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CHARACTERISTIC NUMBERS FOR THE BORDISM OF IMMERSIONS

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1. Introduction

We consider immersions of closed smooth n-manifolds in closed smooth $(n+k)$-manifolds with a G-structure on their normal bundle. The equivalence classes modulo bordism of such immersions form an abelian group under disjoint union which we denote by $I_G(n,k)$. In section 2 we show that the James-Hopf maps give a splitting of $I_G(n,k)$ in terms of bordism groups of vector bundles with a $\Sigma_r \int G$-structure, where $\Sigma_r \int G$ is the wreath product of a symmetric group and G. With this result we give, in section 3, conditions for an immersion to be bordant to an embedding. In section 4 we associate characteristic numbers to a self-transverse immersion which determine the bordism class of the immersion. We use these numbers to show that certain immersions are not bordant to oriented immersions and that certain oriented immersions are not bordant to spin immersions.

2. A splitting for the bordism of immersions

PRELIMINARIES (2.1). Let G be a compact Lie group and let $\varphi : G \to O(k)$ be a continuous homomorphism. We consider immersions $f : M \to N$, where M is a closed smooth n-manifold and N is a closed smooth $(n+k)$-manifold, such that the normal bundle ν_f has a G-structure. Given two immersions $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$, we say that they are bordant if there exists an immersion $F : V \to W$ with a G-structure on ν_F such that i) V is a compact smooth $(n+1)$-manifold whose boundary ∂V is diffeomorphic to the disjoint union $M_1 \amalg M_2$, ii) W is a compact smooth $(n+k+1)$-manifold such that ∂W is diffeomorphic to $N_1 \amalg N_2$, iii) $F\,|M_1 = f_1$ and $F\,|M_2 = f_2$, iv) the G-structure on ν_F induces the given G-structure on ν_{f_1} and ν_{f_2}. We denote by $I_G(n,k)$ the set of equivalence classes and by $[f : M \to N]$ the equivalence class of an immersion. We can make $I_G(n,k)$ into an abelian group by defining $[f_1 : M_1 \to N_1] + [f_2 : M_2 \to N_2] = [f_1 \amalg f_2 : M_1 \amalg M_2 \to N_1 \amalg N_2]$. Every element has order 2 so $I_G(n,k)$ is a $\mathbb{Z}_2$-module.

Let X be a space, define the r-th configuration space $F(X;r)$ of X by $F(X;r) = \{(x_1, x_2, \ldots, x_r) \in X^r \mid x_i \neq x_j \text{ if } i \neq j\}$. The symmetric group Σ_r acts freely on $F(X;r)$ by permuting the factors. If Y is a space then Σ_r acts on Y^r by permuting the factors and we denote by $F(X;r) \underset{\Sigma_r}{\times} Y^r$ the quotient space under the diagonal action.

We say that an immersion $f : M \to N$ is self-transverse if $f^r : M^r \to N^r$ restricted to $F(M;r)$ is transverse to the diagonal $\Delta \subset N^r$, $r \geq 2$. Since any immersion is regularly homotopic to an immersion which is self-transverse then any class in $I_G(n,k)$ can be represented by a self-transverse immersion.

Let $f: M \to N$ be a self-transverse immersion. Then $\mu_r = f^r \mid F(M;r)^{-1}(\Delta)$ is a compact manifold. The free Σ_r-action on $F(M;r)$ restricts to μ_r and μ_r/Σ_r is called the manifold of r-tuple points. We can also define the manifold of based r-tuple points by μ_r/Σ_{r-1}, where Σ_{r-1} acts by permuting the first $(r-1)$ coordinates. Define immersions $f_r : \mu_r/\Sigma_r \to N$ and $\phi_r : \mu_r/\Sigma_{r-1} \to M$ by $f_r[x_1,x_2,\dots,x_r] = f(x_1)$ and $\phi_r[x_1,x_2,\dots,x_r] = x_r$. Their normal bundles are given by $\nu_{f_r} = [(\nu_f)^r \mid \mu_r]/\Sigma_r$ and $\nu_{\phi_r} = [(\nu_f)^{r-1} \times \{0\} \mid \mu_r]/\Sigma_{r-1}$.

Let $f: M \to N$ be a self-transverse immersion of codimension k. We denote by ξ_G the pull-back of the universal vector bundle over $BO(k)$ along the map $B\varphi : BG \to BO(k)$. Let $\overline{\nu} : M \to BG$ be the lifting given by the G-structure on ν_f. Then we have a bundle map

$$\begin{array}{ccc} \nu_f & \xrightarrow{\widehat{\nu}} & \xi_G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\overline{\nu}]{} & BG \end{array}$$

If $e: M \to \mathbb{R}^\infty$ is an embedding then we get a bundle map

$$\begin{array}{ccc} \nu_{f_r} & \xrightarrow{\delta_r} & F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} \xi_G^r \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id} \underset{\Sigma_r}{\times} p^r} \\ \mu_r/\Sigma_r & \xrightarrow{\overline{\delta}_r} & F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} BG^r \end{array}$$

where $\delta_r[v_1,\dots,v_r] = [eq(v_1),\dots,eq(v_r),\widehat{\nu}(v_1),\dots,\widehat{\nu}(v_r)]$ and $\overline{\delta}_r[x_1,\dots,x_r] = [e(x_1),\dots,e(x_r),\overline{\nu}(x_1),\dots,\overline{\nu}(x_r)]$.

We denote by $\Sigma_r \int G$ the semi-direct product of Σ_r and G^r, where Σ_r acts on G^r by permuting the factors.

