correspondence generalizing the case $m = 1$ as written in [10]. In [6], this theory was cast into the foliation context but the deformation theory of such was not considered. In this paper we discuss mainly some aspects of deformation theory in this setting.

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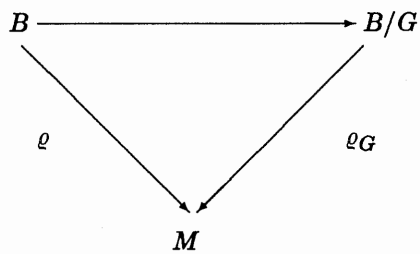
1. G-structures and their deformations

Let M be an n-manifold and $G \subset GL(n, \mathrm{R})$ a Lie group. A G-structure on M is given by a reduction of the principal tangent frame bundle of TM from $GL(n, \mathrm{R})$ to G. More precisely, let B_G denote the manifold of all G-frames. There is a principal fibration $G \hookrightarrow B_G \to M$ from which the principal bundle $GL(n, \mathrm{R}) \to B \to M$ results as the extension of the structure group of B_G to $GL(n, \mathrm{R})$ and so realizes the bundle $GL(n, \mathrm{R}) \hookrightarrow B \to M$ as the principal tangent frame bundle. We have the diagram



with $\varrho_G^{-1}(m) = GL(n, \mathrm{R})/G$.

Consider a local diffeomorphism $f : M \to M$ lifting to a bundle automorphism $f_* : B \to B$. We say that f is a local G-automorphism if $f_*(B_G) \subset B_G$. Let X be a local vector field on M generating a local 1-parameter group $f(t) = \exp(tX)$ of local diffeomorphisms of X. We say that X is an infinitesimal G-automorphism if the $f(t)$ are local G-automorphisms and one may then consider Θ_G the sheaf of germs of infinitesimal G-automorphisms [7].

Let $\mathcal{U} \subset \mathrm{R}^\nu$ be an open neighborhood of 0 in R^ν with parameter $t = (t_1, ..., t_\nu)$ and let $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ be a smooth fibre bundle with fibre M. The structure group of $T\mathcal{W}$ may be described as follows: Consider the group of all matrices

$$\begin{bmatrix} \mathrm{a} & * \\ 0 & \mathrm{b} \end{bmatrix}$$

where $\mathrm{a} \in GL(n, \mathrm{R}), b \in GL(\nu, \mathrm{R})$ and $* \in Hom(\mathrm{R}^\nu, \mathrm{R}^n)$. Let G^* be the linear group of all matrices of this type where $\mathrm{a} \in G$ (let $G' \subset G^*$ consist of the subgroups where $* = 0$).

For a given G^*-structure on $\mathcal{W}$ there is on the fibre $M_t = \omega^{-1}(t)$ an induced G-structure

$$G \to B_G(t) \to M_t.$$

For an open set $\mathcal{U} \subset M$, there is a natural G-structure on $\mathcal{W} \times \mathcal{U}$ induced from that on $\mathcal{W}$. Let $\rho : \mathcal{W} \times \mathcal{U} \to \mathcal{U}$ be the projection. If $\mathcal{W}$ possesses a G^*-structure, then we say that $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ gives a deformation of the G-structure on M if:

1. There exists a G-diffeomorphism between M and $M_0 = \omega^{-1}(0)$.

2. The fibre bundle $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ is locally trivial in the following sense:

Let $A \subset \mathcal{U}$ be an analytic set through the origin 0 in $\mathbf{R}^{\nu}$ and assume that $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ admits a G^*-structure satisfying condition 1. For each $x_0 \in M_0$, there exists a neighborhood $A(x_0)$ of x_0 in $\mathcal{W}$, a neighborhood $W(x_0)$ of x_0 in M_0, a diffeomorphism $f_{x_0} : A(x_0) \to W(x_0) \times \mathcal{U}$ with $\rho \circ f = \omega$ such that $f_{x_0}|\omega^{-1}(t) \cap A(x_0) \to W(x_0) \times \{t\}$ is a G-isomorphism for $t \in$ A. By some abuse of notation, we set $\mathcal{W} = \omega^{-1}(\mathcal{U})$ and the deformation is written as $\{\mathcal{W} \xrightarrow{\omega}$ A$\}$, the existence of the ambient spaces understood.

