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ON THE CHARACTER VARIETY OF GROUP REPRESENTATIONS OF A 2-BRIDGE LINK $p/3$ INTO $PSL(2,\mathbb{C})$

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In studying hyperbolic structures on the complement of a knot or link in S^3, one is led to consider the holonomies of these structures. They are, up to conjugation in $PSL(2,\mathbb{C})$, representations of the group G of the knot or link in $PSL(2,\mathbb{C})$. The set of conjugacy classes of non abelian representations of G into $PSL(2,\mathbb{C})$ is in a natural way a closed algebraic set ([CS], see also [GM]) which is called the *character variety of representations of G into* $PSL(2,\mathbb{C})$. This variety has been intensively studied by a number of authors, starting with the pioneering and beautiful work of Riley ([Ri_1], [Ri_2], [Ri_3]). In the paper [HLM1] this character variety was computed for the group of a 2-bridge knot or link p/q. The result of the computation is a polynomial in two variables (three, if p is even) which represents a curve (surface) in $\mathbb{C}^2$ ($\mathbb{C}^3$ respectively). The computation was carried out using a recursion procedure, essentially due to Burde ([Bu], see also [FK], [He], [Kl]), which consisted of looking at $SL(2,\mathbb{C})$ as the elements of norm 1 in the quaternion algebra $M(2,\mathbb{C})$.

This recursion procedure is very useful for computational purposes. In fact, given p,q it is fairly easy to obtain the corresponding polynomial. However, for purposes of obtaining general results, it would be preferable to have a general formula, in the variables p,q (and in the polynomial indeterminates, as well) giving the different polynomials for the different values of p,q.

Computational evidence shows that such a formula must exist if q is maintained constant. To be more precise, fixing q, it seems possible to write formulas for the polynomials of the links p/q which belong to the same class of $p \mod q$. We have carried out this computation in the simplest case $q=3$. It is not surprising (see [Ri_1], [Ri_2] and [T]) that families of polynomials appear that are closely related to the Morgan-Voyce polynomials.

To obtain these results, which are the contents of the present paper, we have used a different method of computation from the one used in [HLM1]. As Riley says in [Ri_1], it is possible to use the Fricke-character method. (See [CS], [Ma] and [GM]). It has been already used for computing the polynomials of the knot 5/3 ([Wh],[HLM2]) and the link 8/3 [HLM3]. This is the method we will use in this paper. The main results are Theorems (2.2), (2.5) and Corollary (2.4), where formulas for the polynomials are given. An important and easy consequence of these results is that the curves of non abelian representations of $p/3$ into $PSL(2,C)$ are always irreducible.

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The shape of the curve is obtained in section 3. It follows another proof that the number of arithmetic orbifolds $(p/3, n)$, for a fixed p, must be finite, where $(p/3, n)$ has singular set $p/3$ and isotropy cyclic of order n. (We refer to [Du], [HKM], [HLM1], [HLM2], [HLM3], [HLM4], [MR], [Re], [Ta], [Thu] and [V] for the definitions and results concerning orbifolds and arithmeticity.)

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1. Some families of polynomials

We will need some particular families of polynomials, closely related to the Morgan-Voyce polynomials, whose definition and properties are given below.

Definition (1.1). $\mathcal{P} = \{p_m(z) | m \in \mathbb{Z}, m \geq -1\}$ is the family of polynomials defined by the recurrence formula

$$p_m = zp_{m-1} - p_{m-2}$$

with the initial values $p_{-1} = 0, p_0 = 1$.

These polynomials are closely related to the Morgan-Voyce polynomials $B_n(x)$ (see[S]), since $p_n(z) = B_n(z-2)$. They are also related to the polynomials $\overline{\rho}_{n+1}(x,y)$ defined in [T]. In fact $p_n(z) = \overline{\rho}_{n+1}(z, -1)$.

PROPOSITION (1.2). *The polynomials $p_m(z)$ of $\mathcal{P}$ have the following properties:*

(1) $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} p_n & -p_{n-1} \\ p_{n-1} & -p_{n-2} \end{pmatrix}$.

(2) $p_m^2 = 1 + p_{m+1}p_{m-1}, \quad m \geq 1$

(3) $p_{m-1}^2 + p_m^2 = 1 + zp_mp_{m-1}, \quad m \geq 1$

(4) $p_{2n} = p_n^2 - p_{n-1}^2 = \sum_{i=0}^{n} \binom{2n-i}{i} (-1)^i z^{2(n-i)}, \quad n \geq 0$

$$p_{2n+1} = p_n(p_{n+1} - p_{n-1}) = \sum_{i=0}^{n} \binom{2n+1-i}{i} (-1)^i z^{2(n-i)+1}, \quad n \geq 0$$

(5) $p_m(-z) = (-1)^m p_m(z)$

(6) *If* $p_m(z_0) = 0$, *then* $p_{m-1}(z_0) = -p_{m+1}(z_0) = \pm 1$

(7) *For every* $m \geq 0$, $p_m(z)$ *has degree* m, *and has* m *real roots in the interval* (-2,2).

Proof. [S], [T] and easy inductive arguments. □

PROPOSITION (1.3). a) *For* $z \geq 2$

$$p_m(z) > p_{m-1}(z), \quad m \geq 1.$$

b) *If* $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_{m-1}\}$ *are the roots of* $p_m(z)$ *and* $p_{m-1}(z)$ *we have:*

$$-2 < x_1 < y_1 < x_2 < y_2 < \ldots < x_{m-1} < y_{m-1} < x_m < 2$$

Proof. a) Let us show that if $z \geq 2$, $p_m(z) > p_{m-1}(z)$. This is true for $m = 1$:

$$p_1(z) = z \geq 2 > 1 = p_0(z)$$

If $m > 1$:

$$p_m(z) - p_{m-1}(z) = (z-1)p_{m-1} - p_{m-2} > 0$$

(using $z - 1 \geq 1$ and induction).

b) If follows from (4) in Proposition (1.2) that $z = 0$ is a root of p_{2n+1}, $n \geq 0$. Hence to show that the real roots of $p_m(z)$ are intercalated with those of $p_{m-1}(z)$ it is enought ((5) in Proposition (1.2)) to show that p_{2n+1} and p_{2n} have n different positive roots, say $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_n\}$ respectively, such that

$$0 < y_1 < x_1 < y_2 < \ldots < y_i < x_i < y_{i+1} < \ldots < y_{n-1} < x_{n-1} < y_n < 2$$
$$0 < y_1 < z_1 < y_2 < \ldots < y_i < z_i < y_{i+1} < \ldots < y_{n-1} < z_{n-1} < y_n < z_n < 2$$

where $\{x_1, \ldots, x_{n-1}\}$ are the roots of $p_{2n-1}(z)$.

Let us prove this by induction on n. This is clear for $n = 0$. Let $\{x_1, \ldots, x_{n-1}\}$ be the $(n-1)$ positive real roots of p_{2n-1} given by the induction hypothesis.

From Proposition (1.2)(4) we have

$$p_{2n-1}(0) = 0, p_{2n}(0) = (-1)^n, \ p_{2n-2}(0) = (-1)^{n-1}$$

By the induction hypothesis, $p_{2n-2}(z)$ has a root between 0 and x_1, therefore the sign of $p_{2n-2}(x_1)$ is $(-1)^n$. Therefore $p_{2n}(x_1) = x_1 p_{2n-1}(x_1) - p_{2n-2}(x_1) = -p_{2n-1}(x_1) = -(-1)^n a$, where $a > 0$. Since $p_{2n}(0) = (-1)^n$, we conclude that there exists $0 < y_1 < x_1$ such that $p_{2n}(y_1) = 0$.

Similarly, for $1 < i < n-1$, $p_{2n}(x_{i-1}) = -p_{2n-2}(x_{i-1}) = (-1)^{n+i-1} a_{i-1}$, $a_{i-1} > 0$

$$p_{2n}(x_i) = -p_{2n-2}(x_i) = (-1)^{n+i} a_i, a_i > 0$$

because of the induction hypothesis applied to p_{n-2}, as before. Therefore there exists y_i, $x_{i-1} < y_i < x_i$, such that $p_{2n}(y_i) = 0$, $1 < i < n-1$.

Finally, $p_{2n}(x_{n-1}) = (-1)^{2n-1} a_{n-1}, a_{n-1} > 0$.

$$p_{2n}(2) > 0$$

Therefore, there exists y_n between x_{n-1} and 2 such that $p_{2n}(y_n) = 0$. The proof for p_{2n+1} is similar. □

Polynomials $p_m(z)$ are characterized by

$$\begin{cases} p_m(z) = \frac{\sin(m+1)\theta}{\sin\theta} \\ z = 2\cos\theta, \end{cases}$$

see[S]. Hence the roots of $p_m(z) = 0$ are

$$x_k = 2\cos\frac{k\pi}{m+1}, \quad k = 1, 2, \ldots, m.$$

From this Proposition (1.3)(b) can also be obtained.

