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RELATIONS IN THE MOD 3 COHOMOLOGY ALGEBRA OF A SPACE

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Introduction

This paper is concerned with a method for establishing relations among cup products and Steenrod operations in the cohomology of any space. The method was developed in [HZ], with a particular application in mind. The primary motivation for writing this paper is to extend the scope of the ideas developed in [HZ]. Parts of the exposition are simpler if we confine our attention to mod 3 cohomology. At the end of the paper, we give the statements of the results for arbitrary odd primes.

To indicate the type of result sought, we first reformulate the case for $p = 3$ of the main result in [HZ].

THEOREM (1). *Let $x \in H^{2n+1}(X; Z/3)$ and suppose, a) $\mathcal{P}^n x = 0$ b) if $y \in H^{6n+4}$ satisfies $\mathcal{P}^2 y = 0$, then $y = \mathcal{P}^1 z$ for some $z \in H^{6n}$. Then, if $n \not\equiv 2 \bmod 3$, the following relation holds,*

$$x \cup \mathcal{P}^1 x \cup \mathcal{P}^2 x = \mathcal{P}^1 w$$

for some $w \in H^{6n+11}$.

Thus, under certain conditions involving the action of the Steenrod algebra, a relation involving cup products and Steenrod operations holds. We call such a relation a *conditional relation*.

A new result along these lines is

THEOREM (2). *Suppose $x \in H^{2n+1}(X; Z/3)$ is the reduction of an integral class and suppose,*

a) $\mathcal{P}^n x = 0$

b) if $(y_1, y_2) \in H^{6n+4} \oplus H^{6n+5}$ satisfies $\Phi(y_1, y_2) = 0$, then $(y_1, y_2) = (\mathcal{P}^1, \mathcal{P}^1\beta)z$ for some $z \in H^{6n}$, where Φ is the 2×2 matrix

$$\Phi = \begin{pmatrix} \beta\mathcal{P}^1 + \mathcal{P}^1\beta & \mathcal{P}^1 \\ 0 & \beta\mathcal{P}^1 + \mathcal{P}^1\beta \end{pmatrix}.$$

Then if $n \not\equiv 2 \bmod 3$, the following relation holds,

$$x \cup (\beta\mathcal{P}^1 x)^2 + \lambda\mathcal{P}^{n+2}\mathcal{P}^1 x = \beta\mathcal{P}^1 w_1 + \mathcal{P}^1 w_2$$

where $\lambda \in Z/3$ and $\lambda = 0$ if $n \equiv 0 \bmod 3$.

The information provided by conditional relations can be used to study an old problem first raised by Steenrod; the problem of whether a given algebra over the Steenrod algebra can be the cohomology algebra of a topological space. For example, let M be the module

$$M = \{x_{2n+1}, \mathcal{P}^1 x_{2n+1}, \beta\mathcal{P}^1 x_{2n+1}\}.$$

COROLLARY (3). *The algebra $U(M)$ is the cohomology algebra of a space only if $n = 1, 4$ or $n \equiv 2 \bmod 3$.*

Recall that, in this case $U(M) = \Lambda(x,y) \otimes Z/3[z]/(z^3)$ where $x = x_{2n+1}, y = \mathcal{P}^1 x$ and $z = \beta y$. Then Cor. 3 follows easily from Th 2; since the $U(M)$ formulation implies that (a) is automatically satisfied for $n \neq 1$ and the value of λ is superfluous. In case $n = 4$, condition (b) may not hold as $(y_1, y_2) = (z^2, 0)$ is not in the image of $(\mathcal{P}^1, \mathcal{P}^1\beta)$.

The paper has the following organization. In section 1, two results are presented, from which Theorem 2 follows. The first of these is then developed in section 2 while the second is developed in section 3. In the final section 4 statements of the analogues of Th. 1 and 2 for primes ≥ 5 are presented.

Section 1

Here we lay out the homotopy theory underlying Th. 2. We denote the Eilenberg-MacLane space $K(Z/3, n)$ by K_n, and a product $\Pi_i K_{n_i}$ by $K(I)$ where $I = (n_1, n_2, \ldots)$.