2. Foliations and foliated bundles

Let M be an oriented n-dimensional manifold and let $\mathcal{F}$ be an oriented foliation on M of dim $\mathcal{F} = p$. Let $T\mathcal{F}$ be the tangent bundle along the leaves of $\mathcal{F}$ and let Q be the normal bundle to $\mathcal{F}$. Then we have the exact sequence

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \longrightarrow Q \longrightarrow 0.$$

A metric g_M on M is taken to give an identification $T\mathcal{F}^{\perp} \cong Q$ and $g_M = g_L + g_Q$. Let G be a connected Lie group and $\psi : P \longrightarrow M$ a principal G-bundle. We say that P is a foliated G-bundle if P has a foliation $\widetilde{\mathcal{F}}$ 'lifted' from $\mathcal{F}$[9]. More precisely:

i) for each $u \in P$, the differential $\psi_* : T_uP \longrightarrow T_{\psi(u)}M$ maps the tangent space of the leaf of $\widetilde{\mathcal{F}}$ isomorphically onto the tangent space of the leaf of $\mathcal{F}$;

ii) the G-action permutes the leaves of $\widetilde{\mathcal{F}}$ and for $X \in C^{\infty}(T\mathcal{F})$, the lift $\widetilde{X} \in C^{\infty}(T\widetilde{\mathcal{F}})$ such that $\psi_*\widetilde{X} = X$, satisfies $R_{g*}\widetilde{X} = \widetilde{X}$, where R_g is the right action by $g \in G$. This definition adapts in the usual way to any vector bundle V associated to P.

If $\mathfrak{g} = Lie\ G$, the sequence

$$0 \longrightarrow P \times \mathfrak{g} \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0$$

$$(u, \xi) \longrightarrow \xi_u^*$$

is equivalent to the (foliated) Atiyah sequence of P

$$0 \longrightarrow Ad(P) = P \times_{Ad} \mathfrak{g} \longrightarrow TP/G \xrightarrow{\pi_*} TM \longrightarrow 0. \tag{2.1}$$

Suppose now Z is a complex manifold and $T\mathcal{F}$ denotes an integrable holomorphic subbundle of $T^{1,0}Z$ of codimension q. Then $T\mathcal{F}$ is the holomorphic tangent bundle of a holomorphic foliation on Z of codimension q. At the level of sheaves we have

$$0 \longrightarrow \underline{T\mathcal{F}} \longrightarrow T^{1,0}Z \longrightarrow \underline{Q} \longrightarrow 0. \tag{2.2}$$

Equivalently, the holomorphic foliation is given by an integrable distribution (of fixed dimension) on Z, spanned by holomorphic vector fields. The notion of a holomorphic foliated principal (or vector) bundle follows by adapting the above definition to this category. For $G_{\mathbf{C}}$ a connected complex Lie group, we have the exact sequence

$$0 \longrightarrow Ad(P) \longrightarrow T^{1,0}P/G_{\mathbb{C}} \longrightarrow T^{1,0}Z \longrightarrow 0. \tag{2.3}$$