Definition (1.4). Given two arbitrary polynomials $Q_1(z)$, $Q_2(z)$, we define the family of polynomials

$$\mathcal{F}(Q_1, Q_2) = \{f_m = Q_1 p_m + Q_2 p_{m-1} | p_i \in \mathcal{P}\}.$$

It is easy to check that

$$\begin{aligned} f_m &= z f_{m-1} - f_{m-2} \\ f_0 &= Q_1 \\ f_1 &= z Q_1 + Q_2 \end{aligned}$$

We are interested in some particular cases of $\mathcal{F}(Q_1, Q_2)$, namely

$$\begin{aligned} \mathcal{T} &:= \mathcal{F}(1,-1) = \{t_m = p_m - p_{m-1} | p_i \in \mathcal{P}\} \\ \mathcal{S} &:= \mathcal{F}(-1, 3-z) = \{s_m = -p_m + (3-z)p_{m-1} | p_i \in \mathcal{P}\} \\ \mathcal{R} &:= \mathcal{F}(1, 3-2z) = \{r_m = p_m + (3-2z)p_{m-1} | p_i \in \mathcal{P}\} \\ \mathcal{U} &:= \mathcal{F}(-1, 2z-y) = \{u_m = -p_m + (2z-y)p_{m-1} | p_i \in \mathcal{P}, y \in \mathbb{R}\} \\ \mathcal{V} &:= \mathcal{F}(1, z-y) = \{v_m = p_m + (z-y)p_{m-1} | p_i \in \mathcal{P}, y \in \mathbb{R}\} \\ \mathcal{G} &:= \mathcal{F}(-2, 1) = \{g_m = -2p_m + p_{m-1} | p_i \in \mathcal{P}\} \\ \mathcal{H} &:= \mathcal{F}(1, -2) = \{h_m = p_m - 2p_{m-1} | p_i \in \mathcal{P}\} \end{aligned}$$

PROPOSITION (1.5). *The degree of $t_m, s_m, r_m, u_m, v_m, g_m$ and h_m in z is m. The polynomials t_m, s_m, r_m, g_m and h_m have m real roots. Let $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_{m-1}\}$, $\{z_1, \ldots, z_m\}$, $\{\gamma_1, \ldots, \gamma_m\}$, $\{\eta_1, \ldots, \eta_m\}$, $\{\sigma_1, \ldots, \sigma_m\}$, $\{\rho_1, \ldots, \rho_{m+1}\}$, be the ordered roots of p_m, p_{m-1}, t_m, g_m, h_m, s_m, and r_{m+1}, respectively. Then:*

(a) $-2 < x_1 < \gamma_1 < z_1 < \eta_1 < y_1 < x_2 < \cdots < x_i < \gamma_i < z_i < \eta_i < y_i < \cdots < y_{m-1} < x_m < \gamma_m < z_m < 2 < \eta_m < 3$

(b) $-2 < x_1 < \rho_1 < z_1 < \sigma_1 < y_1 < x_2 < \cdots < x_i < \rho_i < z_i < \sigma_i < y_i < \cdots < y_{m-1} < x_m < \rho_m < z_m < \sigma_m < 2 < \rho_{m+1} < 3$

Proof.

Step 1. $-2 < x_1 < z_1 < y_1 < x_2 < z_2 \cdots < y_{m-1} < x_m < z_m < 2$.

By Proposition (1.3), $t_m(z) = p_m(z) - p_{m-1}(z) > 0$, if $z \geq 2$.
By Proposition (1.2) and (1.3),

$$p_{2n}(z) = p_{2n}(-z) > p_0(-z) = 1, \text{ if } z \leq -2;$$

$$-p_{2n+1}(z) = p_{2n+1}(z) > p_0(-z) = 1, \text{ if } z \leq -2.$$

Therefore, for $z \leq -2$, $t_m(z) = p_m(z) - p_{m-1}(z) \neq 0$. Thus, the real roots of $t_m(z)$ belong to $(-2, 2)$.

Now, $t_m(2) > 0$, $t_m(x_m) = p_m(x_m) - p_{m-1}(x_m) = -p_{m-1}(x_m) < 0$, because $y_{m-1} < x_m < 2$ and $p_{m-1}(y_{m-1}) = 0$, $p_{m-1}(2) > 0$ and y_{m-1} is the "last" root of p_{m-1}. Therefore, there exist z_m between x_m and 2 such that $t_m(z_m) = 0 : x_m < z_m < 2$.

$$\begin{aligned} \text{Also, } t_m(x_i) &= p_m(x_i) - p_{m-1}(x_i) = -p_{m-1}(x_i) = -(-1)^{m+i-1}a_i \\ t_m(y_i) &= p_m(y_i) - p_{m-1}(y_i) = p_m(y_i) = (-1)^{m+i-1}b_i \end{aligned}$$

where $a_i, b_i > 0$, implies that there exists $z_i \in (x_i, y_i)$ such that $t_m(z_i) = 0$, for $i = 1, 2, \ldots, m-1$.

Step 2. $-2 < z_1 < \eta_1 < y_1 < z_2 < \cdots < y_{m-1} < z_m < 2 < \eta_m < 3$.

In fact, for $z \geq 3$, $h_m(z) = p_m(z) - 2p_{m-1}(z) = zp_{m-1}(z) - p_{m-2}(z) - 2p_{m-1}(z) = (z-2)p_{m-1}(z) - p_{m-2}(z) > p_{m-1}(z) - p_{m-2}(z) > 0$. For $z \leq -2$, $h_{2n}(z) = p_{2n}(z) + 2p_{2n-1}(-z) > 0$, because $p_{2n}(z) = p_{2n}(-z) > 0$, and $-p_{2n-1}(z) = p_{2n-1}(-z) > 0$; and $h_{2n+1}(z) = -p_{2n+1}(-z) - 2p_{2n}(z) < 0$. Thus, the real roots of h_m belong to $(-2, 3)$.

Now, $h_m(z) = p_m(z) - 2p_{m-1}(z) = p_m(z) - p_{m-1}(z) - p_{m-1}(z) = t_m(z) - p_{m-1}(z)$.

Then, for $i = 1, 2, \ldots, m-1$, we have

$$\begin{aligned} h_m(y_i) &= t_m(y_i) = (-1)^{m+i-1}a_i, & a_i > 0 \\ h_m(z_i) &= -p_{m-1}(z_i) = -(-1)^{m+i-1}b_i, & b_i > 0. \end{aligned}$$

Therefore, there exists $\eta_i \in (z_i, y_i)$ such that $h_m(\eta_i) = 0$.

Also, $h_m(2) = p_m(2) - 2p_{m-1}(2) = 2p_{m-1}(2) - p_{m-2}(2) - 2p_{m-1}(2) = -p_{m-2}(2) < 0$; and $h_m(3) = p_m(3) - 2p_{m-1}(3) = 3p_{m-1}(3) - p_{m-2}(3) - 2p_{m-1}(3) = p_{m-1}(3) - p_{m-2}(3) > 0$, implies that there exists $\eta_m \in (2, 3)$ such that $h_m(\eta_m) = 0$.

Step 3. $-2 < x_1 < \gamma_1 < z_1 < \eta_1 < y_1 < x_2 < \cdots < x_i < \gamma_i < z_i < \eta_i < y_i < \cdots < y_{m-1} < x_m < \gamma_m < z_m < 2 < \eta_m < 3$.

Note that $g_m = -2p_m + p_{m-1} = -(p_m - p_{m-1}) - p_m = -t_m - p_m$.

Then, for $i = 1, 2, \ldots, m$, we have

$$g_m(x_i) = -t_m(x_i) = (-1)^{m+i-1}a_i, \quad a_i > 0$$
$$g_m(z_i) = -p_m(z_i) = -(-1)^{m+i-1}b_i, \quad b_i > 0$$

Therefore, there exists $\gamma_i \in (x_i, z_i)$ such that $g_m(\gamma_i) = 0$. Thus a) is proved.

Step 4. $-2 < z_1 < \sigma_1 < y_1 < z_2 < \cdots < y_{m-1} < z_m < \sigma_m < 2$.

In fact, for $z \geq 2$, $s_m(z) = -p_m(z) + (3-z)p_{m-1}(z) \leq -p_m(z) + p_{m-1}(z) < 0$. For $z \leq -2$, $s_{2n}(z) = -p_{2n}(z) + (3-z)p_{2n-1}(z) < 0$, because $p_{2n}(z) = p_{2n}(-z) > 0$, $(3-z) > 0$, and $p_{2n-1}(z) = -p_{2n-1}(-z) < 0$; $s_{2n+1}(z) = -p_{2n+1}(z) + (3-z)p_{2n}(z) > 0$. Thus, the real roots of s_m belong to (-2,2).