The space $F(\mathbb{R}^\infty;r)$ is contractible [8]. It is the direct limit $\lim\limits_{n} F(\mathbb{R}^n;r)$, where each $F(\mathbb{R}^n;r)$ is a smooth manifold. Hence $F(\mathbb{R}^\infty;r)$ is a numerable CW-complex. The same holds for EG which can be taken as a limit of Stiefel manifolds. Therefore $F(\mathbb{R}^\infty;r) \times EG^r$ is contractible and completly regular. We can define a free action of $\Sigma_r \int G$ on this space by $(a,b_1,\dots,b_r)\cdot(\sigma,g_1,\dots,g_r) = (a\cdot\sigma, b_{\sigma(1)}\cdot g_1,\dots,b_{\sigma(r)}\cdot g_r)$. Since $\Sigma_r \int G$ is a compact Lie group, by a theorem of Gleason the quotient map is a principal $\Sigma_r \int G$-bundle. Therefore $[F(\mathbb{R}^\infty;r) \times EG^r]/\Sigma_r \int G \cong F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} BG^r = B(\Sigma_r \int G)$.

There is a linear action of $\Sigma_r \int G$ on $(\mathbb{R}^k)^r$ given by $(\sigma,g_1,\dots,g_r)\cdot(v_1,\dots,v_r) = (g_{\sigma^{-1}(1)}\cdot v_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(r)}\cdot v_{\sigma^{-1}(r)})$. This action gives a representation $\rho : \Sigma_r \int G \to O(rk)$. One can easily show that the pull-back of the universal rk-vector bundle along the map $B\rho$ is the bundle with projection $\mathrm{id} \underset{\Sigma_r}{\times} p^r$ defined above.

Recall that a coefficient system $\mathcal{C}$ defines a functor C from the category of based spaces to itself [3]. The cubes operad $\mathcal{C}_\infty$ [8] determines a coefficient system by neglect of structure. We denote by C_∞ the functor associated to $\mathcal{C}_\infty$. The functor defined by the system $\mathcal{C}(\mathbb{R}^\infty)$ of configuration spaces of $\mathbb{R}^\infty$ will be denoted by $C_{\mathbb{R}^\infty}$.

Given a pointed space X, we denote by QX the direct limit, under suspensions, $\varinjlim_n \Omega^n S^n(X)$, where $\Omega^n(-)$ denotes the space of n-loops and where $S^n(-)$, denotes the n-th suspension.

We denote by MO the Thom spectrum for unoriented cobordism and by $\mathcal{N}_*(-)$ the associated geometric homology theory [4].

Definition (2.2). We are going to define an isomorphism

$$\beta_k : I_G(n,k) \to \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G).$$

where $T\xi_G$ is the Thom space of the bundle ξ_G.

According to [10] there is an isomorphism $\alpha_k : I_G(n,k) \to \mathcal{N}_{n+k}(QT\xi_G)$ given as follows. If $[f : M \to N] \in I_G(n,k)$, then one can find an embedding $f_o : M \to N \times \mathbb{R}^m$ (taking $m > n-k+1$) regularly homotopic to the composition $M \xrightarrow{f} N \subset N \times \mathbb{R}^m$. Clearly we have that $\nu_{f_o} \cong \nu_f \oplus \varepsilon^m$, where ε^m is the trivial m-bundle. Applying the Thom-Pontryagin construction we get a stable map $t_f : S^\infty(N^+) \to S^\infty(T\nu_f)$, where N^+ denotes $N \amalg \{+\}$. The lifting of the classifying map for ν_f induces a map of Thom spaces $\tau_f : T\nu_f \to T\xi_G$. Taking the adjoint of $S^\infty\tau_f \circ t_f$ we get a map $N \subset N^+ \to QT\xi_G$, representing $\alpha_k[f : M \to N]$.

Let X be a connected CW-complex. There is a weak homotopy equivalence $\gamma_\infty : C_\infty X \to QX$ [8] and a homotopy equivalence $d : C_\infty X \to C_{\mathbb{R}^\infty} X$ [7].

Therefore we can define β_k by the commutativity of the following diagram:

$$\begin{tikzcd}
I_G(n,k) \arrow[r] \arrow[d, "\alpha_k"', "\cong"] & \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G) \\
\mathcal{N}_{n+k}(QT\xi_G) & \mathcal{N}_{n+k}(C_\infty T\xi_G) \arrow[l, "\gamma_{\infty *}"', "\cong"] \arrow[u, "d_*", "\cong"']
\end{tikzcd}$$

Definition (2.3). Given a pointed space X, we denote by $D_r X$ the space $F(\mathbb{R}^\infty; r)^+ \underset{\Sigma_r}{\wedge} X^{(r)}$, where $X^{(r)}$ denotes the smash product of r copies of X. By [3,11] there exist maps of spectra $h_r : S^\infty C_{\mathbb{R}^\infty} X \to S^\infty D_r X$, $r \geq 1$, whose adjoints are called James-Hopf maps, such that the induced map $h : S^\infty C_{\mathbb{R}^\infty} X \to \bigvee_{r \geq 1} S^\infty D_r X$ is a homotopy equivalence. Therefore we can define homomorphisms $\overline{h}_r : \mathcal{N}_*(C_{\mathbb{R}^\infty} T\xi_G, *) \to \mathcal{N}_*(D_r T\xi_G, *), r \geq 1$, by the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{N}_*(C_{\mathbb{R}^\infty}T\xi_G,*) \arrow[r] & \mathcal{N}_*(D_rT\xi_G,*) \\ MO_*(S^\infty C_{\mathbb{R}^\infty}T\xi_G) \arrow[u, "\cong"] \arrow[r, "h_{r_*}"] & MO_*(S^\infty D_rT\xi_G) \arrow[u, "\cong"'] \end{tikzcd}$$

If $\xi = (E,p,B)$ is a vector bundle with Thom space $T\xi$, then the Thom space of the vector bundle $F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} E^r \underset{\underset{\Sigma_r}{\mathrm{id}\times p^r}}{\longrightarrow} F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} B^r$ is given by $D_r(T\xi)$.