THEOREM (1.1). *The following tower of fibrations exists;*

$$\begin{array}{ccc} E_2 & \xrightarrow{k_2} & K_{6n+13} \\ \downarrow & & \\ E_1 & \xrightarrow{k_1} & K(6n+9,\, 6n+10) \\ \downarrow & & \\ K_{6n+1} & \xrightarrow{k_0} & K(6n+5, 6n+6) \end{array}$$

where k_0 is represented by $(\mathcal{P}^1\iota_{6n+1}, \mathcal{P}^1\beta\iota_{6n+1})$. The class k_1 is created by a null- homotopy of the composition

$$K_{6n+1} \xrightarrow{k_0} K(6n+5, 6n+6) \xrightarrow{\Phi} K(6n+10, 6n+11)$$

where Φ is the matrix appearing in Th. 2.b. The class k_2 is created by a null-homotopy of the composition

$$E_1 \xrightarrow{k_1} K(6n+9, 6n+10) \xrightarrow{\varphi} K_{6n+14}$$

where φ is represented by $(\beta\mathcal{P}^1, \mathcal{P}^1)$.

THEOREM (1.2). *Let $Y = K(Z, 2n+1)$ and define the homotopy class of $f_0 : Y \to K_{6n+1}$ by $\mathcal{P}^n\iota_{2n+1}$. Then f_0 lifts through the tower of 1.1 and there* *is a choice* $f_2 : Y \to E_2$ *such that the composition* $k_2 \circ f_2$ *is represented by* $\iota \cup (\beta\mathcal{P}^1\iota)^2 + \lambda\mathcal{P}^{n+2}\mathcal{P}^1\iota$ *where* $\iota = \iota_{2m+1}$ *and* λ *is as in Th. 2.*

Now we obtain Th. 2 from these results. Consider the following diagram

$$\text{(1.3)}\qquad \begin{array}{ccccccccc} & & K(6n+8,6n+9) & \xrightarrow{j_2} & E_2 & \xrightarrow{k_2} & K_{6n+13} \\ & & & & \downarrow & & \\ K_{6n} & \xrightarrow{\Omega k_0} & K(6n+4,6n+5) & \xrightarrow{j_1} & E_1 & \xrightarrow{k_1} & K(6n+9,6n+10) \\ & & & & \downarrow & & \\ X & \xrightarrow{g} & Y & \xrightarrow{f_0} & K_{6n+1} & & \end{array}$$

where $g : X \to Y$ is represented by x of Th. 2, and (to avoid clutter) the lifts of f_0 are understood. Consider first the composition $f_1 \circ g$. Since $f_0 \circ g = \mathcal{P}^n x = 0$ by hypothesis (a), there is $g_1 : X \to K(6n+4, 6n+5)$ such that $j_1 \circ g_1 \sim f_1 \circ g$. The composition $k_1 \circ j_1$ is $\Omega\Phi$ from 1.1, and $k_1 \circ j_1 \circ g_1 \sim k_1 \circ f_1 \circ g \sim *$. Therefore, by the hypothesis in part (b), g_1 factors through Ωk_0. Consequently $j_1 \circ g_1$ is null. Now consider the composition $f_2 \circ g$. This map lifts $f_1 \circ g$ which is known from the above argument to be null. Hence there is a map $g_2 : X \to K(6n+8, 6n+9)$ such that $j_2 \circ g_2 \sim f_2 \circ g$. Since the composition $k_2 \circ j_2$ is represented by $(\beta\mathcal{P}^1, \mathcal{P}^1)$, the conclusion of Th. 2 follows from 1.2 and the equation $k_2 \circ f_2 \circ g \sim k_2 \circ j_2 \circ g$.

Remark. Theorems 1.1 and 1.2 assert the existence and evaluation on $\mathcal{P}^n\iota$ of a tertiary cohomology operation associated with a Toda bracket

$$\text{(1.4)}\qquad \langle(\beta\mathcal{P}^1, \mathcal{P}^1), \Phi, \begin{pmatrix} \mathcal{P}^1 \\ \mathcal{P}^1\beta \end{pmatrix}\rangle.$$

By fixing the maps f_1 and f_2, we can regard the diagram in (1.3) as a universal example into which X can be mapped. This formulation achieves precision of calculation at the expense of those parts of the theory of tertiary operations which depend on naturality.