Now, $s_m(2) < 0$, $s_m(z_m) = -t_m(z_m) + (2-z_m)p_{m-1}(z_m) = (2-z_m)p_{m-1}(z_m) > 0$, because $z_m < 2$, $p_{m-1}(z_m) > 0$. Hence, there exist $\sigma_m \in (z_m, 2)$ such that $s_m(\sigma_m) = 0$.

Also, for $i = 1, 2, \ldots, m-1$, we have

$$s_m(y_i) = -p_m(y_i) = (-1)^{m+i-1}a_i, \quad a_i > 0$$
$$s_m(z_i) = (2-z_i)p_{m-1}(z_i) = (2-z_i)(-1)^{m+i-1}b_i, \quad b_i > 0$$

Therefore, there exists $\sigma_i \in (z_i, y_i)$ such that $s_m(\sigma_i) = 0$.

Step 5. $-2 < x_1 < \rho_1 < z_1 < \sigma_1 < y_1 < x_2 < \cdots < x_i < \rho_i < z_i < \sigma_i < y_i < \cdots < y_{m-1} < x_m < \rho_m < z_m < \sigma_m < 2 < \rho_{m+1} < 3$.

Note that $r_{m+1} = p_{m+1} + (3-2z)p_m = (3-z)p_m - p_{m-1}$. For $z \geq 3$, $r_{m+1}(z) = (3-z)p_m(z) - p_{m-1}(z) \leq -p_{m-1}(z) < 0$. For $z \leq -2$, $r_{2n+1}(z) = (3-z)p_{2n}(z) - p_{2n-1}(z) = (3-z)p_{2n}(-z) + p_{2n-1}(-z) > 0$; $r_{2n}(z) = (3-z)p_{2n-1}(z) - p_{2n-2}(z) = (z-3)p_{2n-1}(-z) - p_{2n-2}(-z) < 0$. Thus, the real roots of r_{m+1} belong to $(-2, 3)$.

Now, for $i = 1, \ldots, m$, we have

$$\begin{aligned} r_{m+1}(y_i) &= (3-y_i)p_m(y_i) = (-1)^{m+i}a_i, \quad a_i > 0 \\ r_{m+1}(z_i) &= (3-z_i)p_m(z_i) - p_{m-1}(z_i) = (2-z_i)p_m(z_i) \\ &= (2-z_i)(-1)^{m+i-1}b_i, \quad b_i > 0; \end{aligned}$$

$2 - z_i > 0$.

Hence, there exists $\rho_i \in (y_i, z_i)$, $i = 1, \ldots, m$, such that $r_{m+1}(\rho_i) = 0$.

Also $r_{m+1}(3) < 0$, $r_{m+1}(2) = p_m(2) - p_{m-1}(2) > 0$, implies the existence of $\rho_{m+1} \in (2, 3)$ such that $r_{m+1}(\rho_{m+1}) = 0$. □

Remark. A different approach to prove the above Proposition is to compute the actual roots of the polynomials involved. For instance one has

$$t_m(z) = \frac{\sin(m+1)\theta - \sin m\theta}{\sin\theta} = \frac{2\sin(m+1/2)\theta\cos(\theta/2)}{\sin\theta}$$

2. The character variety of representations of $[p/3]$ into $PSL(2,\mathbb{C})$

In [HLM1;§2] the character varieties of representations of $[p/q]$ into $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$ are defined. We recall here the notation and definitions.

The group $[p/q]$ of the 2-bridge link p/q (p, q relatively prime, q odd) has the following presentation (see [BZ], for instance):

$$[p/q] = |a, b : r|$$

where $r = \omega b \omega^{-1} a^{-1}$, $\omega = b^{\varepsilon_1} a^{\varepsilon_2} b^{\varepsilon_3} \dots a^{\varepsilon_{p-1}}$, if p is odd (knot), and $r = \omega a \omega^{-1} a^{-1}$, $\omega = b^{\varepsilon_1} a^{\varepsilon_2} b^{\varepsilon_3} \dots b^{\varepsilon_{p-1}}$, if p is even (two component link), and in both cases ε_i is the sign (plus of minus 1) of iq reduced mod $2p$ in the interval $(-p, p)$.

For a representation $\rho : [p/q] \to SL(2,\mathbb{C})$ define $t(\rho) : [p/q] \to \mathbb{C}$ by $t(\rho)(g) = t(\rho(g))$. For each element $g \in G$ the correspondence

$$\rho \longmapsto t(\rho)g$$

defines a function t_g from the set R of conjugacy classes of representations of $[p/q]$ into $SL(2,\mathbb{C})$ to $\mathbb{C}$. This function, can be expressed as a polynomial with integer coefficients in the three variables $y_1 = t_a, y_2 = t_b, z = t_{ab}$. In the case that $[p/q]$ is a knot, the generators a and b are conjugate, thus $t_a = t_b$ and it is enough to consider the two variables $y = y_1 = y_2, z$.

Let $i : R \longrightarrow \mathbb{C}^3$ be the map $i(\rho) = (y_1(\rho), y_2(\rho), z(\rho))$. The image $S([p/q])$ by i of the set of non-abelian representations is called the *surface of representations* of $[p/q]$ into $SL(2,\mathbb{C})$.

Let R_1 be the set of conjugacy classes of representations of $[p/q]$ into $SL(2,\mathbb{C})$, such that $t_a = t_b$. Note that $R_1 = R$ in the case of a knot.

Let $i : R_1 \longrightarrow \mathbb{C}^2$ be the map $i(\rho) = (y(\rho), z(\rho))$. The image $\widehat{C}([p/q])$ by i of the set of non-abelian representations is called the *curve of representations* of $[p/q]$ into $SL(2,\mathbb{C})$.

Let $j : RP \longrightarrow \mathbb{C}^2$ be the map on the set RP of conjugacy of representations of $[p/q]$ into $PSL(2,\mathbb{C})$ such that $t_a = t_b$, given by $j(\rho) = (x(\rho), z(\rho))$, were $x(\rho) := t_{a^2}$. The image $C([p/q])$ by j of the set of non-abelian representations is called the *curve of representations* of $[p/q]$ into $PSL(2,\mathbb{C})$.

Remark (2.1) on $S([p/q])$, $\widehat{C}([p/q])$ and $C([p/q])$.

(1) The character varieties of representations $i(R)$ (and $i(R_1)$ in the case of a knot) are algebraic varieties defined by the polynomials

$$t_r = 2, \quad t_{ra} = t_a, \quad t_{rb} = t_b \qquad \text{(See [GM]).}$$

(2) If $[p/q]$ is a knot, the polynomial $t_{ra} = t_a$ is $t_{\omega b \omega^{-1}} = t_a$, which implies that $t_b = t_a$, as we already know. If $[p/q]$ is a two component link, the polynomial $t_{ra} = t_a$ is $t_{\omega a \omega^{-1}} = t_a$. Thus, $i(R)$ is the subset of $\mathbb{C}^3$ given by the

points (t_a, t_b, t_{ab}) and which are zeroes of the polynomials $t_r = 2$, $t_{rb} = t_b$ in the variables t_a, t_b, t_{ab}. Similarly $i(R_1)$ is the subset of $\mathbb{C}^2$ given by the pairs (t_a, t_{ab}) which are zeroes of the polynomials $t_r = 2$, $t_{rb} = t_b$ in the variables t_a, t_{ab}, and $j(RP)$ is the subset of $\mathbb{C}^2$ given by the pairs (t_{a^2}, t_{ab}) which are zeroes of the polynomials $t_r = 2$, $t_{rb} = t_b$ in the variables t_{a^2}, t_{ab}.

(3) We remark that if $\rho : [p/q] \longrightarrow SL(2,\mathbb{C})$ is a representation such that $\rho(a) = A$, $\rho(b) = B$, so is its conjugate $\overline{\rho}(a) = \overline{A}, \overline{\rho}(b) = \overline{B}$. Therefore if $(y_1,y_2,z) \in S([p/q])$, so does $(\overline{y}_1,\overline{y}_2,\overline{z})$: the equations of $S([p/q])$ have real coefficients. Therefore the equations of $\widehat{C}([p/q])$ and $C([p/q])$ also have real coefficients.

(4) The points of the character variety of representations which correspond to the reducible representations are obtained as follows. Assume $\rho : G \longrightarrow SL(2,\mathbb{C})$ is reducible. Then [CS], $t(ABA^{-1}B^{-1}) = 2$ where $A = \rho(a)$, $B = \rho(b)$. Therefore the reducible representation $(y_1,y_2,z) \in i(R)$ correspond to points satisfying

$$D(y_1,y_2,z) = 0$$

where $D(y_1,y_2,z) = z^2 - zy_1y_2 + y_1^2 + y_2^2 - 4$.