PROPOSITION (2.4). *Consider the homomorphism given by the composition*

$$I_G(n,k) \xrightarrow{\beta_k} \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty}T\xi_G) \xrightarrow{j_*} \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty}T\xi_G,*) \xrightarrow{\overline{h}_r} \mathcal{N}_{n+k}(D_rT\xi_G,*) \xrightarrow{\Phi}$$
$$\mathcal{N}_{n-(r-1)k}(F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} BG^r),$$

where j is the inclusion and Φ is the Thom isomorphism. If $r > 1$, then this homomorphism maps the class of a self-transverse immersion $f : M \to N$ to the class $[\mu_r/\Sigma_r, \overline{\delta}_r]$. If $r = 1$, then it maps $[f : M \to N]$ to $[M, \overline{\nu}]$.

Proof. Let us denote by $[N,\psi]$ the class of $j_*\beta_k[f : M \to N]$. Let $i : (N,\emptyset) \to (N^+,+)$ be the inclusion. We denote by $\psi^+ : (N^+,+) \to (C_{\mathbb{R}^\infty}T\xi_G,*)$ the extension of ψ to N^+. Clearly we have that $[N,\psi] = \psi_*^+[N,i]$. Since N is compact the stable map $h_r \circ S^\infty\psi^+$ should be given by a map $g : S^m(N^+) \to S^m(D_rT\xi_G)$, with m large enough. Therefore $\overline{h}_r[N,\psi] = (\Sigma^m)^{-1}g_*\Sigma^m[N,i]$, where $\Sigma^m : \mathcal{N}_{n+k}(N^+,+) \to \mathcal{N}_{n+k+m}(S^m(N^+),*)$ and $(\Sigma^m)^{-1} : \mathcal{N}_{n+k+m}(S^mD_rT\xi_G,*) \to \mathcal{N}_{n+k}(D_rT\xi_G,*)$ are the suspension isomorphisms. The composition $\Phi^\circ(\Sigma^m)^{-1} \equiv \overline{\Phi}$ is the Thom isomorphism for the bundle $[F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} \xi_G] \oplus \varepsilon^m$.

On the other hand, if $r > 1$, then the composition $N \xrightarrow{\psi} C_{\mathbb{R}^\infty}T\xi_G \xrightarrow{\mathrm{adj}\, h_r} QD_rT\xi_G$ represents, under the Thom-Pontryagin construction, the immersion $f_r : \mu_r/\Sigma_r \to N$ [7]. By the naturality of adjointness we have that $\mathrm{adj}\, h_r \circ \psi^+ \simeq \mathrm{adj}\,(h_r \circ S^\infty\psi^+)$. Hence the map g inducing $h_r \circ S^\infty\psi^+$ is the Thom-Pontryagin map for the embedding $\mu_r/\Sigma_r \to N \times \mathbb{R}^m$, whose projection is the immersion f_r.

Let I^m be the product of m copies of the unit interval. Then one can easily show that $\Sigma^m[N,i] = [N \times I^m, q]$, where $q : N \times I^m \to S^m(N^+)$ is the identification map.

Using the geometric definition of the Thom isomorphism [1] we see that $\Phi\overline{h}_r[N,\psi] = \overline{\Phi}g_*[N \times I^m, q] = [\mu_r/\Sigma_r, \overline{\delta}_r]$.

If $r = 1$, then we get $N \xrightarrow{\psi} C_{\mathbb{R}^\infty}T\xi_G \xrightarrow{\overline{d}} C_\infty T\xi_G \xrightarrow{\gamma_\infty} QT\xi_G$, where $\overline{d}$ is a homotopy inverse for d. Since this map corresponds to the immersion f, its image is $[M,\overline{\nu}]$ □

Definition (2.5). We can give an interpretation for the group $\mathcal{N}_m(B(\Sigma_r \int G))$ (for $r = 1, \Sigma_1 \int G = G$) as follows. We consider pairs $(\eta \to M, \varphi)$, where η is an (rk)-vector bundle over a closed smooth m-manifold and $\varphi : \eta \to F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} \xi_G^r$ is a bundle map. We say that $(\eta_1 \to M_1, \varphi_1)$ and $(\eta_2 \to M_2, \varphi_2)$ are bordant if : i) there is a compact smooth $(m+1)$-manifold V such that $\partial V \cong M_1 \amalg M_2$, ii) there is an (rk)-vector bundle $\gamma \to V$ with a bundle map $\psi : \gamma \to F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} \xi_G^r$, iii) there is an isomorphism $h_i : \eta_i \cong \iota_i^* \gamma$ such that $\psi \circ \widehat{\iota}_i \circ h_i = \varphi_i$ $(i = 1, 2)$, where $\widehat{\iota}_i : \iota_i^* \eta \to \gamma$ is the bundle map induced by the embedding $\iota_i : M_i \to V$. Notice that if two bundle maps $\varphi, \varphi' : \eta \to F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} \xi_G^r$ are homotopic then $(\eta \to M, \varphi)$ and $(\eta \to M, \varphi')$ are bordant. The definition of structure using bundle maps is equivalent to the definition using liftings [2].