3. Quaternionic manifolds and their twistor spaces

Let us now take $\dim_{\mathbf{R}} M = 4m (m > 1)$. We shall say that M is *almost quaternionic* if M is equipped with a

$$GL(m,\mathbb{H})GL(1,\mathbb{H}) := GL(m,\mathbb{H}) \times_{\mathbf{R}} GL(1,\mathbb{H})$$

structure. We shall denote by G_M such a structure on M. Note that this structural group is the same as $GL(m,\mathbb{H})Sp(1) := GL(m,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$. Equivalently, there is a distinguished rank 3 sub-bundle $\mathcal{G} \subset \text{End } TM$ for which there exists a local basis $\{I, J, K\}$ satisfying the usual quaternion identities

$$I^2 = J^2 = K^2 = -1, \qquad IJ = K = -JI, \text{ etc.}$$

Let E and H respectively denote the vector bundles associated to the fundamental representations of $GL(m,\mathbb{H})$ and Sp(1) respectively on $\mathbb{C}^{2m}$ and $\mathbb{C}^2$. Then T^*M as a bundle associated to a $GL(m,\mathbb{H})Sp(1)$-module is $T^*_{\mathbf{C}}M \cong E \otimes H$ and

$$\Lambda^2 T^*_{\mathbf{C}}M \cong S^2E + \Lambda^2 E \otimes S^2 H. \tag{3.1}$$

In this context, if $g_M(IX, IY) = g_M(X,Y)$ for any local section I of $\mathcal{G}$, with $I^2 = -1$, then M is said to be *quaternion-Hermitian*. Here, the structure G_M of M reduces to $Sp(m)Sp(1)$ and

$$\Lambda^2 T^*_{\mathbf{C}}M \cong S^2H + S^2E + \Lambda^2_0 E \otimes S^2 H \tag{3.2}$$

where $\Lambda^2 E \cong \mathbf{R} + \Lambda_0^2 E$ is the decomposition into irreducible $Sp(m)$-modules. *The fundamental* 4*-form* of M is a global 4-form Ω defined locally by

$$\Omega = \omega_I \Lambda\, \omega_I + \omega_J \Lambda\, \omega_J + \omega_K \Lambda\, \omega_K \tag{3.3}$$

for I, J, K a local basis as before and $\omega_I(X,Y) = g_M(X, IY)$ is the local 2-form associated to I, etc. Given a connection $\widehat{A}$ on a bundle on M, let $F_{\widehat{A}}$ denote its curvature: $F_{\widehat{A}} = d\widehat{A} + [\widehat{A}, \widehat{A}]$. We consider the *generalized self-duality* equations

$$^*F_{\widehat{A}} = c_i F_{\widehat{A}} \Lambda\, \Omega^{m-1} \tag{3.4}$$

where * is the Hodge-star operator on M and for $i = 1, 2, 3$, the c_i are constants corresponding to the eigenspaces in (3.2) and were calculated in [4]. We shall restrict attention to $c_2 = c_{S^2E}$ whereby the c_2-self-duality of the connection $\widehat{A}$ gives $F_{\widehat{A}} \in \Omega^2(M, Ad(P) \otimes S^2E)$ which we simply write as $F_{\widehat{A}} \in S^2E$. This condition turns out to be solely determined by the almost quaternionic structure G_M.

A vector field X on M is said to be *quaternionic* if, via its infinitesimal automorphisms, it preserves the structure G_M of M. In the case $m = 1$, $GL(1, \mathbb{H})GL(1, \mathbb{H})$ is simply the conformal group CO(4) on a 4-manifold.

The *twistor space* $\pi : Z \to M$ is the sphere bundle of $\mathcal{G}$ consisting of endomorphisms of square -1. This is an $S^2 \cong \mathbb{C}P^1$ bundle on M and is endowed with an almost complex structure I for each choice of a G_M-connection D on M. The connection D is determined by choice of a horizontal distribution on the principal frames B_{G_M}. Then Z is the bundle associated to the G_M structure of M via the adjoint action of the $GL(1, \mathbb{H})$ factor on S^2. Note that Z itself has a canonical $GL(2m+1, \mathbb{C})$ structure G_Z whereby the horizontal distribution on B_{G_M} determines that on Z. The almost complex structure I at I ϵ Z is defined to be I on horizontal vectors and multiplication by i on vertical vectors, namely those tangent to the fibre S^2. The almost complex structure I depends on the torsion of theG_M-connection D alone where the integrability of I is equivalent to D being a torsion-free connection. Following [12], M is said to be (*'integrable'*) *quaternionic* if M admits a torsion-free G_M-connection. In this way, I on Z determines the quaternionic structure on M.