If $y_1 = y_2 = y$, the above polynomial factors as follows

$$(z-2)(z-y^2+2) = 0$$

We have let $x = t(A^2) = y^2 - 2$. Therefore the reducible representations $(x,z) \in j(RP)$ correspond to points satisfying

$$(z-2)(z-x) = 0$$

The points (x,z) for which $x = z$ correspond to the representations $\rho(a) = \rho(b) = A$ any arbitrary matrix, because $z = t(AB) = t(A^2) = t(A)^2 - 2 = x$ in this case. Therefore $j(RP)$ always contains the irreducible component corresponding to the equation $x = z$. In [HLM1] we showed (Proposition 1.7) that the image under i (or j) of the set of classes of abelian representations coincides with the image of the set of classes of reducible representations. Therefore to obtain the equations of $S([p/q])$, $\widehat{C}([p/q])$ and $C([p/q])$ we only have to delete the algebraic component $D = 0$.

(5) If $(x,z) \in C([p/q])$ and $x = t_{a^2} = -2$, this corresponds to representations $\rho : G \longrightarrow SL(2,\mathbb{C})$ such that $\rho(a) = A$ satisfies $A^2 - t(A)A + I = 0$, hence $t(A^2) - t(A)^2 + 2 = 0$, hence $t(A)^2 = 0$, hence $t(A) = 0$. Up to conjugation A equals

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and therefore $B^2 = -I$ also. Then ρ induce a homomorphism

$$\widehat{\rho} : \frac{|a,b : r|}{\langle a^2 \rangle} \longrightarrow PSL(2,\mathbb{C}).$$

The group $(\widehat{[p/q]}) = |a,b : R, a^2|$ is the group of the orbifold $(p/q, 2)$ with underlying space S^3, singular set p/q and isotropy cyclic of order 2. This has universal cover S^3. Therefore $(\widehat{[p/q]})$ is finite (dihedral of order $2p$, as a matter of fact). As a consequence $\widehat{\rho}(\widehat{[p/q]})$ is finite. Therefore $\rho([p/q])$ is finite. Hence the image $\rho([p/q])$ is composed of elliptic elements. Therefore the set of traces of the elements of $\rho([p/q])$ is real. In particular $t(AB) = z$ is real, for every such ρ. We conclude that *the line $x = -2$ cuts $C([p/q])$ in points $(-2,z)$ whose second coordinate z is always real.* (Compare [Bu].)

THEOREM (2.2). *Let $[p/3]$ be the group of the link $p/3$ (p even). Then the algebraic variety $S([p/3])$ is defined by the equation $q((y_1,y_2,z)) = 0$, where*

$$(2.1)\quad q = \begin{cases} v_m(z,y_1y_2) + D(y_1,y_2,z)[p_{m-1}(z)]^2 p_m(z); & \text{if } p = 6m+2 \\ u_m(z,y_1y_2) + D(y_1,y_2,z)[p_{m-1}(z)]^2 p_{m-2}(z); & \text{if } p = 6m-2 \end{cases}$$

and where $D(y_1,y_2,z) = z^2 - zy_1y_2 + y_1^2 + y_2^2 - 4.$

Proof. The two polynomials $t_r - 2 = 0$, $t_{rb} - t_b = 0$ defining $S([p/q])$ reduce to

$$(2.2)\quad \begin{cases} p_1 \equiv t_{a\omega}^2 + t_a^2 + t_\omega^2 - t_{a\omega}t_\omega t_a - 4 = 0 \\ p_2 \equiv t_{a\omega}t_{ab\omega} - t_{a\omega}t_{b\omega}t_a + t_{b\omega}t_\omega + t_a^2 t_b - t_a t_{ab} - 2t_b = 0 \end{cases}$$

In fact, $t_r = t_{\omega a\omega^{-1}a^{-1}} = t_{\omega a}t_{\omega^{-1}a^{-1}} - t_{\omega aa\omega}$ and $t_{\omega aa\omega} = t_{aa\omega\omega} = t_{aa\omega}t_\omega - t_{aa}$ which yields p_1; and similarly for p_2.

In the calculations that follow we will find that p_1, p_2 above have the form

$$(2.3)\quad \begin{cases} p_1 = D(y_1,y_2,z)[q(y_1,y_2,z)]^2 \\ p_2 = D(y_1,y_2,z)\cdot q_2(y_1,y_2,z)\cdot q(y_1,y_2,z) \end{cases}$$

Therefore $S([p/q])$ is the union of $D(y_1,y_2,z) = 0$ and $q(y_1,y_2,z) = 0$. The component $D = 0$ corresponds to the abelian representations. Therefore we are interested in the other component, given by the equation $q(y_1,y_2,z) = 0$.

Use the following notation

$$\begin{aligned} E_{m,n} &= t_{(\bar{b}\bar{a})^m(ba)^n}, \quad E_{1,1} = D+2. \\ G_{m,n} &= t_{b(ba)^n(\bar{b}\bar{a})^m} \\ F_{m,n} &= t_{a(ab)^n(\bar{a}\bar{b})^m} \end{aligned}$$

The following Lemma follows easily by induction:

LEMMA (2.3). *We have:*

(1) $E_{m,0} = E_{0,m} = zp_{m-1}(z) - 2p_{m-2};\ m \geq 1$
$E_{1,0} = E_{0,1} = z$

(2) $E_{m,1} = E_{1,m} = (D+2)p_{m-1}(z) - zp_{m-2};\ m \geq 1$

(3) $E_{m,n} = E_{m,1}p_{n-1}(z) - E_{m,0}p_{n-2}$; $m, n \geq 1$

(4) $G_{m,0} = G_{1,0}p_{m-1}(z) - G_{0,0}p_{m-2}(z)$; $m \geq 1$
$G_{0,0} = y_2$, $G_{1,0} = zy_2 - y_1$, $G_{0,1} = y_1$

(5) $G_{m,1} = G_{1,1}p_{m-1}(z) - y_1p_{m-2}(z)$; $m \geq 1$

(6) $G_{m,n} = G_{m,1}p_{n-1}(z) - G_{m,0}p_{n-2}$; $m, n \geq 1$

(7) $F_{m,0} = F_{1,0}p_{m-1}(z) - F_{0,0}p_{m-2}(z)$; $m \geq 1$
$F_{0,0} = y_1$, $F_{1,0} = zy_1 - y_2$, $F_{0,1} = y_2$

(8) $F_{m,1} = F_{1,1}p_{m-1}(z) - y_2p_{m-2}(z)$; $m \geq 1$

(9) $F_{m,n} = F_{m,1}p_{n-1}(z) - F_{m,0}p_{n-2}$; $m, n \geq 1$ □

Since p is an even number prime to 3 two cases are, in fact, possible $p = 6m + 2$ or $p = 6m - 2$. We study these two cases separately.

Case 1. $p = 6m + 2$. In this case $w = (ba)^m(\overline{b}\overline{a})^m\overline{b}(ab)^m$ gives
$t_w = E_{m,m}G_{0,m} - G_{m,2m}$
$t_{aw} = G_{m,0}F_{2m,0} - E_{3m+1,0}$
$t_{bw} = G_{m,m}G_{0,m} + E_{2m,m} - G_{m,2m}y_2$
$t_{abw} = E_{m,m}G_{0,m+1} - G_{m,2m+1}$

Replacing these values in (2.2), using Lemma (2.3), and (1.2)(3), we obtain the two equations (2.3) defining $S([p/3])$, where

$$q(y_1, y_2, z) = v_m(z) + D(y_1, y_2, z)[p_{m-1}(z)]^2 p_m(z)$$

Case 2. $p = 6m - 2$. Here $w = b(ab)^{m-1}\overline{a}(\overline{b}\overline{a})^{m-1}b$. Hence
$t_w = (G_{m-1,0}F_{m-1,0} - E_{0,2m-1})G_{m-1,0} - F_{m-1,0}$
$t_{aw} = G_{m-1,0}y_2 - E_{m,2m-1}$
$t_{bw} = t_w y_2 - G_{2m-2,0}F_{m-1,0} + E_{0,3m-2}$
$t_{abw} = (G_{m-1,0}F_{m-1,0} - E_{0,2m-1})G_{m,0} - F_{m,0}$

Computing as before one obtains

$$q(y_1, y_2, z) = u_m(z) + D(y_1, y_2, z)[p_{m-1}(z)]^2 p_{m-2}(z) \qquad \square$$

COROLLARY (2.4). *Let $[p/3]$ be the group of the link $p/3$ (p even). Then the algebraic variety $C([p/3])$ is defined by the equation $q((X, z)) = 0$, where*

$$(2.4) \qquad q = \begin{cases} h_m(z) - X[t_m(z)]^2 p_{m-1}(z); & \text{if } p = 6m + 2 \\ g_{m-1}(z) - X[t_{m-1}(z)]^2 p_{m-1}(z); & \text{if } p = 6m - 2 \end{cases}$$

and where $X = y^2 - z - 2$.