Section 2

In this section we develop the ideas for Th. 1.1. This is just a matter of recollecting some of the ideas from the construction of the classical Adams spectral sequence. For us, the first ingredient is a *stable complex of unstable A-modules*, where A is the mod p Steenrod algebra;

$$\text{(2.1)}\qquad \begin{array}{ccccccccc} P_0 & \xleftarrow{\theta_0} & P_1 & \xleftarrow{\theta_1} & \cdots & \xleftarrow{\theta_{p-1}} & P_p & \xleftarrow{\theta_p} & P_{p+1} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ V_0 & \xleftarrow[\tilde{\theta}_0]{} & V_1 & \xleftarrow[\tilde{\theta}_1]{} & \cdots & \xleftarrow[\tilde{\theta}_{p-1}]{} & V_p & \xleftarrow[\tilde{\theta}_p]{} & V_{p+1} \end{array}$$

where each P_i is a free unstable A-module, V_i is a free A-module mapping surjectively to P_i, each composition $\tilde{\theta}_i \circ \tilde{\theta}_{i+1} = 0$ and the diagram commutes. The portion of (2.1) through stage p is used to construct a tower. The additional stage is used to refine the evaluation of maps into the tower.

We next require a realizability property which could be summarized by saying that the higher Toda bracket $\langle \tilde{\theta}_{p-1}, \ldots, \tilde{\theta}_0 \rangle$ exists and contains 0, where the $\tilde{\theta}_i$ are regarded as maps of Eilenberg-MacLane spectra. One way to achieve realizability is by means of classical Adams resolutions. If there is a stable complex with cohomology isomorphic to coker $\tilde{\theta}_0$, and the stable complex of (2.1) is an initial segment of a resolution for H^*X as an A-module, then the Adams resolution for X can be constructed. From this a suitable tower of spaces can be extracted.

PROPOSITION (2.2). *There exists a stable complex X with* mod p *(reduced) cohomology of the form* $\{x, \beta x, \mathcal{P}^p x, \beta\mathcal{P}^p x\}$ *with* $\mathcal{P}^p\beta x = \beta\mathcal{P}^p x$.

Proof. (due to D. Ravenel) The complex is the mapping cone of a certain map of (stable) Moore spaces, $(q = 2p - 2)$

$$b : P^{pq+1} \to P^2.$$

Recall that $\beta_1 : S^{pq-2} \to S^0$ has order p. So, after raising dimension by 1, it factors through $\beta_1' : S^{pq-1} \to P^1$. Since P^1 is a ring spectrum, we have

$$P^{pq} = S^{pq-1} \wedge P^1 \to P^1 \wedge P^1 \to P^1.$$

Taking $\bar{\beta}_1$ as the composition of the above map followed by the pinch map $P^1 \to S^1$, we have an extension of β, which is annihilated by post-multiplication by p,

$$\begin{array}{ccccc} P^{pq} & \xrightarrow{\bar{\beta}_1} & S^1 & \xrightarrow{\times p} & S^1 \\ & & \uparrow & & \\ & & P^1 & & \end{array}$$

The desired map b is the coextension. Its mapping cone has the stated cohomology by a form of the mod p Hopf invariant one result.

By direct calculation (for $p = 3$) our initial segment of a resolution for H^*X can be constructed with maps

$$\begin{aligned} \tilde{\theta}_0 &= \begin{pmatrix} \mathcal{P}^1 \\ \mathcal{P}^1\beta \end{pmatrix} \\ \tilde{\theta}_1 = \Phi &= \begin{pmatrix} \beta\mathcal{P}^1 + \mathcal{P}^1\beta & \mathcal{P}^1 \\ 0 & \beta\mathcal{P}^1 + \mathcal{P}^1\beta \end{pmatrix} \\ \tilde{\theta}_2 &= \begin{pmatrix} \beta\mathcal{P}^1, & \mathcal{P}^1 \\ 0 & \beta\mathcal{P}^1 \end{pmatrix} \end{aligned} \tag{1.3}$$

$$\tilde{\theta}_3 = \begin{pmatrix} \mathcal{P}^1\beta & -\mathcal{P}^1 \\ 0 & \mathcal{P}^1\beta \end{pmatrix}$$

We shall employ the notation of [HM] to denote the spaces in the tower associated with a stable complex of unstable A-modules

$$(2.4)\qquad \begin{array}{ccc} E_{p-1} & \xrightarrow{k_{p-1}} & K(\Omega^{p-1}P_p) \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ E_1 & \xrightarrow{k_1} & K(\Omega P_2) \\ \downarrow & & \\ K(P_0) & & \end{array}$$