Let us briefly state several important classes of quaternionic manifolds:

1. If there exists a global basis $\{I, J, K\}$ of $\mathcal{G}$ satisfying the quaternion identities with I, J, K integrable, then M is said to be *hypercomplex* and in particular *hyperkähler* if M also admits closed 2-forms $\omega_I, \omega_J, \omega_K$.

2. A quaternionic manifold is said to be *quaternionic–Kähler* if its linerar holonomy lies in $Sp(m)Sp(1)$. The model example is $M = \mathbb{H}P^m$, quaternionic projective space, with $Z = \mathbb{C}P^{2m+1}$.

For a quaternionic manifold M with twistor space Z, a point $I \in Z$ determines a decomposition of 2-forms $\Lambda^2 T^*_{\mathbb{C}}M = \Lambda_I^{2,0} + \Lambda_I^{1,1} + \Lambda_I^{0,2}$ and this is related to the $GL(m, \mathbb{H})Sp(1)$ decomposition by $S^2E = \bigcap_{I \in Z} \Lambda_Z^{1,1}$. At $I \in Z$, the complex structure I on Z is equivalent to I on horizontal vectors relative to π. As a consequence, if V is a complex vector bundle associated to a principal G-bundle $G \hookrightarrow P \longrightarrow M$ with connection $\widehat{A}$ satisfying $F_{\widehat{A}} \in S^2E$, then π^*V is a holomorphic vector bundle on Z. Here the complex structure $\widetilde{I}$ on π^*V is obtained by taking $\pi^*\widehat{A}$ to give the local splitting $T(\pi^*V) = TZ \oplus \mathbb{C}^r(r =$

rank V) and then take $\widetilde{I} = (I, i)$ where i denotes the usual almost complex structure on $\mathbb{C}^r$.

In particular, if $G = GL(s, \mathbb{H})$, then V is said to be *quaternionic* [12] when $F_{\widehat{A}} \in S^2E$. This generalizes the notion of a vector bundle with self-dual connection ($^*F_{\widehat{A}} = F_{\widehat{A}}$) on a self-dual 4 manifold (see *e.g.* [3],[12]).

4. Foliations on quaternionic manifolds

Let M be a quaternionic manifold. A fundamental observation noted in [6] is that an automorphism f preserving the $G = GL(m, \mathbb{H})GL(1, \mathbb{H})$ structure also preserves $\mathcal{G}$. In this way, we induce an automorphism f_* on Z. The map determined by f_* maps a torsion-free G-connection to the same in the space of G-connections and therefore f_* preserves the almost complex structure I on Z. Taking now a 1-parameter family leads to the following:

LEMMA (4.1) [6]. *Let M be a quaternionic manifold and X a quaternionic vector field on M. Then there exists an induced holomorphic vector field Y on Z such that $\pi_*(Y) = X$.*

The converse to this is given by the twistor correspondence which says that a π-projectable holomorphic vector field Y on Z projects to a quaternionic vector field X on M.

Furthermore, when a Lie group G acts freely on M and preserves the G-structure, we obtain a holomorphic distribution on Z covering the foliation by orbits of the G-action. As in [6], we might ask when a foliation on M locally generated by quaternionic vector fields gives rise to a holomorphic foliation on Z. We state now a general result in this direction as established in [6]:

THEOREM (4.2) [6]. *Suppose $\mathcal{F}$ is a foliation on a quaternionic manifold M such that $\mathcal{F}$ is locally spanned by quaternionic vector fields and such that* $\dim < \mathcal{F}, I\mathcal{F} > = 2 \dim \mathcal{F}$ *for all local sections I of Z. Then lifting the quaternionic vector fields spanning $\mathcal{F}$ to holomorphic vector fields on Z and taking the complex linear span of the resulting distribution defines a holomorphic foliation F on Z of complex dimension* $\dim_{\mathbf{R}} \mathcal{F}$. *Let $\psi : (P, \widetilde{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$ be a foliated principal G-bundle and let V be a complex vector bundle on M associated to P. Suppose that P has a c_{S^2E}-self-dual-connection $\hat{A}$ such that the Lie derivative condition $L_{\widetilde{X}}\widehat{A} = 0$ is satisfied for any quaternionic vector field $X \in C^\infty(U, T\mathcal{F})$. Then π^*V is a holomorphic vector bundle on Z with a holomorphic foliation obtained via pull-back from $\widetilde{\mathcal{F}}$ and $\mathcal{F}$.*

Remarks.

1. In the case of a flow $\mathcal{F} = < X >$ where X is a quaternionic vector field on M, the flow on M lifts directly to a holomorphic flow on Z. Theorem 4.2 applies to a c_{S^2E}-self-dual connection $\widehat{A}$ satisfying the Lie invariance condition $L_{\widetilde{X}}\widehat{A} = 0$, to give a holomorphic vector bundle on Z with holomorphic flow.

2. If in addition to the hypotheses of Theorem 4.2 we have $\dim \mathcal{F} < 2m$, then $T\mathcal{F}$ is a flat bundle *i.e.* $T\mathcal{F}$ admits transition functions which are constants.

3. Generally, the lift to Z of such a quaternionic foliation on M, will induce a holomorphic distribution on Z that may be a *'singular'* foliation in the sense that the leaf-dimensions may vary (cf. the notion of *'singular'* in [2] and [6, Remark 7.16]).

Let $\mathfrak{F}$ be a holomorphic foliation on Z with leaves transverse to the fibre of $\pi : Z \longrightarrow M$ and F is Sp(1)-invariant. Then $\bigcap_{I \in Z_x} \pi_*(T\mathfrak{F})$ gives a foliation on M locally generated by quaternionic vector fields.

5. Deformations of quaternionic structures

Let M be an almost quaternionic manifold with twistor space $\pi : Z \longrightarrow M$. We assume henceforth that M, and therefore Z, are compact. In the notation of §1, let

$$\widetilde{\mathcal{W}} \xrightarrow{\tilde{\omega}} \widetilde{\mathcal{U}} \text{ be a } G_Z\text{-deformation}$$

and

$$\mathcal{W} \xrightarrow{\omega} \mathcal{U} \text{ be a } G_M\text{-deformation.}$$

From the definition of a quaternionic vector field and Lemma 4.1, the following is immediate:

LEMMA (5.1). *An infinitesimal G_M-automorphism of M lifts via π to an infinitesimal G_Z-holomorphic automorphism on Z, and conversely on π-related vector fields.*

PROPOSITION (5.2). *A G_M-deformation induces a G_Z-deformation and conversely on π-related vector fields. In particular, the diagram below is commutative:*

$$\begin{array}{ccc} \widetilde{\mathcal{W}} & \xrightarrow{\tilde{\omega}} & \widetilde{\mathcal{U}} \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{W} & \xrightarrow{\omega} & \mathcal{U} \end{array} \tag{5.1}$$

Proof. From Lemma 5.1, we see that a local G_M-diffeomorphism on M induces a local G_Z-holomorphic automorphism on Z and conversely on π related vector fields. Consequently, we obtain an induced map on principal frames

$$\begin{array}{ccc} B_{G_Z} & \xrightarrow{\ \widetilde{\pi}\ } & B_{G_M} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\ \pi\ } & M \end{array} \tag{5.2}$$

and thus an induced map on the principal frames of $T\mathcal{W}$ and $T\widetilde{\mathcal{W}}$. Recalling from § 1 the structure groups of G_Z^* and G_M^*, we obtain an induced morphism $G_M^* \longrightarrow G_Z^*$.