Proof. The result follows from (2.1), when $y_1 = y_2 = y$. In this case $D = (y^2 - 2 - z)(2 - z)$ Call $x = y^2 - 2 - z$. Then

Case 1. $p = 6m + 2$.
$q(y,z) = v_m(z) + D(y,z)[p_{m-1}(z)]^2 p_m(z) = v_m(z) + (y^2 - 2 - z)(2 - z)[p_{m-1}(z)]^2 p_m(z)$.
Using the definition of v_m and substituting y^2 by $X + z + 2$, we get
$q(X,z) = p_m - (X+2)p_{m-1} + X(2-z)p_{m-1}^2 p_m = p_m - 2p_{m-1} + Xp_{m-1}((2-z)p_{m-1}p_m - 1) =$
$h_m + Xp_{m-1}(2p_{m-1}p_m - zp_{m-1}p_m - 1) = h_m + Xp_{m-1}(2p_{m-1}p_m - p_{m-1}^2 - p_m^2) = h_m - Xp_{m-1}(p_m - p_{m-1})^2 = h_m - Xp_{m-1}t_m^2$

Case 2. $p = 6m - 2$.
$q(y,z) = u_m(z) + D(y,z)[p_{m-1}(z)]^2 p_{m-2}(z) = u_m(z) + (y^2 - 2 - z)(2 - z)[p_{m-1}(z)]^2 p_{m-2}(z)$.
Using the definition of u_m and substituting y^2 by $X + z + 2$, we get
$q(X,z) = -p_m + (z - X - 2)p_{m-1} + X(2-z)p_{m-1}^2 p_{m-2} =$
$-p_m + (z-2)p_{m-1} + Xp_{m-1}((2-z)p_{m-1}p_{m-2} - 1) =$
$-(zp_{m-1} - p_{m-2}) + (z-2)p_{m-1} + Xp_{m-1}(2p_{m-1}p_{m-2} - zp_{m-1}p_{m-2} - 1) =$
$-2p_{m-1} + p_{m-2} + Xp_{m-1}(2p_{m-1}p_{m-2} - p_{m-1}^2 - p_{m-2}^2) =$
$g_m - Xp_{m-1}(p_{m-1} - p_{m-2})^2 = g_m - Xp_{m-1}t_{m-1}^2$. □

THEOREM (2.5). *Let $[p/3]$ be the group of the knot $p/3$. Then the algebraic variety $C([p/3])$ is defined by the equation $q(X,z) = 0$, where*

$$(2.5) \quad q = \begin{cases} s_m(z) + X[p_{m-1}(z)]^2(z-2)t_m(z); & \text{if } p = 6m+1 \\ r_m(z) + X[p_{m-1}(z)]^2(-z+2)t_{m-1}(z); & \text{if } p = 6m-1 \end{cases}$$

and where $X = y^2 - z - 2$.

Proof. If p/q is a knot, the two polynomials $t_r - 2 = 0$, $t_{rb} - t_b = 0$ defining $C([p/q])$ reduce to

$$(2.6) \quad \begin{cases} p_1 \equiv t_{a\omega}^2 - t_{ba\omega}t_\omega + t_{ab} - 2 = 0 \\ p_2 \equiv t_{a\omega}[t_{a\omega}t_a - t_{ba\omega} - t_\omega] = 0 \end{cases}$$

In fact, $t_r = t_{\omega b \omega^{-1} a^{-1}} = t_{\omega b}t_{\omega^{-1}a^{-1}} - t_{\omega ba\omega}$ and $t_{\omega ba\omega} = t_{ba\omega\omega} = t_{ba\omega}t_\omega - t_{ba}$ which yields p_1; and similarly for p_2.

Since p is an odd number primer to 3 two cases are, in fact, possible $p = 6m+1$ or $p = 6m-1$. We study these two cases separately.

Case 1. $p = 6m + 1$. In this case $w = (ba)^m(\overline{b}\overline{a})^m(ba)^m$ gives
$t_w = E_{m,n}E_{0,m} - E_{m,0}$
$t_{baw} = E_{m,m+1}E_{0,m} - E_{m,1}$
$t_{aw} = E_{m,m}F_{m,0} - F_{m,0}$
Replacing these values in (2.6), using Lemma (2.3), and 1.2(3), and substituting $z + X + 2$ for y^2, we get the two equations defining $C([p/3])$:

$$\begin{cases} X \cdot q(X,z)^2 = 0 \\ X \cdot q_1(X,z) \cdot q(X,z) = 0 \end{cases}$$

where $q(X,z) = s_m(z) + X[p_{m-1}(z)]^2(z-2)t_m(z)$

Case 2. $p = 6m-1$. Here $w = (ba)^{m-1}b(\bar{a}\bar{b})^m a(ba)^{m-1}$. Hence

$t_w = E_{m,m}E_{0,m-1} - E_{m,1}$
$t_{baw} = E_{m,m}E_{0,m} - E_{m,0}$
$t_{aw} = E_{m,m}F_{m-1,0} - F_{m-1,0}$

Computing as before one obtains

$$q(X,z) = r_m(z) + X[p_{m-1}(z)]^2(z-2)(-t_{m-1}(z)) \qquad \square$$

COROLLARY (2.6). *Let* $[p/3]$ *be the group of the knot or link* $p/3$. *Then the algebraic variety of non abelian representations in* $PSL(2,\mathbb{C})$ *is always irreducible.*

Proof. The algebraic variety of non abelian representations in $PSL(2,\mathbb{C})$ is defined by $q(X,z) = 0$, where $q(X,z)$ is given in (2.4) and (2.5). Suppose $q(X,z) = A(z) + B(z)X$ is the product of two polynomials q_a and q_b in the variables X and z. Necessarily one of then, say q_a, has degree one in X and the other, q_b, has degree zero in X. Then $q_b(z)$ is a common factor of $A(z)$ and $B(z)$, i. e., the roots of $q_b(z)$ are roots of both $A(z)$ and $B(z)$. But Proposition 1.5 shows that, in the four possible cases, $A(z)$ and $B(z)$ have no common roots. $\square$

3. The shape of the curve $C([p/3])$

Graphs in this section are depicted, not in $\mathbb{C}^2$, but in $\mathbb{R}^2$.

Case 1. $p = 6m+2$. Here $q = h_m(z) - X[t_m(z)]^2 p_{m-1}(z)$. Let $\{y_1,\dots,y_{m-1}\}$, $\{z_1,\dots,z_m\}$, $\{\eta_1,\dots,\eta_m\}$ be the roots of p_{m-1}, t_m, h_m, respectively. One has (Proposition (1.5)):

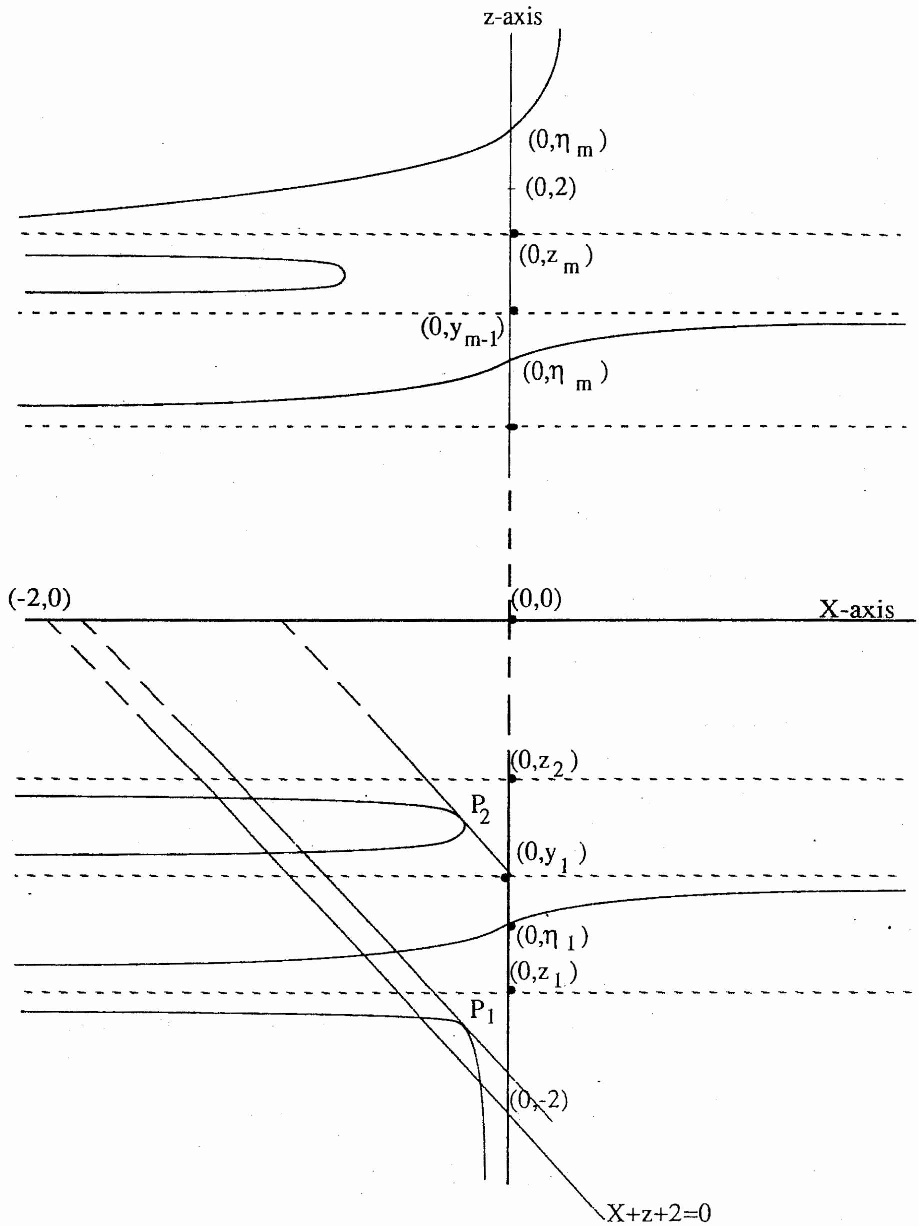
$$-2 < z_1 < \eta_1 < y_1 < z_2 < \dots < y_{m-1} < z_m < 2 < \eta_m < 3.$$

