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CALCULATING THE GENUS OF CERTAIN NILPOTENT GROUPS

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1. Introduction

Let $\mathcal{N}_0$ be the class of finitely generated, but not finite, nilpotent groups N with finite commutator subgroup $[N,N]$. Then for any N in $\mathcal{N}_0$ the (Mislin) genus $\mathcal{G}(N)$ (see [M, HM]) has the structure of a finite abelian group. This *genus-group* was calculated in [CH] in the case that N belongs to a certain subclass $\mathcal{N}_1$ of $\mathcal{N}_0$.

Thus consider the short exact sequence (valid for any nilpotent group N)

(1.1) $$TN \rightarrowtail N \xrightarrow{\pi} FN$$

where TN is the torsion subgroup of N, and FN is the torsionfree quotient. Then $N \in \mathcal{N}_0$ if and only if TN is finite and FN is free abelian of finite rank. We say that $N \in \mathcal{N}_1$ if, additionally,

(a) TN is commutative;

(b) (1.1) splits on the right, so that N is the semidirect product for an action $\omega : FN \to \operatorname{Aut} TN$, of FN on TN;

(c) the action ω satisfies $\omega(FN) \subseteq Z(\operatorname{Aut} TN)$, where Z is the center.

Note that, in the presence of (a), (c) is equivalent to the condition that, for each $\xi \in FN$, there exists an integer u, such that $\xi.a = ua$ for all $a \in TN$ (written additively).

Now let t be the *height* of $\ker \omega$ in FN; here the height of a (non-trivial) subgroup R of a free abelian group F is the largest positive integer h such that $R \subseteq hF$. Then the authors prove in [CH]

THEOREM (1.1). $\mathcal{G}(N) \cong (Z/t)^*/\{\pm 1\}$ *if* $N \in \mathcal{N}_1$.

Let N^k be the k^{th} direct power of N, $k \geq 2$. There is then a surjective homomorphism

$$\rho : \mathcal{G}(N) \to \mathcal{G}(N^k),$$

given by $\rho(M) = M \times N^{k-1}$ and the authors also prove in [CH]

THEOREM (1.2). *Let TN be a cyclic p-group, for some prime p, and let FN be cyclic. Then ρ is an isomorphism.*

Our object in this paper is to calculate $\mathcal{G}(N^k)$ for any $N \in \mathcal{N}_1$ and any $k \geq 2$. We know, by the principal result of [HS], that $\mathcal{G}(N) = 0$ if FN is not cyclic, so that $\mathcal{G}(N^k) = 0$ under the same hypothesis, so that we may assume FN cyclic. To state our result, let

(1.2) $$\exp TN = n = p_1^{m_1} p_2^{m_2} \cdots p_\lambda^{m_\lambda},\ p_1 < p_2 < \cdots < p_\lambda, m_i \geq 1.$$

We say n is of Type 1 if $p_1 = 2$, $m_1 = 1$; otherwise it is of Type 2. It is known that t must have the form

(1.3) $$t = p_1^{l_1} p_2^{l_2} \cdots p_\lambda^{l_\lambda},\ 0 \leq l_i < m_i,\ i = 1, 2, \ldots, \lambda.$$

Now we may ignore the case that n is of Type 1 with $\lambda = 1$ since then $t = 1$ and $\mathcal{G}(N) = \mathcal{G}(N^k) = 0$. Thus the following theorem constitutes a complete statement of our result, generalizing Theorem 1.2; but note that we place no restriction on the structure of the finite abelian group TN. In stating our theorem, we identify $\mathcal{G}(N)$ with $(Z/t)^*/\{\pm 1\}$, according to Theorem 1.1. We repeat that, to avoid triviality, we assume FN cyclic.

THEOREM (1.3). *For any $k \geq 2$, we obtain $\mathcal{G}(N^k)$ from $\mathcal{G}(N)$ by factoring out those residues m mod t such that (see (1.3))*

$$m \equiv \varepsilon_i \bmod p_i^{l_i},\ \varepsilon_i = \pm 1,\ i = 1, 2, \ldots, \lambda. \tag{1.4}$$

Thus $\mathcal{G}(N^k) = \mathcal{G}(N)/H$, where H is an elementary abelian 2-group, and

$$\text{rank } H = \begin{cases} \lambda - 2, & \text{if } n \text{ is of Type 1;} \\ \lambda - 1, & \text{if } n \text{ is of Type 2.} \end{cases}$$

It is not difficult to prove that if $N \in \mathcal{N}_1$ and $k \geq 2$, then $N^k \in \mathcal{N}_1$ if and only if N is abelian. The condition which fails is, of course, condition (c). Thus we see how vital condition (c) is to the validity of Theorem 1.1; for clearly $t(N^k) = t(N)$.