Also, a G_M-deformation induces the same between M and $M_0 = \omega^{-1}(0)$. Consequently the G_M-connection D on TM varies within its G_M-class. Following the Salamon correspondence in § 3, the associated G_Z-diffeomorphism between Z and $Z_0 = \widetilde{\omega}^{-1}(0)$ deforms I to I_0 with the deformation integrable if D varies through torsion-free G_M-connections. For $t \in \widetilde{\mathcal{U}}$, $s \in \mathcal{U}$, we thus obtain the induced fibre maps

$$\begin{array}{ccc} Z & \xrightarrow{\ \widetilde{g}\ } & Z_t \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{t,s}} \\ M & \xrightarrow{\ g\ } & M_s \end{array} \tag{5.3}$$

The map $\pi_{t,s}$ induces the map $u : \widetilde{\mathcal{U}} \longrightarrow \mathcal{U}$ and $u \circ \widetilde{\omega}$ gives a deformation of the G_M-structure for a given deformation of the G_Z-structure. But at the level of principal frames, this is precisely $\omega \circ w$. □

6. Deformations of quaternionic bundles on M versus deformations of holomorphic vector bundles on Z

Let Z be a compact complex manifold, $G_{\mathbf{C}}$ a connected complex Lie group and $G_{\mathbf{C}} \hookrightarrow P \longrightarrow Z$ a (complex) principal $G_{\mathbf{C}}$-bundle. By a holomorphic family of complex principal bundles on Z parametrized by a complex space $\mathcal{M}$, we mean a holomorphic principal bundle $\mathcal{P} \longrightarrow \mathcal{M} \times Z$. For $t \in \mathcal{M}$, $P_t = \mathcal{P}|_{Z \times \{t\}}$ gives a holomorphic family of principal bundles $\{P_t\}$ on Z.

If, given such a bundle $G_{\mathbb{C}} \hookrightarrow P \longrightarrow Z$, there exists a family $\mathcal{P} = \{P_t\}_{t \in \mathcal{M}}$ such that $G_{\mathbb{C}} \hookrightarrow P \longrightarrow Z$ is isomorphic to $G_{\mathbb{C}} \hookrightarrow P_{t_0} \longrightarrow Z$ for some $t_0 \in \mathcal{M}$, then the family is said to be a *holomorphic family of deformations* of the given bundle. We say that this family is *complete* at t_0 if for any holomorphic family $\mathcal{R} = \{R_s\}_{s \in \mathcal{N}}$ of deformations of $P_{t_0} \approx R_{s_0}$, $s_0 \in \mathcal{N}$, there exists an open neighborhood $\mathcal{U}$ of s_0 in $\mathcal{N}$ and a holomorphic map $f : \mathcal{U} \longrightarrow \mathcal{M}$ such that $\mathcal{P}$ induces $\mathcal{R}$ on $Z \times \mathcal{U}$ by the map $Z \times \mathcal{U} \longrightarrow Z \times \mathcal{M}$ defined by $(z, r) \longrightarrow (z, f(r))$.

Given a deformation of P over $(\mathcal{M}, t_0)$ and a map $f : (\mathcal{N}, s_0) \longrightarrow (\mathcal{M}, t_0)$, we obtain, by pull back, an induced deformation of P over $(\mathcal{N}, s_0)$. The corresponding notion is defined at the level of germs. A given deformation of P parametrized by a complex space germ $(\mathcal{M}, t_0)$ is said to be *versal* if every other deformation of P parametrized by a complex space germ $(\mathcal{N}, s)$ can be induced by a map $f : (\mathcal{N}, s_0) \longrightarrow (\mathcal{M}, t_0)$. If the inducing map is unique, this versal family is said to be *universal.*

In the above notions of versal and universal deformations, the parameter-complex spaces are assumed to be general complex spaces including singularities and nilpotent elements. It is known ([8], [13]) that for any holomorphic principal bundle over a compact complex manifold, there exists a versal family of deformations. The proof given in [13] for reduced base spaces extends to the case of general nonreduced spaces. For the proof in the case of holomorphic vector bundles see [11]. See also Proposition 6.43, page 239 in [3]. More generally for any transversely holomorphic foliation (and hence for any holomorphic foliation) on compact smooth (complex) manifold, there exists a versal space of deformations; see [5]. Under very restrictive conditions, for G-structures there is locally complete family of deformation, see [7]. Let us now recall the Atiyah sequence for complex principal bundles

$$0 \longrightarrow Ad(P) \longrightarrow TP/G_{\mathbb{C}} \longrightarrow TZ \longrightarrow 0.$$