The roots of h_m define the intersection of $q = 0$ with the axis $X = 0$.

The roots of $[t_m(z)]^2 p_{m-1}(z)$ correspond to the asymptotes of $q = 0$, as follows: $z = z_i, (i = 1,\dots,m)$ are double asymptotes; $z = y_i, (i = 1,\dots,m-1)$ are simple asymptotes.

Moreover, the line $L \equiv X+z+2 = 0$ intersects $q = 0$ in $3m$ *real* points ($3m$ is the degree of q), because the change of variables $x = z + X$, transforms L in the line $x = -2$, which interesects $q(x,z) = 0$ only in real points (Remark 2.1 (5)).

Therefore $q = 0$ looks like Figure 1 in coordinates (X,z). Figure 2 represents $q = 0$ in coordinates $x = X + z$, $z = z$.



z-axis
(0,η m)
(0,2)
(0,z m)
(0,y m-1)
(0,η m)
(-2,0)
(0,0)
X-axis
(0,z 2)
P2
(0,y 1)
(0,η 1)
(0,z 1)
P1
(0,-2)
X+z+2=0

Figure 1

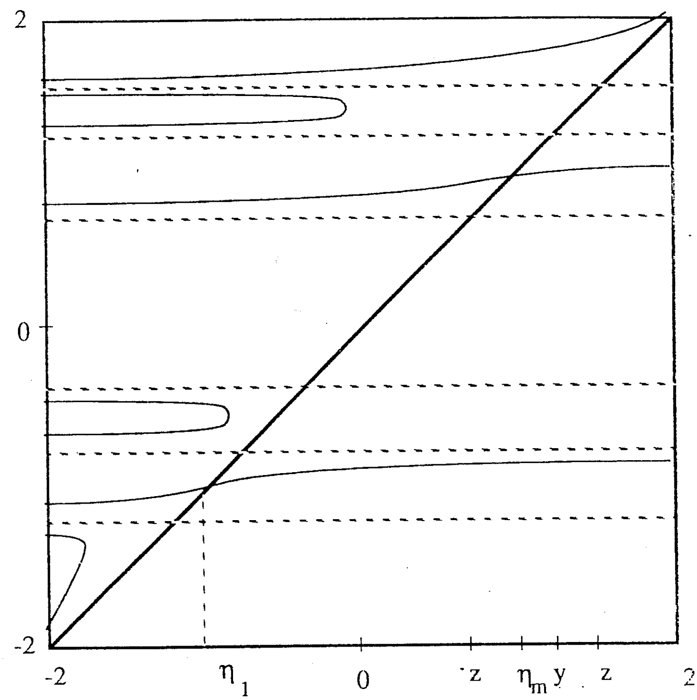


Figure 2, p=6m+2

Case 2. $p = 6m - 2$. Here $q = g_{m-1}(z) - X[t_{m-1}(z)]^2 p_{m-1}(z)$. Let $\{\gamma_1, \ldots, \gamma_{m-1}\}$ be the roots of g_{m-1}, which correspond to the intersections of $q = 0$ with the z-axis.

The asymptotes are given by the roots of $[t_{m-1}(z)]^2 p_{m-1}(z)$. $z = x_i (i = 1, \ldots, m-1)$ are simple, and $z = z_i$, $(i = 1, \ldots, m-1)$ are double asymptotes. Here $\{x_1, \ldots, x_{m-1}\}$ and $\{z_1, \ldots, z_{m-1}\}$ are the roots of p_{m-1} and t_{m-1} respectively. We know that (Proposition (1.5)):

$$-2 < x_1 < \gamma_1 < z_1 < x_2 < \ldots < x_{m-1} < \gamma_{m-1} < z_{m-1} < 2$$

Again, we also know that $X + Z + 2 = 0$ cuts $q = 0$ in $3m - 2$ real points (the degree of q). Thus the curve $q = 0$ looks like Figure 3 in coordinates (X, z). Figure 4 represents $q = 0$ in coordinates $x = X + z$, $z = z$.

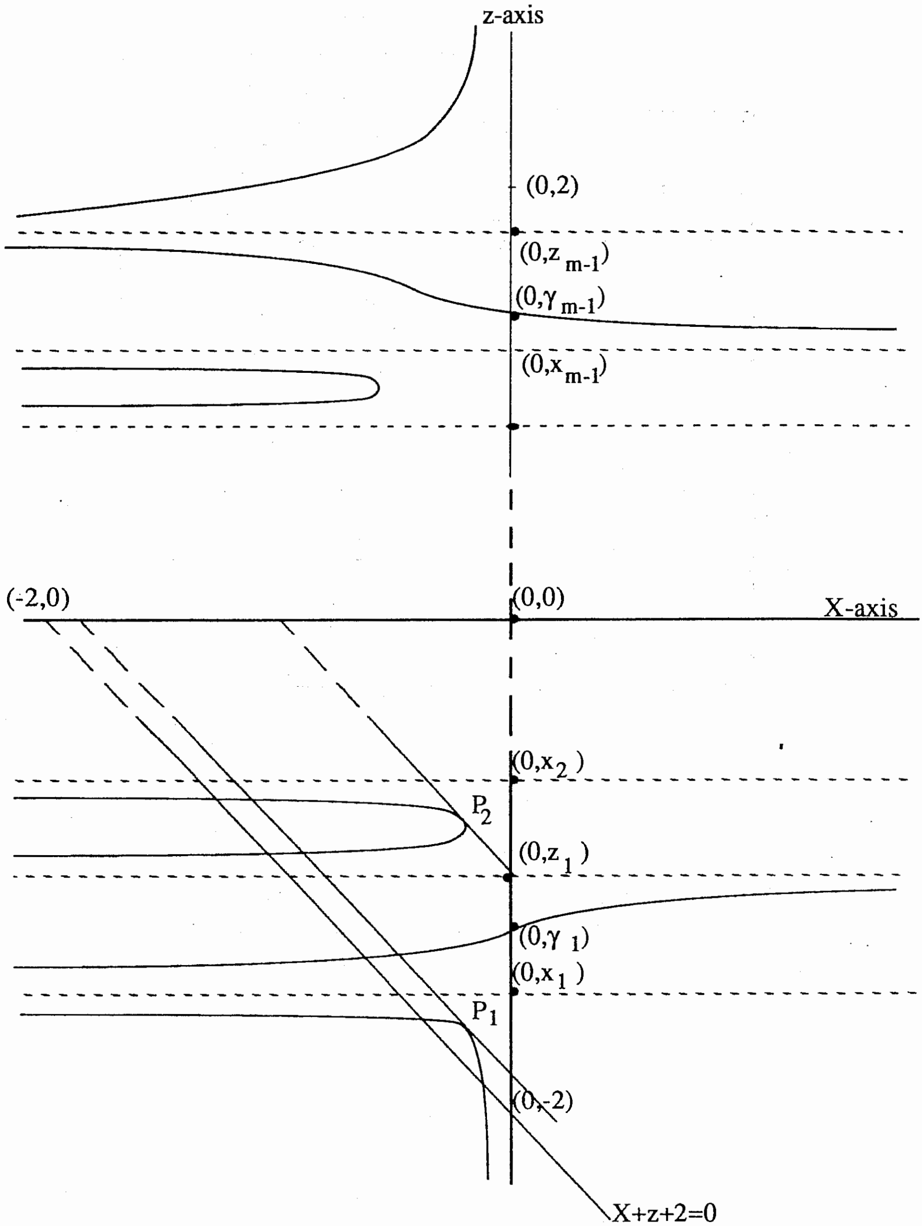

z-axis
(0,2)
(0,z m-1)
(0,γ m-1)
(0,x m-1)
(-2,0)
(0,0)
X-axis
(0,x 2)
P2
(0,z 1)
(0,γ 1)
(0,x 1)
P1
(0,-2)
X+z+2=0