In Section 2, we establish, or recall, some preliminary results; and in Section 3 we prove Theorem 1.3. In Section 4 we give a typical, illustrative example. The content of this paper forms part of the Ph.D. dissertation of the second-named author at the State University of New York at Binghamton, written under the direction of the first-named author.

2. Preliminaries

The key sequence for calculating $\mathcal{G}(N)$, for any $N \in \mathcal{N}_0$, is (see [HM])

$$T\text{-Aut}\, N \xrightarrow{\theta} (Z/e)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(N). \tag{2.1}$$

Here T is the set of prime divisors of $n = \exp TN$, and a T-automorphism $\varphi : N \to N$ is an endomorphism such that, localizing at T, φ_T is an automorphism of N_T. We refer to [CH] for the definition of e, since it plays a minor role in our argument, but we will explain how θ acts. If $d = \exp TZN$, then dZN is a free abelian group which is called the *free center* of N. It is shown in [M, HM] that any T-automorphism φ sends FZN to itself, so we may associate with φ the integer $\det \varphi\,|FZN$. Then $\theta(\varphi)$ is the residue class, modulo ± 1, of this integer. We are now ready for our first lemma, valid for any N in $\mathcal{N}_0$.

LEMMA (2.1). *Let $\varphi : N \to N$ be an endomorphism. Then φ induces $\psi : FN \to FN$* (see (1.1)). *Moreover, if $\varphi(FZN) \subseteq FZN$, then $det(\varphi|FZN) = det\,\psi$.*

Proof. A famous theorem of I. Schur asserts that, if N is a group such that N/ZN is finite, then $[N, N]$ is finite. It is not difficult to prove that the converse holds if N is finitely generated, nilpotent. Thus if $N \in \mathcal{N}_0$ then N/ZN

is finite, and hence N/FZN is finite. It follows that $\pi : N \twoheadrightarrow FN$ maps FZN onto a subgroup $\pi(FZN)$ of FN of maximal rank. Thus

$$\det(\varphi|FZN) = \det(\psi|\pi(FZN)) = \det(\psi|\pi(FZN)\otimes \mathcal{Q}) = \det(\psi\otimes\mathcal{Q}) = \det\psi.$$

□

Now (2.1) may be embedded in the commutative diagram

$$(2.2)\qquad \begin{array}{ccccc} T\text{-Aut}N & \xrightarrow{\theta} & (Z/e)^*/\{\pm1\} & \twoheadrightarrow & \mathcal{G}(N) \\ \downarrow\sigma & & \| & & \downarrow\rho \\ T\text{-Aut}N^k & \xrightarrow{\bar\theta} & (Z/e)^*/\{\pm1\} & \twoheadrightarrow & \mathcal{G}(N^k) \end{array}$$

(see (4.1) of [CH]), where $\sigma(\varphi) = \varphi\times \mathrm{Id}$, Id being the identity on N^{k-1}. (It is easy to see that T and e remain unchanged when one passes from N to N^k). We claim then that it follows from (2.2) that Theorem 1.3 will be proved when we have established the following proposition.

PROPOSITION (2.2). *Let $N\in\mathcal{N}_1$ and let us adopt the notation and data of Section 1. Then, in (2.2), Im $\bar\theta$ consists of those residue classes [m], modulo ±1, such that*

$$(2.3)\qquad m\equiv\varepsilon_i \bmod p_i^{l_i},\ \varepsilon_i=\pm1,\ i=1,2,\dots,\lambda.$$

This was, of course, precisely the approach taken in [CH] to prove Theorem 1.2. In that special case, however, no problem of realizability arose. Once it was shown that any $[m]$ in Im $\bar\theta$ satisfied $m\equiv\pm1 \bmod t$, Theorem 1.2 followed immediately. Here we must also show that all m given by (2.3) can be realized by some T-automorphism of N^k.