At the level of sheaves

$$0 \longrightarrow \underline{Ad(P)} \longrightarrow \underline{TP/G_{\mathbb{C}}} \longrightarrow \underline{TZ} \longrightarrow 0, \tag{6.1}$$

we take complex forms of type $(0, r)$ on Z :

$$A^r = \underline{Ad(P)} \otimes A^{0,r}, \ B^r = \underline{TP/G_{\mathbb{C}}} \otimes A^{0,r}, \ C^r = \underline{TZ} \otimes A^{0,r}$$

and set

$$\mathcal{A} = \sum_{r \geq 0} A^r, \ \mathcal{B} = \sum_{r \geq 0} B^r, \ \mathcal{C} = \sum_{r \geq 0} C^r.$$

Then we have an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \xrightarrow{h} \mathcal{C} \longrightarrow 0. \tag{6.2}$$

Recall that an almost complex principal bundle structure on $G_{\mathbb{C}} \hookrightarrow P \longrightarrow Z$ is an almost complex structure $\widetilde{I}$ on P such that

i) $\widetilde{I}$ is $G_{\mathbb{C}}$-invariant;

ii) the almost complex structure on $P/G_{\mathbb{C}}$ induced by $\widetilde{I}$ is I;

iii) $\widetilde{I}$ restricted to a fiber gives the integrable almost complex structure on $G_{\mathbb{C}}$.

'Integrability' equations such as

$$\overline{\partial}\omega - \frac{1}{2}[\omega,\omega] = 0 \tag{6.3}$$

for ω close to zero in one of A^1, B^1 or C^1, correspond bijectively to deformations of almost complex structures I on Z and almost complex principal bundle structures $\widetilde{\mathcal{I}}$ or $(\widetilde{I}, I)$ on $G_{\mathbb{C}} \hookrightarrow P \longrightarrow Z$, sufficiently close to some $\mathcal{I}_0, \widetilde{\mathcal{I}}_0$ on $(\widetilde{\mathcal{I}}_0, \widetilde{\mathcal{I}})$.

Using the results of [14, pp. 172-174], we can relate these equations on Z via the Salamon correspondence to deformations of the structures G_M of M and quaternionic principal bundles ($F_{\widehat{A}} \in S^2E$)

$$GL(s,H) \hookrightarrow P_H \longrightarrow M.$$

Hence we obtain the following:

THEOREM (6.1). *For deformations sufficiently close to some fixed structure G_M of M and sufficiently close to some fixed quaternionic structure on P_H, we have:*

i) *a deformation through torsion-free G_M-connections on M corresponds to the equation $\overline{\partial}\varphi - \frac{1}{2}[\varphi,\varphi] = 0$ on Z, where $\varphi \in C^1$ is close to zero;*

ii) *a deformation as in i) and a deformation of the quaternionic structure on P_H, correspond to the equation $\overline{\partial}\psi - \frac{1}{2}[\psi,\psi] = 0$ on Z, where $\psi \in B^1$ is close to zero and $h(\psi) = \varphi$.*

iii) *a deformation of the quaternionic structure on P_H corresponds to the equation $\overline{\partial}\theta - \frac{1}{2}[\theta,\theta] = 0$ on Z, for $\theta \in A^1$ close to zero.*

On considering the foliated Atiyah sequence (2.3), equations such as (6.3) may be considered and Theorem 6.1 can apply to deformations of foliated holomorphic principal (or vector) bundles on Z.

We propose to investigate, in a subsequent paper, the relationships between the various versal families using the above theorem.

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