Figure 3

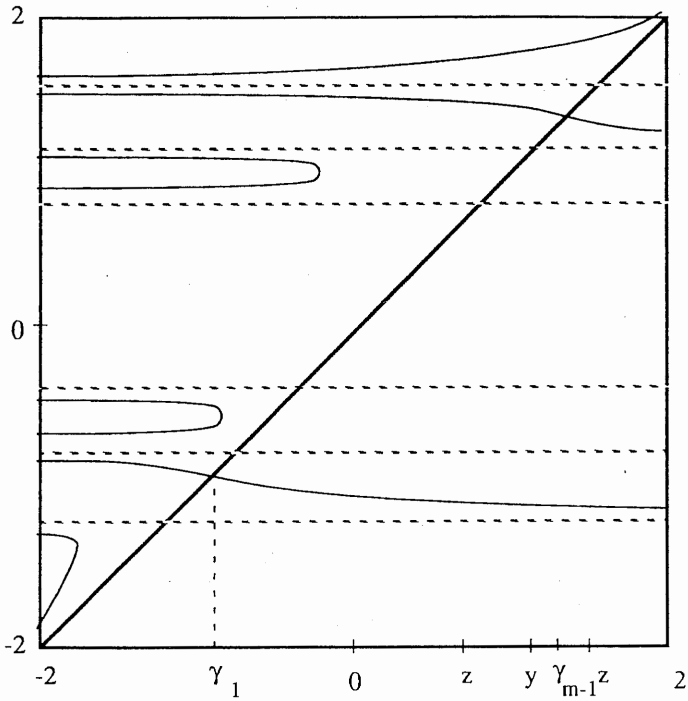


Figure 4, p=6m-2

Case 3. $p = 6m+1$. Here $q = s_m + X(z-2)p_{m-1}^2)t_m$. Let $\{y_1, \ldots, y_{m-1}\}$, $\{z_1, \ldots, z_m\}$, $\{\sigma_1, \ldots, \sigma_m\}$ be the roots of p_{m-1}, t_m, s_m, respectively. One has (Proposition (1.5)):

$$-2 < z_1 < \sigma_1 < y_1 < z_2 < \ldots < y_{m-1} < z_m < \sigma_m < 2$$

The roots of s_m define the intersection of $q = 0$ with the axis $X = 0$.

The roots of $(z-2)p_{m-1}^2 t_m$ correspond to the asymptotes of $q = 0$, as follows: $z = y_i$, $(i = 1, \ldots, m-1)$ are double asymptotes; $z = zi$, $(i = 1, \ldots, m)$ and $z = 2$ are simple asymptotes.

Moreover, the line $L \equiv X + z + 2 = 0$ intersects $q = 0$ in $3m$ *real* points ($3m$ is the degree of q), because the change of variables $x = z + X$, transforms L in the line $x = -2$, which interesects $q(x,z) = 0$ only in real points (Remark (2.1) (5)).

Therefore $q = 0$ looks like Figure 5 in coordinates (X, z). Figure 6 represents $q = 0$ in coordinates $x = X + z$, $z = z$.

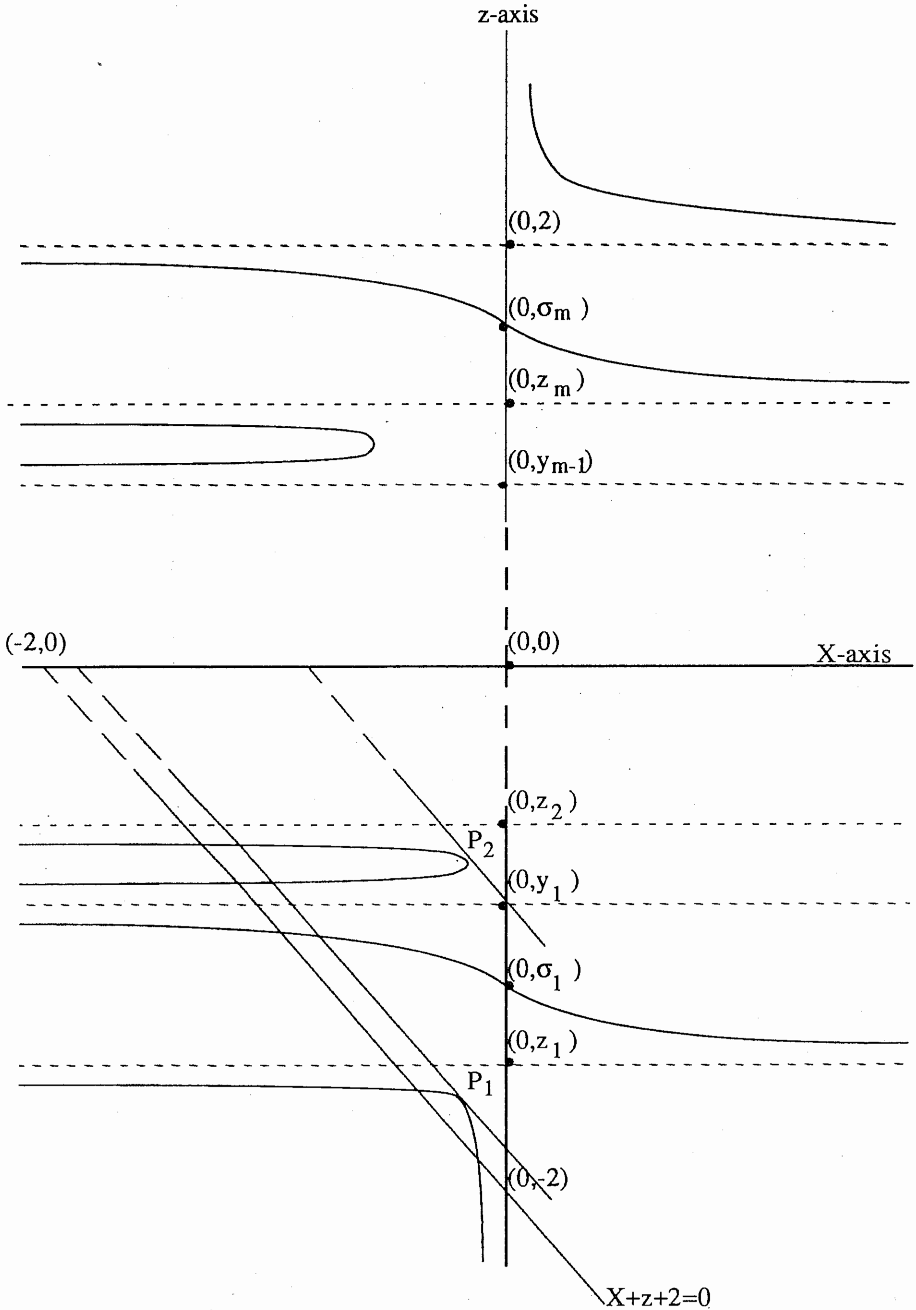


Figure 5

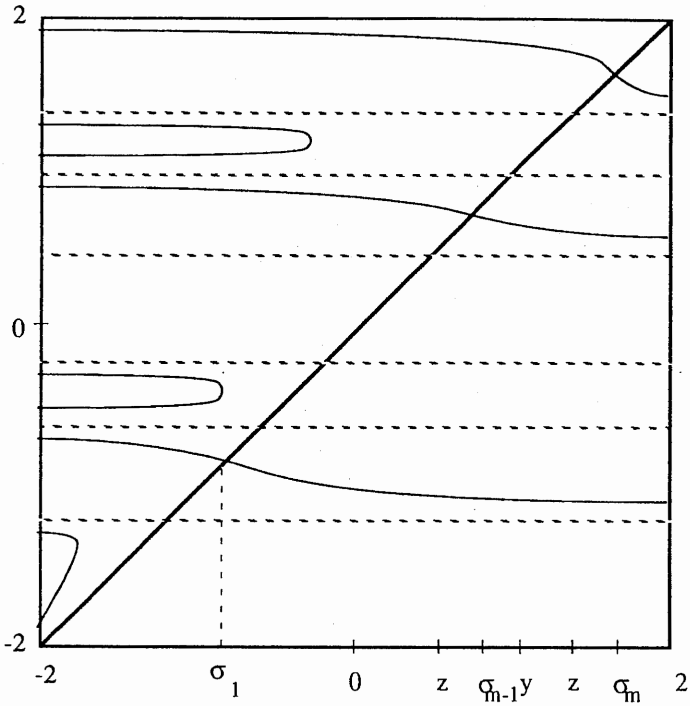


Figure 6, p=6m+1

Case 4. $p = 6m - 1$. Here $q = r_m - X(z-2)p_{m-1}^2 t_{m-1}$. Let $\{\rho_1, \ldots, \rho_m\}$ be the roots of r_m, which correspond to the intersections of $q = 0$ with the Z-axis.