Recall that in this situation, the k-invariant

$$E_s \xrightarrow{k_s} K(\Omega^s P_{s+1})$$

is determined by a null-homotopy for the composition $\theta_s \circ k_{s-1}$,

$$E_{s-1} \xrightarrow{k_{s-1}} K(\Omega^{s-1}P_s) \xrightarrow{\theta_s} K\,(\Omega^{s-1}P_{s+1}).$$

The main reason for the condition $\tilde{\theta}_k \circ \tilde{\theta}_{k+1} = 0$ is to have k-invariants created by null-homotopies after one de-looping, as well as in the tower (2.4).

Applying this construction with the data in (2.3) and then looping down yields the tower

$$(2.5)\qquad \begin{array}{ccccc} E_2 & \xrightarrow{k_2} & K(6n+13, 6n+14) & \xrightarrow[\theta_3]{} & K(6n+18, 6n+19) \\ \downarrow & & & & \\ E_1 & \xrightarrow{k_1} & K(6n+9, 6n+10) & \xrightarrow[\theta_2]{} & K(6n+14, 6n+15) \\ \downarrow & & & & \\ K_{6n+1} & \xrightarrow[\theta_0 = k_0]{} & K(6n+5, 6n+6) & \xrightarrow[\theta_1]{} & K(6n+10, 6n+11) \end{array}$$

and the tower in Th. 1.1 is obtained by restriction to the first row in $\tilde{\theta}_2$ and ignoring $\tilde{\theta}_3$.

Section 3

In this section we construct the maps in Th. 1.2. This is one of the novel features of the argument in [HZ]. The main step is the reduction of the evaluation problem to a problem of pure algebra. This algebraic problem is described in detail here and called the *zig-zag equations*.

The construction is based on the Milnor filtration for a classifying space and the associated spectral sequence for cohomology [RS]. We write $BY = \bigcup_{k\geq 0} B_k$ with $B_0 = *$ and $B_1 = \Sigma Y$. In the spectral sequence for cohomology $E_1^{s,t} = H^{s+t}(\Sigma^s Y^{(s)})$ and (E_1, d_1) is isomorphic to the cobar construction on H^*Y, if this is of finite type, with a degree shift. If H^*Y is a primitively generated Hopf algebra, we can pick a basis of monomials for H^*Y and define *weight* for a monomial as cup length. There results a gradation of E_1 by weights, written $E_1^{s,t,m}$. The differential d_1 preserves weights, although the higher differentials do not.

Our calculation depends on the following phenomenon. The composition

$$Y \xrightarrow{f_0} K(P_0) \xrightarrow{\theta_0} K(P_1)$$

must be null, while after one de-looping, there is a factorization

$$\begin{array}{ccccc} & & B_p & \longrightarrow & \Sigma^p Y^{(p)} \\ & & \downarrow & & \downarrow{\scriptstyle u_p} \\ BY & \xrightarrow[Bf_0]{} & BK(P_0) & \xrightarrow[\theta_0]{} & BK(P_1) \end{array} \tag{3.1}$$

with u_p having $\Sigma^p[\iota|\dots|\iota]$ in one component and 0 elsewhere. This is achieved in Th. 1.2 by taking $Y = K(Z, 2n+1)$ and $n \not\equiv 2 \bmod 3$, since

$$\begin{pmatrix} \mathcal{P}^1 \\ \mathcal{P}^1\beta \end{pmatrix} \mathcal{P}^n = \begin{pmatrix} (n+1)\mathcal{P}^{n+1} \\ n\beta\mathcal{P}^{n+1} + \mathcal{P}^{n+1}\beta \end{pmatrix},$$

and $\mathcal{P}^{n+1}\iota_{2n+2}$ has Milnor filtration p. We can now describe the *zig-zag equation*. We seek a sequence $\{u_1, \dots, u_p\}$ of elements, extending u_p above,

$$u_r \in E_1^{r,*,p}$$

where $*$ may be a multi-degree, and the weight is p. Each u_r is to be regarded as a map

$$u_r : \Sigma^r Y^{(r)} \longrightarrow BK(\Omega^{p-r} P_{p-r+1})$$

where the targets are as in 2.4. These elements are required to satisfy the "zig-zag equations"

$$d_1 u_r = \theta_{p-r} u_{r+1} \tag{3.2}$$

where θ_k are as in (2.1).