We denote by $\mathcal{N}_m[\Sigma_r \int G]$ the set of bordism classes and by $[\eta \to M, \varphi]$ the equivalence class of a pair. This set is a group under disjoint union.

One can easily show that there is an isomorphism:

$$\mathcal{N}_m(B(\Sigma_r \int G)) \xrightarrow{\cong} \mathcal{N}_m[\Sigma_r \int G]$$

given by

$$[M, f] \mapsto [f^*(F(\mathbf{R}^\infty; r) \underset{\Sigma_r}{\times} \xi_G^r) \to M, \widehat{f}\,].$$

THEOREM (2.6). *There is an isomorphism*

$$I_G(n,k) \cong \mathcal{N}_{n+k} \oplus \mathcal{N}_n[G] \oplus \bigoplus_{r \geq 2} \mathcal{N}_{n-(r-1)k}[\Sigma_r \int G]$$

given by

$$[f : M \to N] \mapsto ([N], [\nu_f \to M, \widehat{\nu}], \sum_{r \geq 2} [\nu_{f_r} \to \mu_r/\Sigma_r, \delta_r]) ,$$

for $n \geq 0, k > 0$.

Proof. By 2.2 we have an isomorphism $\beta_k : I_G(n,k) \cong \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G)$. If $j : (C_{\mathbb{R}^\infty} T\xi_G, \emptyset) \to (C_{\mathbb{R}^\infty} T\xi_G, *)$ denotes the inclusion, then we have an isomorphism

$$\mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G) \cong \mathcal{N}_{n+k} \oplus \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G, *) ,$$

given by $[N, \psi] \mapsto ([N], j_*[N, \psi])$. The spectrum MO satisfies the wedge axiom so by 2.3 and the Thom isomorphism we have that

$$\mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} T\xi_G, *) \cong \bigoplus_{r \geq 1} \mathcal{N}_{n-(r-1)k}(F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} BG^r).$$

Therefore the theorem follows from 2.4 and 2.5. □

This result is based on the author's thesis (University of Warwick, 1984). A different approach which does not use stable homotopy, based on unpublished work by P. Schweitzer can be found in [6].

3. Bordism of embeddings

Let $E_G(n,k)$ be the set of bordism classes of embedings $e : M \to N$, where M is a closed smooth n-manifold and N is a closed smooth $(n+k)$-manifold, such that the normal bundle ν_e has a G-structure. The definition of $E_G(n,k)$ is the same as that of $I_G(n,k)$ but we now take embeddings instead of immersions. By taking disjoint union of embeddings we can make $E_G(n,k)$ into a $\mathbb{Z}_2$-module. The structure of $E_G(n,k)$ is given by the following

THEOREM (3.1) [14]. *The Thom-Pontryagin construction defines an isomorphism* $\overline{\alpha}_k : E_G(n,k) \to \mathcal{N}_{n+k}(T\xi_G)$. □

THEOREM (3.2) *Let* $f : M \to N$ *be a self-tranverse immersion. Then* f *is bordant to an embedding if and only if, for each* $r \geq 2$, *the bundle* $\nu_{f_r} \to \mu_r/\Sigma_r$ *bords.*

Proof. By 3.1, the Thom isomorphism and 2.5, we have an isomophism $E_G(n,k) \cong \mathcal{N}_{n+k} \oplus \mathcal{N}_n[G]$ given by $[g : M \to N] \mapsto ([N], [\nu_g \to M, \widehat{\nu}])$.

Let $\lambda : E_G(n,k) \to I_G(n,k)$ be the homomorphism which maps the bordism class of an embedding to its class in $I_G(n,k)$. Then an immersion is bordant to an embedding if and only if it belongs to the image of λ. Using the isomorphism in 2.6 we have the following diagram:

$$\begin{array}{ccc} E_G(n,k) & \xrightarrow{\cong} & \mathcal{N}_{n+k} \oplus \mathcal{N}_n[G] \\ \lambda \big\downarrow & & \big\downarrow \iota \\ I_G(n,k) & \xrightarrow{\cong} & \mathcal{N}_{n+k} \oplus \mathcal{N}_n[G] \oplus \bigoplus_{r\geq 2} \mathcal{N}_{n-(r-1)k}[\Sigma_r \int G] \end{array},$$

where ι in the inclusion. The result now follows from the commutativity of this diagram. □

4. Characteristic numbers

Definition (4.1). Let $[\eta \to M, \varphi]$ be an element of $\mathcal{N}_m[\Sigma_r \int G]$. We define characteristic numbers for η as follows. For each cohomology class $c \in H^j(B(\Sigma_r \int G); \mathbb{Z}_2)$ and each partition ρ of $m-j$ (i.e., a sequence $0 \leq i_1 \leq \ldots \leq i_s$ such that $i_1 + \ldots + i_s = m-j$) there is associated a Whitney number $\langle w_\rho(M)\overline{\varphi}^*(c), \sigma(M)\rangle$, where $w_\rho(M)$ is the product $w_{i_1}(M) \ldots w_{i_s}(M)$, $\sigma(M)$ is fundamental class of M and $\overline{\varphi} : M \to B(\Sigma_r \int G)$ is the map induced by φ. Notice that when $c = 1$ we get the ordinary Stiefel-Whitney numbers of M.

We associate to each self-transverse immersion $f : M \to N$ the characteristic numbers of N and of each of the normal bundles $\nu_f, \nu_{f_2}, \nu_{f_3}, \ldots, \nu_{f_t}$, $(t \leq (n/k) + 1)$, which we call the characteristic numbers of f.