We come now to our final set of preliminary observations before proceeding to the proof of Theorem 1.3. Let $M\in\mathcal{N}_0$ satisfy the supplementary conditions (a), (b) defining the subclass $\mathcal{N}_1$, but not necessarily (c); we will be applying our forthcoming remarks to the case $M=N^k$. Let T be defined as before. If φ is an endomorphism of M, then φ induces a commutative diagram

$$(2.4)\qquad \begin{array}{ccccc} TM & \rightarrowtail & M & \xrightarrow{\pi} & FM \\ \downarrow\alpha & & \downarrow\varphi & & \downarrow\psi \\ TM & \rightarrowtail & M & \xrightarrow{\pi} & FM \end{array}$$

LEMMA (2.3). *φ is a T-automorphism if, and only if, α is an automorphism and ψ is a T-automorphism.*

Proof. It follows from the standard properties of localization that φ is a T-automorphism if and only if α and ψ are T-automorphisms; but, since TM is itself T-local, α is a T-automorphism if and only if it is an automorphism. □

LEMMA (2.4.) *(i)* $\alpha(\xi.a)=\psi\xi.\alpha a$

(ii) Suppose, conversely, that a diagram

$$\begin{array}{ccccc} TM & \rightarrowtail & M & \xrightarrow{\pi} & FM \\ \downarrow \alpha & & & & \downarrow \psi \\ TM & \rightarrowtail & M & \xrightarrow{\pi} & FM \end{array}$$

is given such that $\alpha(\xi.a) = \psi\xi.\alpha a$. *Then we may find* $\varphi : M \to M$ *making a commutative diagram*(2.4).

Proof. This argument was given in [H]; note that conclusion (i) requires that TM be commutative, but conclusion (ii) does not. Of course, it is crucial that M be a semidirect product. □

3. Proof of Theorem 1.3

Let $\varphi : N^k \to N^k$ be a T-automorphism. Then (see (2.4) and Lemma 2.3) φ gives rise to the commutative diagram

$$\text{(3.1)} \qquad \begin{array}{ccccc} TN^k & \rightarrowtail & N^k & \twoheadrightarrow & FN^k \\ \cong\downarrow \alpha & & \downarrow \varphi & & \downarrow \psi \\ TN^k & \rightarrowtail & N^k & \twoheadrightarrow & FN^k \end{array}$$

where ψ is a T-automorphism, so that $\det\psi$ is prime to T. Note that, by Lemma 2.1, $\det\psi = \overline{\theta}(\varphi)$ in (2.2). Let $n = p_1^{m_1}p_2^{m_2}\cdots p_\lambda^{m_\lambda}$, $t = p_1^{l_1}p_2^{l_2}\cdots p_\lambda^{l_\lambda}$ as in (1.2), (1.3). Let p be a typical prime occurring in the prime factorization of t with exponent[1] l, and let $TN_p = \langle a_1, a_2, \ldots, a_r\rangle = \oplus_{i=1}^r Z/p^{d_i}$, with $m = d_1 \geq d_2 \geq \cdots \geq d_r$. Now if $FN = \langle\xi\rangle$, then $\xi.a = ua$, $a \in TN_p$, where u is of order p^l mod p^m. Write, in an obvious notation,

$$\text{(3.2)} \qquad \left.\begin{array}{ll} TN_p^k = \langle a_{i(s)}\rangle, & i = 1, 2, \ldots, r; s = 1, 2, \ldots, k. \\ FN^k = \langle \xi_{(s)}\rangle, & s = 1, 2, \ldots, k. \end{array}\right\}$$

Let

$$\text{(3.3)} \qquad \alpha a_{i(s)} = \sum \alpha_{i(s)j(v)}a_{j(v)},\ \psi\xi_{(s)} = \sum \beta_{sf}\xi_{(f)}.$$

We now exploit the key relationship (Lemma 2.4(i)),

$$\text{(3.4)} \qquad \alpha(\xi_{(w)}.a_{i(s)}) = \psi\xi_{(w)}.\alpha a_{i(s)}.$$

First, set $w = s$. Then $u\sum\alpha_{i(s)j(v)}a_{j(v)} = \prod\xi_{(f)}^{\beta_{sf}}\sum\alpha_{i(s)j(v)}a_{j(v)}$ $= \sum u^{\beta_{sv}}\alpha_{i(s)j(v)}a_{j(v)}$. We conclude that

$$\text{(3.5)} \qquad \text{if } p \nmid \alpha_{i(s)1(v)}, \text{ then } \beta_{sv} \equiv 1 \bmod p^l$$

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[1] We allow $l = 0$ as a possibility rendering the argument trivial.