The asymptotes are given by the roots of $(z-2)p_{m-1}^2 t_{m-1}$. $z = 2$, and $z = z_i (i = 1, \ldots, m-1)$ are simple, and $z = x_i$, $(i = 1, \ldots, m-1)$ are double asymptotes. Here $\{x_1, \ldots, x_{m-1}\}$ and $\{z_1, \ldots, z_{m-1}\}$ are the roots of p_{m-1} and t_{m-1} respectively. We know that (Proposition (1.5)):

$$-2 < x_1 < \rho_1 < z_i < x_2 < \ldots < x_{m-1} < \rho_{m-1} < z_{m-1} < 2 < \rho_m < 3$$

Again, we also know that $X + Z + 2 = 0$ cuts $q = 0$ in $3m - 1$ real points (the degree of q). Thus the curve $q = 0$ looks like Figure 7 in coordinates (X, z). Figure 8 represents $q = 0$ in coordinates $x = X + z$, $z = z$.

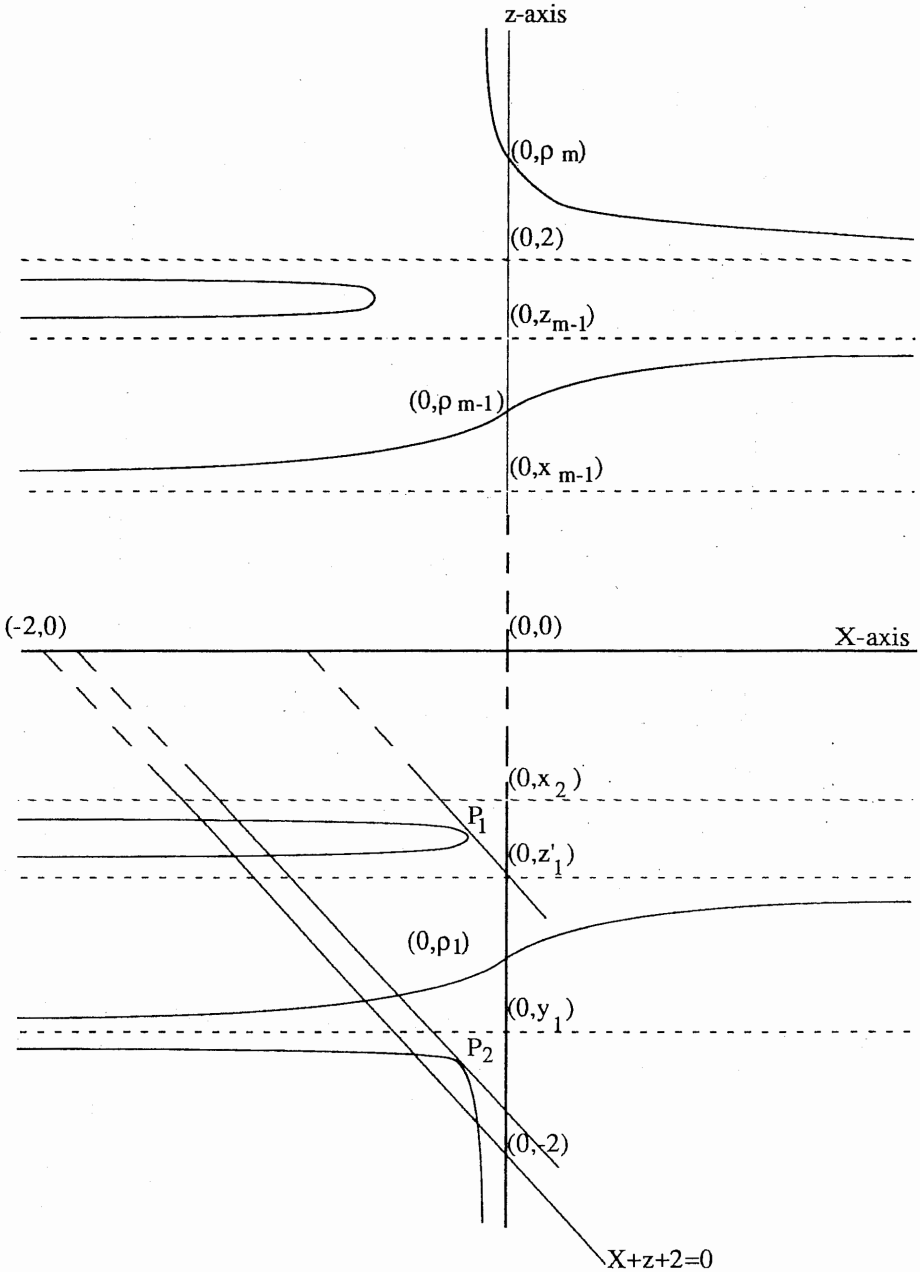

z-axis
(0,ρ m)
(0,2)
(0,z m-1)
(0,ρ m-1)
(0,x m-1)
(-2,0)
(0,0)
X-axis
(0,x 2)
P1
(0,z'1)
(0,ρ1)
(0,y 1)
P2
(0,-2)
X+z+2=0

Figure 7

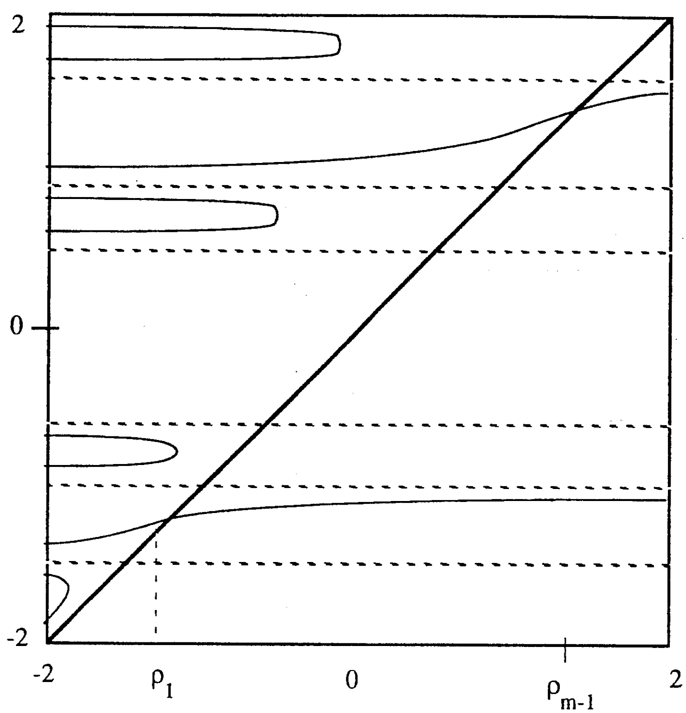


Figure 8, p=6m-1

Let w be ν_m, γ_{m-1}, σ_{m-1}, or ρ_{m-1} in cases 1,2,3, or 4 respectively. Let $\Theta(w)$ be $\cos^{-1}(\frac{w}{2} \cap [0,\pi]$. Write $\Theta(w) = \frac{2\pi}{n(w)}$, $2 \leq n(w) \leq \infty$. Given $m > 2$ say that $\frac{m}{i}$ is a conjugate of m if i is an integer such that $1 < i < \frac{m}{2}$ and $(i,m) = 1$. Define $L_w = \{m|$ each conjugate of m is $\leq n(w)\}$. Propositions 8.2 and 8.3 of [HLM1] implies that if $(p/3,n)$ is arithmetic, then $n \in L_w$. As the set L_w is finite, we can apply an algorithm to find the values n for which $(p/3,n)$ is arithmetic, once p is given. In particular we have

THEOREM (3.1). *Let $(p/3,n)$ be the orbifold with singular set $p/3$ and isotropy cyclic of order $n > 2$. Then, for each p, the set of values n for which $(p/3,n)$ is arithmetic is finite.* □

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