We illustrate these equations with the data in (2.3). To begin with, we have

$$\begin{array}{ccccc} & & B_3 & \longrightarrow & \Sigma^3 Y^{(3)} \\ & & \downarrow & & \downarrow{\scriptstyle u_3} \\ BY & \xrightarrow[\mathcal{P}^n]{} & K_{6n+2} & \xrightarrow[\theta_0]{} & K_{6n+6} \times K_{6n+7} \end{array}$$

with $u_3 = (\Sigma^3[\iota|\iota|\iota], 0)$. The first zig-zag equation is

$$d_1 u_2 = \theta_1 u_3.$$

Since $\beta\iota = 0$, a solution is given by

$$u_2 = (\Sigma^2\{[\iota \cdot \beta\mathcal{P}^1\iota|\iota] - [\iota|\iota \cdot \beta\mathcal{P}^1\iota]\}, 0)$$

The next zig-zag equation is

$$d_1 u_1 = \theta_2 u_2 ,$$

and a solution is given by

$$u_1 = (\Sigma^2\iota \cdot (\beta\mathcal{P}^1\iota)^2, 0) .$$

The solutions to these formal equations are not unique; u_2 can vary by elements in $\ker d_1$ while u_1 can vary by elements in $\ker d_1 \cup \mathrm{im}\theta_2$. But there is the following invariance.

THEOREM (3.3). *If Y is an Eilenberg-MacLane space, then in any set of solutions to (3.2) by elements of weight p, and no component of the multi-degree $*$ is congruent to $0 \bmod 2p$, the value of u_1 is unique modulo elements in the image of θ_{p-1} and primitives of H^*Y.*

We sketch the proof after showing the connection between the sequence $\{u_r\}$ and the lifts of $f_0 : Y \to K(P_0)$ into the tower (2.4). This connection is based on the following elementary observation. In the diagram below, of based objects,

(3.4)
$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & & \\ & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} & & \\ & & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

suppose $\beta\alpha$ and gf are null and $fr \sim s\beta$. Let ℓ_1, ℓ_2 and H denote respective homotopies;

ℓ_1 from $*$ to $\beta\alpha$
H from $s\beta$ to fr
ℓ_2 from gf to $*$.

We have a map

$$\Sigma A \longrightarrow C_\beta \to Z$$

where the first map is the coextension to the mapping cone of β using ℓ_1, and the second map extends gs using H and ℓ_2. We also have a map

$$A \to F_f \to \Omega Z$$

where the first map is a lifting of r to the homotopy fibre of f using ℓ_1 and H while the second map is determined by the homotopy ℓ_2. One checks directly that these maps are adjoint, up to reparametrization.

We now turn to the construction of maps from Y into the tower. This work makes use of the fact that $E_2 = E_\infty$ in the cohomology spectral sequence for BY, where Y is an Eilenberg-MacLane space. To begin, we enlarge diagram 3.1 to

$$(3.5p)\qquad \begin{array}{ccccccc} B_{p-1} & \longrightarrow & B_p & \longrightarrow & \Sigma^p Y^{(p)} & \longrightarrow & \Sigma B_{p-1} \\ & & \downarrow{\scriptstyle \tilde{u}_p} & & \downarrow{\scriptstyle u_p} & & \\ BE_1 & \longrightarrow & BK(P_0) & \underset{\theta_0}{\longrightarrow} & BK(P_1) & \underset{\theta_1}{\longrightarrow} & BK(P_2) \end{array}$$

to reveal the pattern of (3.4). Thus $(3.5)_p$ can be filled in on the right by

$$\hat{u}_p : \Sigma B_{p-1} \to BK(P_2)$$

and on the left by

$$B_{p-1} \xrightarrow{\hat{u}_{p-1}} BE_1 \xrightarrow{Bk_1} K(P_2)$$

such that the displayed maps are adjoint, up to homotopy. Then $f_1 : Y \to E_1$, is taken as the adjoint of the composition of the inclusion of ΣY in B_{p-1} with $\tilde{u}_{p-1}$.