Now let $w \neq s$. Then $\sum \alpha_{i(s)j(v)} a_{j(v)} = \prod \xi_{(f)}^{\beta_{wf}} \sum \alpha_{i(s)j(v)} a_{j(v)}$
$= \sum u^{\beta_{wv}} \alpha_{i(s)j(v)} a_{j(v)}$. We conclude that

(3.6) if $p \not| \alpha_{i(s)1(v)}$, and $w \neq s$, then $\beta_{wv} \equiv 0 \bmod p^l$

Fix v. Then $\exists i(s)$ such that $p \not| \alpha_{i(s)1(v)}$, since α is an automorphism. With s chosen from such an $i(s)$, $\beta_{sv} \equiv 1 \bmod p^l, \beta_{wv} \equiv 0 \bmod p^l, w \neq s$, by (3.5), (3.6). Thus the matrix of ψ, reduced mod p^l, reads

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

in each column. Now ψ is a T-automorphism, so that $p \not| \det \psi$. Thus $\det \psi \not\equiv 0 \bmod p^l$. This implies that, in the matrix of ψ, reduced mod p^l, the non-zero entries occupy different rows. Thus $\det \psi \equiv \pm 1 \bmod p^l$. Thus

$$\bar{\theta}(\varphi) \equiv \pm 1 \bmod p_i^{l_i},$$

for all primes p_i in the factorization of t as $\prod p_i^{l_i}$. This shows that Proposition 2.2 holds in one direction.

We now move to the converse. We consider a residue m mod t such that

(3.7) $m \equiv \varepsilon_i \bmod p_i^{l_i}$, $i = 1, 2, \ldots, \lambda$, where $\varepsilon_i = \pm 1$

and we show the existence of a T-automorphism $\varphi : N^k \to N^k$ inducing $\psi : FN^k \to FN^k$ with $\det \psi = m$. Once again we fix a particular p among the prime factors of n and we describe α and ψ explicitly; actually, we determine α completely, but are content to determine the matrix of ψ mod t; once again, any prime p for which the exponent l in (1.3) is 0 plays essentially no role. We write ε_p for ε_i in (3.7), if $p = p_i$. Then if $\varepsilon_p = 1$, $\alpha(a_{i(s)}) = a_{i(s)}$, all $i(s)$; while, if $\varepsilon_p = -1$, $\alpha(a_{i(1)}) = a_{i(2)}, \alpha(a_{i(2)}) = a_{i(1)}, \alpha(a_{i(s)}) = a_{i(s)}, s \geq 3$. Plainly α is an automorphism. We subject the matrix (β_{sf}) of ψ, $1 \leq s, f \leq k$, to the conditions

$$(3.8) \qquad \begin{cases} \beta_{11} = \beta_{22} \equiv 1 \bmod p^l; \beta_{12} = \beta_{21} \equiv 0 \bmod p^l; \\ \beta_{ss} = 1, s \geq 3; \beta_{sf} = 0, \text{ otherwise if } \varepsilon_p = 1 \\ \beta_{11} = \beta_{22} \equiv 0 \bmod p^l; \beta_{12} = \beta_{21} \equiv 1 \bmod p^l; \\ \beta_{ss} = 1, s \geq 3; \beta_{sf} = 0, \text{ otherwise if } \varepsilon_p = -1 \end{cases}$$

It is clear from the Chinese Remainder Theorem that these conditions can be satisfied simultaneously for all p entering the factorization of t and that the matrix of ψ is determined mod t. It is also plain from (3.7) and (3.8) that $\det \psi \equiv m \bmod t$. Of course, m is prime to t, so ψ is a T-automorphism of FN^k.

It remains to verify the key relationship (3.4). For then, by Lemma 2.4(ii), we can find $\varphi : N^k \to N^k$ making the diagram (3.1) commutative and, by Lemma 2.3, φ will be a T-automorphism; finally, by Lemma 2.1, $\bar{\theta}(\varphi)$ is the residue class of m, modulo ± 1, so that we have realized m. Thus Proposition 2.2 will have been proved and, with it, Theorem 1.3.

Thus we must verify that

$$\alpha(\xi_{(w)}.a_{i(s)}) = \psi\xi_{(w)}.\alpha a_{i(s)}. \tag{3.9}$$

It is plain that we need only concern ourselves with $w = 1, 2$; $s = 1, 2$, and that we can look at (3.9) at each prime p appearing in the factorization (1.3) of t.