THEOREM (4.2) *Let $f : M \to N$ and $g : M' \to N'$ be self-transverse immersions, then f and g are bordant if and only if their characteristic numbers are equal.*

Proof. By a theorem of Thom [13] the Stiefel-Whitney numbers of a manifold determine its bordism class. The homology groups $H_*(B(\Sigma_r \int G); \mathbb{Z}_2)$ are of finite type so by [4] and 2.5, the characteristic numbers of each normal bundle determine its bordism class. Therefore the theorem follows from 2.6. □

Now we will use these characteristic numbers to show that certain immersions are not bordant to oriented immersions and that certain oriented immersions are not bordant to spin immersions.

Definition (4.3). Consider the antipodal action of Σ_2 on the sphere S^n. If X is any space then we denote by $S^n \underset{\Sigma_2}{\times} X \times X$ the quotient space under the action $(a, x_1, x_2) \mapsto (-a, x_2, x_1)$. If X is a smooth manifold then $S^n \underset{\Sigma_2}{\times} X \times X$ has a smooth structure such that the projection $p : S^n \times X \times X \to S^n \underset{\Sigma_2}{\times} X \times X$ is a local diffeomorphism. Let $\mathbb{R}P^m$ be the real projective m-space and let $e : \mathbb{R}P^m \to \mathbb{R}P^{m+1}$ be the canonical embedding. We define an immersion $f : S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}$ by $f(a, y, x) = [a, y, e(x)]$. Similarly if $\mathbb{C}P^m$ denotes the complex projective m-space and $e' : \mathbb{C}P^m \to \mathbb{C}P^{m+1}$ the canonical embedding then we have an immersion $g : S^n \times \mathbb{C}P^{m+1} \times \mathbb{C}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{C}P^{m+1} \times \mathbb{C}P^{m+1}$ given by $g(a, y', x') = [a, y', e'(x')]$.

PROPOSITION (4.4) i) *For each $m \geq 0$, $n \geq 0$, the immersion $f : S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}$ is self-transverse, with normal bundle $\nu_f = S^n \times \mathbb{R}P^{m+1} \times \gamma_m$, where γ_m is the canonical real line bundle over $\mathbb{R}P^m$. ii) The manifold of double points is diffeomorphic to $S^n \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m$ and the normal bundle ν_{f_2} is isomorphic to $S^n \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m$.*

Proof. i) The immersion f is the composition of the embedding

$$\mathrm{id} \times \mathrm{id} \times e : S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^m \to S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1},$$

with the covering projection

$$p : S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1} \to S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}.$$

Using the differential of the involution on $S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}$, one can define a Σ_2-action on the tangent bundle $T(S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1})$, making this bundle into a Σ_2-bundle. The morphism of vector bundles

$$F : T(S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1})/\Sigma_2 \to T(S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1})$$

given by $F[w] = dp(w)$ is an isomorphism. Therefore the composition

$$\begin{aligned} T(S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}) \quad &\xrightarrow{dp} \quad T(S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}) \cong \\ &\cong \quad T(S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1})/\Sigma_2 \end{aligned}$$

maps an element w to $[w]$. Since this equivalence relation identifies an element of the form (z, α, β) with $(dA(z), \beta, \alpha)$, where A is the antipodal map, then the images of the tangent spaces over $(a, e(x_1), x_2)$ and $(-a, e(x_2), x_1)$ are in general position. The normal bundle of the embedding $e : \mathbb{R}P^m \to \mathbb{R}P^{m+1}$ is γ_m, hence $\nu_f = S^n \times \mathbb{R}P^{m+1} \times \gamma_m$.

ii) Since the immersion f has no r-tuple points for $r > 2$, we have that $\mu_2/\Sigma_2 \cong S^n \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m$. From the definition of the classifying map for ν_{f_2} given in 2.1, it is clear that $\nu_{f_2} \cong S^n \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m$ □

Since the normal bundle of the embedding $e' : \mathbb{C}P^m \to \mathbb{C}P^{m+1}$ is the canonical complex line bundle over $\mathbb{C}P^m$, we also have the following result.

PROPOSITION (4.5)

i) *For each $m \geq 0$, $n \geq 0$, the immersion $g : S^n \times \mathbb{C}P^{m+1} \times \mathbb{C}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{C}P^{m+1} \times \mathbb{C}P^{m+1}$ is self-transverse, with normal bundle $\nu_g = S^n \times \mathbb{C}P^{m+1} \times \lambda_m$, where λ_m is the canonical complex line bundle over $\mathbb{C}P^m$.*

ii) *The manifold of double points is diffeomorphic to $S^n \underset{\Sigma_2}{\times} \mathbb{C}P^m \times \mathbb{C}P^m$ and the normal bundle ν_{g_2} is isomorphic to $S^n \underset{\Sigma_2}{\times} \lambda_m \times \lambda_m$* □

In order to calculate the characteristic numbers we need the following results about the quadratic construction.

We denote by B_* the normalized Bar resolution for $\mathbb{Z}_2$ over Σ_2, therefore each B_n is a free $\mathbb{Z}_2[\Sigma_2]$-module in one generator e_n. Let $B_*^{(n)}$ be the n-skeleton of B_*.