Next, we note that $E_2 = E_\infty$ implies that the composition

$$\Sigma B_{p-2} \to \Sigma B_{p-1} \xrightarrow{\hat{u}_p} BK(P_2)$$

is null, since otherwise the cohomology class represented by $\theta_1 u_p$ would be the target of a differential d_r with $r > 1$. Thus $\hat{u}_p$ factors as a composition

$$\hat{u}_p : \Sigma B_{p-1} \longrightarrow \Sigma^p Y^{(p-1)} \xrightarrow{u^*_{p-1}} BK(P_2)$$

The self-map theory in [HZ] is used to show that u^*_{p-1} can be chosen with weight $\equiv 1 \bmod (p-1)$. By construction, $d_1 u^*_{p-1} = \theta_1 u_p$. We can assemble this information in

$$(3.5_{p-1})\qquad \begin{array}{ccccccc} B_{p-2} & \longrightarrow & B_{p-1} & \longrightarrow & \Sigma^{p-1} Y^{(p-1)} & \longrightarrow & \Sigma B_{p-2} \\ & & \downarrow{\scriptstyle \tilde{u}_{p-1}} & & \downarrow{\scriptstyle u_{p-1}} & & \\ BE_2 & \longrightarrow & BE_1 & \longrightarrow & K(P_2) & \underset{\theta_2}{\longrightarrow} & K(P_3) \end{array}$$

to reveal the pattern of 3.4, where u_{p-1} is the adjoint of u^*_{p-1}.

The construction continues inductively, using the pair

$$(\tilde{u}_r, u_r) : (B_r, \Sigma^r Y^{(r)}) \to (BE_{p-r}, BK(\Omega^{p-r} P_{p-r+1}))$$

to produce

$$\hat{u}_r : \Sigma B_{r-1} \to BK(\Omega^{p-r} P_{p-r+2})$$

and

$$\tilde{u}_{r-1} : B_{r-1} \to BE_{p-r+1}.$$

Again $E_2 = E_\infty$ implies that $\hat{u}_r$ factors through $\Sigma^r Y^{(r-1)}$, to produce u^*_{r-1} and the theory in [HZ] guarantees a factorization of weight $\equiv 1 \bmod (p-1)$. Taking adjoints yields $(\tilde{u}_{r-1}, u_{r-1})$. At each stage, $f_r : Y \to E_r$ lifting f_{r-1} is obtained as the adjoint of

$$\Sigma Y \longrightarrow B_{p-r} \xrightarrow{\tilde{u}_{p-r}} BE_r.$$

By construction, we obtain a geometrically induced solution to the zig-zag equations from the u^*_{r-1} extending $\hat{u}_r$.

Now we turn to the invariance of solutions to the zig-zag equations stated in Theorem 3.3. We have introduced weights into (E_1, d_1) of the spectral sequence for H^*BY. Having taken Y to be an Eilenberg-MacLane space, $E_2 = E_\infty$ and [RS]

$$E_2^{s,t} = Ext^{s,t}_{H_*Y}(Z/p, Z/p).$$

The resulting algebra generators for H^*BY are in tri-degrees $(1, *, 1), (2, 2kp, p)$ or $(p, (2k-1)p, p)$. Thus, the following sequence is exact

$$E_1^{r-1,*,p} \xrightarrow{d_1} E_1^{r,*,p} \xrightarrow{d_1} E_1^{r+1,*,p} \tag{3.6}$$

if $2 < r < p$ and also for $r = 2$ if $*$ is not congruent to $0 \bmod 2p$.

We can now prove (3.3). Let $\{u_r\}$ and $\{u^1_r\}$ be a pair of solutions to (3.2) with $u_p = u^1_p$. Then $u^1_{p-1} = u_{p-1} + w_{p-1}$ with $d_1 w_{p-1} = 0$. By (3.6), $w_{p-1} = d_1 v_{p-2}$. Thus

$$d_1 u^1_{p-2} = \theta_2 u^1_{p-1} = d_1(u_{p-2} + \theta_2 v_{p-2}).$$

Inductively, we obtain

$$u^1_r = u_r + \theta_{p-r} v_r + w_r$$

with $d_1 w_r = 0, u_r, v_r, w_r \in E_1^{r,*,p}$. The case $r = 1$ is (3.3).