Assume first that $\varepsilon_p = +1$. Then

$$\alpha(\xi_{(1)}.a_{i(1)}) = \xi_{(1)}.a_{i(1)}, \psi\xi_{(1)}.\alpha a_{i(1)} = \xi_{(1)}^{\beta_{11}}.a_{i(1)} = \xi_{(1)}.a_{i(1)}$$
$$\text{since } \beta_{11} \equiv 1 \bmod p^l;$$
$$\alpha(\xi_{(1)}.a_{i(2)}) = a_{i(2)}, \psi\xi_{(1)}.\alpha a_{i(2)} = \xi_{(1)}^{\beta_{12}}.a_{i(2)} = a_{i(2)}, \text{ since } \beta_{12} \equiv 0 \bmod p^l;$$
$$\alpha(\xi_{(2)}.a_{i(1)}) = a_{i(1)}, \psi\xi_{(2)}.\alpha a_{i(1)} = \xi_{(2)}^{\beta_{21}}.a_{i(1)} = a_{i(1)}, \text{ since } \beta_{21} \equiv 0 \bmod p^l;$$
$$\alpha(\xi_{(2)}.a_{i(2)}) = \xi_{(2)}.a_{i(2)}, \psi\xi_{(2)}.\alpha a_{i(2)} = \xi_{(2)}^{\beta_{22}}.a_{i(2)} = \xi_{(2)}.a_{i(2)},$$
$$\text{since } \beta_{22} \equiv 1 \bmod p^l.$$

Now assume that $\varepsilon_p = -1$. Then

$$\alpha(\xi_{(1)}.a_{i(1)}) = \xi_{(2)}.a_{i(2)}, \psi\xi_{(1)}.\alpha a_{i(1)} = \xi_{(2)}^{\beta_{12}}.a_{i(2)} = \xi_{(2)}.a_{i(2)},$$
$$\text{since } \beta_{12} \equiv 1 \bmod p^l;$$
$$\alpha(\xi_{(1)}.a_{i(2)}) = a_{i(1)}, \psi\xi_{(1)}.\alpha a_{i(2)} = \xi_{(1)}^{\beta_{11}}.a_{i(1)} = a_{i(1)}, \text{ since } \beta_{11} \equiv 0 \bmod p^l;$$
$$\alpha(\xi_{(2)}.a_{i(1)}) = a_{i(2)}, \psi\xi_{(2)}.\alpha a_{i(1)} = \xi_{(2)}^{\beta_{22}}.a_{i(2)} = a_{i(2)}, \text{ since } \beta_{22} \equiv 0 \bmod p^l;$$
$$\alpha(\xi_{(2)}.a_{i(2)}) = \xi_{(1)}.a_{i(1)}, \psi\xi_{(2)}.\alpha a_{i(2)} = \xi_{(1)}^{\beta_{21}}.a_{i(1)} = \xi_{(1)}.a_{i(1)},$$
$$\text{since } \beta_{21} \equiv 1 \bmod p^l.$$

Thus (3.9) (or (3.4)) is verified and the proof of Theorem 1.3 is complete.

4. An example

Let $N = \langle x, y; x^{225} = 1, yxy^{-1} = x^{16} \rangle$. It is then easy to see that $N \in \mathcal{N}_1$; indeed $TN = Z/225 = \langle a \rangle, FN = Z = \langle \xi \rangle$, and $\xi.a = 16a$. Moreover $t = 15$ and, for any m prime to t, we obtain a group N_m in the genus of N corresponding to $[m] \in (Z/t)^*/\{\pm 1\}$ by replacing 16 by 16^m in the second relation for N. Note that $(Z/t)^*/\{\pm 1\} \cong Z/4$, generated by the residue class [2]. Thus $\mathcal{G}(N) = Z/4$, but $\mathcal{G}(N^k) = Z/2$ if $k \geq 2$. We pass from $\mathcal{G}(N)$ to $\mathcal{G}(N^k)$ by killing the residue class m, mod 15, such that $m \equiv +1 \bmod 3, m \equiv -1 \bmod 5$, that is, by killing $m = 4$. Thus $\mathcal{G}(N^k)$ is generated by $N_2 \times N^{k-1}$ and we have the non-cancellation phenomenon

$$N_4 \times N \cong N \times N, \; N_4 \ncong N \tag{4.1}$$

Note that

$$\begin{cases} N_2 = \langle x, y; x^{225} = 1, yxy^{-1} = x^{31} \rangle \\ N_4 = \langle x, y; x^{225} = 1, yxy^{-1} = x^{61} \rangle \end{cases} \tag{4.2}$$

Of course, the situation and phenomena described in this example are quite typical.

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