THEOREM (4.6) [5]. i) *There is a natural isomorphism*

$$H_*(S^n \underset{\Sigma_2}{\times} X \times X; \mathbb{Z}_2) \cong H_*(B_*^{(n)} \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2}),$$

where $H_*(X)^{\otimes 2} = H_*(X; \mathbb{Z}_2) \otimes H_*(X; \mathbb{Z}_2)$ is a chain complex with trivial boundary and with Σ_2 acting by permuting the factors.

ii) *Let $\{a_j\}_{j \in J}$ be an ordered basis for $H_*(X; \mathbb{Z}_2)$.*

A $\mathbb{Z}_2$-basis for $H_*(B_* \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2})$ is given by the following elements

$$\begin{cases} e_r \underset{\Sigma_2}{\otimes} a_j \otimes a_j & , \quad r \geq 0, j \in J \\ e_0 \underset{\Sigma_2}{\otimes} a_j \otimes a_k & , \quad j < k \end{cases}$$

□

Definition (4.7). By 4.6 we can define natural functions

$$q_r^n : H_i(X;\mathbb{Z}_2) \to H_{2i+r}(S^n \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2), \quad 0 \leq r \leq n \leq \infty$$

by the composition

$$H_i(X;\mathbb{Z}_2) \xrightarrow{h_r^n} H_{2i+r}(B_*^{(n)} \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2}) \xrightarrow{\cong} H_{2i+r}(S^n \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2),$$

where $h_r^n(a) = e_r \underset{\Sigma_2}{\otimes} a \otimes a$.

Definition (4.8). Let $P : H^m(X;\mathbb{Z}_2) \to H^{2m}(S^\infty \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2)$ be Steenrod's external power operation [12] and let $\pi : S^\infty \underset{\Sigma_2}{\times} X \times X \to \mathbb{R}P^\infty$ be the projection. We define a transformation

$$P_n : H^m(X;\mathbb{Z}_2) \to H^{2m+n}(S^\infty \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2)$$

by $P_n(\alpha) = \pi^*(w^n) \smile P(\alpha)$, where $w = w_1(\gamma_\infty)$. Let $f : X \to Y$ be a map, then from the naturality of the power operation we obtain the commutativity of the following diagram:

$$\begin{array}{ccc} H^m(Y;\mathbb{Z}_2) & \xrightarrow{P_n} & H^{2m+n}(S^\infty \underset{\Sigma_2}{\times} Y \times Y;\mathbb{Z}_2) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{id} \underset{\Sigma_2}{\times} f \times f)^*} \\ H^m(X;\mathbb{Z}_2) & \xrightarrow{P_n} & H^{2m+n}(S^\infty \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2) \end{array}$$

Let $\{a_0, a_1, \ldots, \}$ and $\{\alpha_0, \alpha_1, \ldots\}$ be dual bases for $H_*(X;\mathbb{Z}_2)$ and $H^*(X;\mathbb{Z}_2)$ respectively. By 4.6 a basis for $H_*(S^\infty \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2)$ is given by $\{q_i^\infty(a_j) \mid i,j \geq 0\} \cup \{a_{jk} \mid j < k\}$, where a_{jk} is the image of $e_0 \underset{\Sigma_2}{\otimes} a_j \otimes a_k$ under the isomorphism 4.6 i). If

$$\{q_i^\infty(a_j)^* \mid i,j \geq 0\} \cup \{a_{jk}^* \mid j < k\}$$

is the dual basis for $H^*(S^\infty \underset{\Sigma_2}{\times} X \times X;\mathbb{Z}_2)$, then we have the following

PROPOSITION (4.9) [9]. $q_i^{\infty}(a_j)^* = P_i(\alpha_j)$ □

The relation between the characteristic numbers and the geometric properties that we need is given by the following

THEOREM (4.10) [4]. *Let $f : X \to Y$ be a map between spaces of finite type. The necesary and sufficient condition that $[M, \varphi] \in \mathcal{N}_n(Y)$ lie in the image of $f_* : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is that every characteristic number of $[M, \varphi]$ associated with an element in the kernel of $f^* : H^*(Y; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ must vanish.* □

THEOREM (4.11). *The immersions*

$$f : S^n \times \mathbb{R}P^{m+1} \times \mathbb{R}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}, \text{ for } n \geq 0 \text{ and } m > 0,$$

are not bordant to immersions with oriented normal bundle.

Proof. By 2.6 we have the following commutative diagram:

$$\begin{array}{ccccccc}
I_{SO(1)}(2m+n+1,1) \cong & \mathcal{N}_{2m+n+2} & \oplus & \mathcal{N}_{2m+n+1}[SO(1)] & \oplus & \mathcal{N}_{2m+n}[\Sigma_2 \int SO(1)] & \oplus \cdots \\
\Big\downarrow F & \Big\downarrow \mathrm{id} & & \Big\downarrow F_1 & & \Big\downarrow F_2 & \\
I_{O(1)}(2m+n+1,1) \cong & \mathcal{N}_{2m+n+2} & \oplus & \mathcal{N}_{2m+n+1}[O(1)] & \oplus & \mathcal{N}_{2m+n}[\Sigma_2 \int O(1)] & \oplus \cdots
\end{array}$$

To prove the theorem we have to show that the class of f is not in the image of F. By 4.4, the image of $[f]$ under the isomorphism is

$$([S^n \underset{\Sigma_2}{\times} \mathbb{R}P^{m+1} \times \mathbb{R}P^{m+1}], [S^n \times \mathbb{R}P^{m+1} \times \gamma_m], [S^n \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m]).$$

We will see that the class of the bundle $S^n \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m$ is not in the image of F_2 (notice that since S^n is a boundary, $[S^n \times \mathbb{R}P^{m+1} \times \gamma_m] = 0$). There is a Σ_2-equivariant homotopy equivalence $F(\mathbb{R}^\infty; 2) \simeq S^\infty$, so the homomorphism F_2 is induced by the map $\mathrm{id} \underset{\Sigma_2}{\times} Bi \times Bi : S^\infty \underset{\Sigma_2}{\times} BSO(1) \times BSO(1) \to S^\infty \underset{\Sigma_2}{\times} BO(1) \times BO(1)$, where $i : SO(1) \to O(1)$.