For the case of Theorem 1.2, we have calculated u_2 of tri-degree $(2, (6n + 8, 6n + 9), 3)$, so the hypotheses necessary to invoke (3.3) are fulfilled. Thus we obtain the conclusion to Theorem 1.2 in the form, $k_2 \circ f_2$ is represented by

$$\iota \cdot (\beta \mathcal{P}^1 \iota)^2 + \alpha \iota$$

for some element of degree $4n + 12$ in the mod 3 Steenrod algebra.

Before analyzing this situation, we point out a feature of the constructions in $(3.5)_r$. At the end of this sequence, $(3.5)_2$, we have

$$\begin{array}{ccccccc} B_2 & \longrightarrow & \Sigma^2 Y^{(2)} & \longrightarrow & \Sigma B_1 & \longrightarrow & \Sigma B_2 \\ \downarrow{\scriptstyle \tilde{u}_2} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle \hat{u}_2} & & \\ BE_{p-2} & \longrightarrow & BK(\Omega^{p-2}P_{p-1}) & \longrightarrow & BK(\Omega^{p-2}P_p) & & \end{array}$$

and $u_1^* = \hat{u}_2$. The element $\hat{u}_2$ arises from the construction in 3.4 involving homotopies for maps to the left of $\hat{u}_2$. While $\hat{u}_2$ can certainly be altered by maps from ΣB_2, such alterations need not arise from the process in 3.4 and thus cannot be invoked in the construction of the map f_{p-1}, as the evaluation of the map depends on the adjoint relationship in 3.4.

To obtain information about the element α, we first enlarge the top of the diagram in Th. 1.1 using all of the data from (2.3), to obtain

$$(3.7)\qquad \begin{array}{ccccc} E_2 & \xrightarrow{k_2} & K(6n+13, 6n+14) & \xrightarrow{\theta_3} & K(6n+18, 6n+19) \\ \downarrow & & & & \\ E_1 & \xrightarrow{k_1} & K(6n+9, 6n+10) & & \end{array}$$

with $\theta_3 k_2$ null, because the tower comes from an Adams resolution for the space constructed in Prop 2.2. Thus the value of $u_1 = k_2 \circ f_2$ is determined up to elements of the form $\alpha = (\alpha_1 \iota, \alpha_2 \iota)$ with $\deg \alpha_1 = 4n+12$, $\deg \alpha_2 = 4n+13$ and

$$\begin{pmatrix} \mathcal{P}^1\beta, -\mathcal{P}^1 \\ 0\ \mathcal{P}^1\beta \end{pmatrix} \begin{pmatrix} \alpha_1 \iota \\ \alpha_2 \iota \end{pmatrix} = 0$$

Now the cohomology of $K(Z, 2n+1)$ is of the form $U(F'_{2n+1})$ and our situation can be represented as

$$(3.8)\qquad \begin{array}{ccccc} \Omega^2 P_4 & \xrightarrow{\theta_3} & \Omega^2 P_3 & \xrightarrow{\theta_2} & \Omega^2 P_2 \\ & & \downarrow{\scriptstyle \alpha} & & \\ & & F'_{2n+1} & & \end{array}$$

with $\alpha \circ \theta_3 = 0$. If we are able to factor α through θ_2, then we can adjust the map f_2 to remove this term. To see what can be done, we use a result from [MP] which states that the mod p Steenrod algebra A is injective as a self-module. The same is true of $\Sigma^{2n+1}A$, which maps surjectively to F'_{2n+1}, and we can lift α to $\alpha' : \Omega^2 P_3 \to \Sigma^{2n+1}A$. If we can choose α' to satisfy the equation $\alpha' \circ \theta_3 = 0$, then α' and hence α, factors through θ_2. So we have to understand the role of excess in the equation $\alpha \circ \theta_3 = 0$.

Now $\mathcal{P}^1\beta$ raises excess by at most 3. So the influence of excess is confined to summands in α_1 or α_2 coming from

$$\operatorname{span}\{\mathcal{P}^{n+1}\mathcal{P}^1\iota\}, \operatorname{span}\{\beta\mathcal{P}^{n+2}\mathcal{P}^1\iota, \mathcal{P}^{n+2}\beta\mathcal{P}^1\iota\}$$

respectively. Using the Adem relations and excess considerations, we have

$$\begin{aligned}\mathcal{P}^1\beta(\mathcal{P}^{n+2}\mathcal{P}^1\iota) &= (n+2)\beta\mathcal{P}^{n+3}\mathcal{P}^1\iota + \mathcal{P}^{n+3}\beta\mathcal{P}^1\iota \\ &= \mathcal{P}^{n+3}\beta\mathcal{P}^1\iota\end{aligned}$$

and

$$\mathcal{P}^1(\mathcal{P}^{n+2}\beta\mathcal{P}^1\iota) = (n+3)\mathcal{P}^{n+3}\beta\mathcal{P}^1\iota.$$

Thus, if $n \equiv 0 \bmod 3$, we can choose α' so that $\alpha' \circ \theta_3 = 0$, while if $n \equiv 1 \bmod 3$, we are prevented only by a summand of α_1 involving $\mathcal{P}^{n+2}\mathcal{P}^1\iota$. This completes the proof of Th. 1.2.

Section 4. Primes ≥ 5

Let p be an odd prime and $q = 2p-2$. To state the corresponding version of Theorem 1, we first introduce the following integers. Fix n and define a_k, b_k, c_k for $1 \leq k \leq \frac{1}{2}(p-3)$ by the equations

$$\begin{aligned}a_1 &= 2np, b_k = a_k + q, c_k = b_k + (p-1)q - 1 \\ a_{k+1} &= c_k - 1.\end{aligned}$$

THEOREM (4.1). [HZ]. *Let $x \in H^{2n+1}(X : Z/p)$ and suppose*

a) $\mathcal{P}^n x = 0$

b) $\ker \mathcal{P}^{p-1}|H^{b_k} = \operatorname{im}\mathcal{P}^1|H^{a_k}$
$\ker \mathcal{P}^1|H^{c_k} = \operatorname{im}\mathcal{P}^{p-1}|H^{b_k-1}, 1 \leq k \leq \frac{1}{2}(p-3)$
$\ker \mathcal{P}^{p-1}|H^t = \operatorname{im}\mathcal{P}^1 H^s$ *where*
$t = 2np + 1 + p(p-2)^2, s = t - q$. *Then, if $n \not\equiv -1 \bmod p$,*

$$x \cup \mathcal{P}^1 x \cup \ldots \cup \mathcal{P}^{p-1}x = \mathcal{P}^1 y.$$

To state the corresponding version of Theorem 2, we use the following integers. Fix n and define a_k, b_k, c_k, d_k for $1 \leq k \leq p-1$ by;

$$\begin{aligned}a_k &= 2np + qk, b_k = a_k + 1, c_k = a_k - (q+1) \\ d_k &= a_k - q.\end{aligned}$$

We also define 2×2 matrices over A;

$$\Phi_k = \begin{pmatrix} \varphi_k, (-1)^{k+1}\mathcal{P}^1 \\ 0, \varphi_k \end{pmatrix}$$

with $\varphi_k = k\beta\mathcal{P}^1 - (k+1)\mathcal{P}^1\beta$. We have $\varphi_k \circ \varphi_{k-1} = 0$ and $\mathcal{P}^1\varphi_{k-1} = \varphi_k \circ \mathcal{P}^1$.

THEOREM (4.2). *Let $x \in H^{2n+1}(X; Z/p)$ be the reduction of an integral class and suppose*

a) $\mathcal{P}^n x = 0$

b) $\ker\ \Phi_1 | H^{a_1} \oplus H^{b_1} = \operatorname{im}\ (\mathcal{P}^1, \mathcal{P}^1\beta) | H^{2np}$
$\ker\ \Phi_k | H^{a_k} \oplus H^{b_k} = \operatorname{im}\ \Phi_{k-1} | H^{c_k} \oplus H^{d_k}$
$2 \leq k \leq p-2$. *Then, if* $n \not\equiv -1 \bmod p$

$$x \cup (\beta\mathcal{P}^1 x)^{p-1} + \lambda\mathcal{P}^{n+p-1}\mathcal{P}^1 x = \beta\mathcal{P}^1 w_1 + \mathcal{P}^1 w_2$$

with $\deg w_1 = a_{p-1}$ *and* $\deg w_2 = b_{p-1}$ *and* $\lambda \in Z/p, \lambda = 0$ *unless* $n \equiv 1 \bmod p$.

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