If we take the class $P_n(w_1(\gamma_\infty)^m) \in H^{2m+n}(S^\infty \underset{\Sigma_2}{\times} \mathbb{R}P^\infty \times \mathbb{R}P^\infty)$, with $m > 0$, then by (4.8), we have $(\mathrm{id} \underset{\Sigma_2}{\times} Bi \times Bi)^*(P_n(w_1(\gamma_\infty)^m) = P_n Bi^*(w_1(\gamma_\infty)^m) = 0$, since $Bi^*(w_1(\gamma_\infty)) = 0$. Therefore $P_n(w_1(\gamma_\infty)^m) \in \ker(\mathrm{id} \underset{\Sigma_2}{\times} Bi \times Bi)^*$. Consider the following pull-back diagram:

$$\begin{array}{ccccc}
S^n \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m & \longrightarrow & S^\infty \underset{\Sigma_2}{\times} \gamma_m \times \gamma_m & \longrightarrow & S^\infty \underset{\Sigma_2}{\times} \gamma_\infty \times \gamma_\infty \\
\downarrow & & \downarrow & & \downarrow \\
S^n \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m & \xrightarrow{i_n \underset{\Sigma_2}{\times} \mathrm{id} \times \mathrm{id}} & S^\infty \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m & \xrightarrow{\mathrm{id} \underset{\Sigma_2}{\times} j_m \times j_m} & S^\infty \underset{\Sigma_2}{\times} \mathbb{R}P^\infty \times \mathbb{R}P^\infty
\end{array},$$

where i_n and j_m are the canonical inclusions. We want to calculate the characteristic number $\langle \varphi^*(P_n(w_1(\gamma_\infty)^m)), \sigma(S^n \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m)\rangle$, where φ is the map given by the composition at the bottom. If M is a connected m-manifold then $S^n \underset{\Sigma_2}{\times} M \times M$ is also connected, hence

$$q^n_n : H_m(M;\mathbb{Z}_2) \to H_{2m+n}(S^n \underset{\Sigma_2}{\times} M \times M;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Since $(i_n \underset{\Sigma_2}{\times} \mathrm{id} \times \mathrm{id})_* \circ q^n_n = q^\infty_n$, then $(i_n \underset{\Sigma_2}{\times} \mathrm{id} \times \mathrm{id})_* q^n_n(\sigma(M)) = q^\infty_n(\sigma(M))$ which is a generator by (4.6). Therefore

$$(*) \qquad q^n_n(\sigma(M)) = \sigma(S^n \underset{\Sigma_2}{\times} M \times M).$$

Using the definition of φ, (*) and the naturality of P_n (4.8), we have:
$\langle \varphi^*(P_n(w_1(\gamma_\infty)^m)), \sigma(S^n \underset{\Sigma_2}{\times} \mathbb{R}P^m \times \mathbb{R}P^m)\rangle =$
$= \langle (i_n \underset{\Sigma_2}{\times} \mathrm{id} \times \mathrm{id})^*(\mathrm{id} \underset{\Sigma_2}{\times} j_m \times j_m)^*(P_n(w_1(\gamma_\infty)^m)), q^n_n(\sigma(\mathbb{R}P^m))\rangle$
$= \langle (\mathrm{id} \underset{\Sigma_2}{\times} j_m \times j_m)^* P_n(w_1(\gamma_\infty)^m)), q^\infty_n(\sigma(\mathbb{R}P^m))\rangle$
$= \langle P_n(w_1(\gamma_m)^m), q^\infty_n(\sigma(\mathbb{R}P^m))\rangle$
$= \langle q^\infty_n(\sigma(\mathbb{R}P^m))^*, q^\infty_n(\sigma(\mathbb{R}P^m))\rangle = 1$, by 4.9, since $w_1(\gamma_m)^m$ and $\sigma(\mathbb{R}P^m)$ are duals.

The theorem now follows from 4.10. □

THEOREM (4.12). *The immersions*

$$g : S^n \times \mathbb{C}P^{m+1} \times \mathbb{C}P^m \to S^n \underset{\Sigma_2}{\times} \mathbb{C}P^{m+1} \times \mathbb{C}P^{m+1}, \ \textit{for}\ n \geq 0$$

and $m > 0$, *are not bordant to immersions whose normal bundle has a* Spin *(2)-structure.*

Proof. By 2.6 we have the following commutative diagram:

$$\begin{array}{ccccccccc}
I_{\mathrm{Spin}(2)}(4m+n+2,2) & \cong & \mathcal{N}_{4m+n+4} & \oplus & \mathcal{N}_{4m+n+2}[\mathrm{Spin}(2)] & \oplus & \mathcal{N}_{4m+n}(\Sigma_2 \int \mathrm{Spin}(2)) & \oplus \dots \\
\downarrow F & & \downarrow \mathrm{id} & & \downarrow F_1 & & \downarrow F_2 & \\
I_{SO(2)}(4m+n+2,2) & \cong & \mathcal{N}_{4m+n+4} & \oplus & \mathcal{N}_{4m+n+2}[SO(2)] & \oplus & \mathcal{N}_{4m+n}(\Sigma_2 \int SO(2)) & \oplus \dots
\end{array}$$

To prove the theorem we have to show that the class of g is not in the image of F. By (4.5), the image of [g] under the isomorphism is

$$([S^n \underset{\Sigma_2}{\times} \mathbb{C}P^{m+1} \times \mathbb{C}P^{m+1}], [S^n \times \mathbb{C}P^{m+1} \times \lambda_m], [S^n \underset{\Sigma_2}{\times} \lambda_m \times \lambda_m]).$$

We will see that the class of the bundle $S^n \underset{\Sigma_2}{\times} \lambda_m \times \lambda_m$ is not in the image of F_2 (notice that since S^n is a boundary, $[S^n \times \mathbb{C}P^{m+1} \times \lambda_m] = 0$). The homomorphism F_2 is induced by the map

$$\mathrm{id} \underset{\Sigma_2}{\times} Bp \times Bp : S^\infty \underset{\Sigma_2}{\times} B\mathrm{Spin}(2) \times B\mathrm{Spin}(2) \to S^\infty \underset{\Sigma_2}{\times} BSO(2) \times BSO(2),$$

where $p : \mathrm{Spin}(2) \to SO(2)$.

If we take the class $P_n(w_2(\lambda_\infty)^m) \in H^{4m+n}(S^\infty \underset{\Sigma_2}{\times} \mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, with $m > 0$, then by (4.8) we have $(\mathrm{id} \underset{\Sigma_2}{\times} Bp \times Bp)^*(P_n(w_2(\lambda_\infty)^m)) = P_n(Bp^*(w_2(\lambda_\infty)^m)) = 0$, since $Bp^*(w_2(\lambda_\infty)) = 0$. Therefore $P_n(w_2(\lambda_\infty)^m) \in \ker(\mathrm{id} \underset{\Sigma_2}{\times} Bp \times Bp)^*$.

Consider the following pull-back diagram:

$$\begin{array}{ccccc}
S^n \underset{\Sigma_2}{\times} \lambda_m \times \lambda_m & \longrightarrow & S^\infty \underset{\Sigma_2}{\times} \lambda_m \times \lambda_m & \longrightarrow & S^\infty \underset{\Sigma_2}{\times} \lambda_\infty \times \lambda_\infty \\
\downarrow & & \downarrow & & \downarrow \\
S^n \underset{\Sigma_2}{\times} \mathbb{C}P^m \times \mathbb{C}P^m & \xrightarrow{i_n \underset{\Sigma_2}{\times} \mathrm{id}\times\mathrm{id}} & S^\infty \underset{\Sigma_2}{\times} \mathbb{C}P^m \times \mathbb{C}P^m & \xrightarrow{\mathrm{id} \underset{\Sigma_2}{\times} j_m\times j_m} & S^\infty \underset{\Sigma_2}{\times} \mathbb{C}P^\infty \times \mathbb{C}P^\infty
\end{array},$$

where i_n and j_m are the canonical inclusions. Then as in the proof of (4.11) we have:

$\langle (i_n \underset{\Sigma_2}{\times} \mathrm{id} \times \mathrm{id})^*(\mathrm{id} \underset{\Sigma_2}{\times} j_m \times j_m)^*(P_n(w_2(\lambda_\infty)^m)), \sigma(S^n \underset{\Sigma_2}{\times} \mathbb{C}P^m \times \mathbb{C}P^m)\rangle$
$= \langle P_n(w_2(\lambda_m)^m), q_n^\infty(\sigma(\mathbb{C}P^m))\rangle$
$= \langle q_n^\infty(\sigma(\mathbb{C}P^m))^*, q_n^\infty(\sigma(\mathbb{C}P^m))\rangle = 1$, by (4.9), since $w_2(\lambda_m)^m$ and $\sigma(\mathbb{C}P^m)$ are duals.

The theorem now follows from (4.10). □

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References

[1] Bröcker, T. and tom Dieck T., *Kobordismentheorie*. Lecture Notes in Math. **178**, Berlin-Heidelberg-New York: Springer. 1970.

[2] BROWDER, W., *The Kervaire invariant of framed manifolds and its generalizations*. Ann. of Math. **90** (1969), 157-186.

[3] COHEN, F. R., MAY, J. P., TAYLOR, L. *The splitting of certain spaces CX*. Math. Proc. Camb. Phil. Soc. **84** (1978), 465-496.

[4] CONNER, P. E., FLOYD, E. E., *Differentiable periodic maps*. Ergeb. Math. Grenzgeb. **33**. Berlin-New York: Springer. 1964.

[5] DYER, E. LASHOF, R. K. *Homology of iterated loop spaces*. Amer. J. Math. **84** (1962), 35-88.

[6] KOSCHORKE, U. *Multiple point invariants of link maps*. Lecture Notes Math. **1350**. Berlin-Heidelberg-New York: Springer. 1988.

[7] KOSCHORKE, U., SANDERSON, B. J., *Self-intersections and higher Hopf invariants*. Topology **17** (1978), 283-290.

[8] MAY, J. P., *The geometry of iterated loop spaces*. Lecture Notes in Math. **271**. Berlin-Heidelberg-New York: Springer. 1972.

[9] NISHIDA, G. *Cohomology operations in iterated loop spaces*. Proc. Japan Acad. **44** (1968), 104-109.

[10] SCHWEITZER, P.A., *Joint cobordism of immersions*. Lecture Notes in Math. **168**. Berlin-Heidelberg-New York: Springer. 1970, pp. 269-282.

[11] SNAITH, V.P., *A stable decomposition of $\Omega^n S^n X$*. J. London Math. Soc. **7** (1974), 577-583.

[12] STEENROD, N., EPSTEIN, D. B. A. *Cohomology operations*. Ann. of Math. Studies, **50**. Princeton, N. J.: Princeton Univ. Press. (1962).

[13] THOM, R. *Quelques propriétés globales des variétés différentiables*. Comment. Math. Helv. **28** (1954), 17-86.

[14] WALL, C.T.C. *Cobordism of pairs*. Comment. Math. Helv. **35** (1961), 136-145.