Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

SUSPENSION SPECTRA ARE HARMONIC

BY MICHAEL J. HOPKINS AND DOUGLAS C. RAVENEL*

In memory of José Adem

The notion of a harmonic spectrum in stable homotopy theory was introduced by the second author in [Rav84]. Roughly speaking, a spectrum is harmonic if its homotopy is computable with the chromatic spectral sequence. It was shown in [Rav84, Theorem 4.4] that a large class of connective spectra (including all finite spectra) are harmonic. On the other hand, the Eilenberg-Mac Lane spectrum for a torsion group was shown to be dissonant (the opposite of harmonic), i.e., chromatic methods give no information at all about its homotopy type.

The purpose of this paper is to show that *all suspension spectra are harmonic* (Theorem (8)), thereby giving an affirmative answer to a question raised in [Rav84]. This result has some interesting corollaries. One is a new proof of the Whitehead theorem for Morava K-theory, Theorem (2). (This was originally proved, although not explicitly stated by Bousfield in [Bou82].) Another (Theorem (12)) is that a suspension spectrum is uniquely determined by any of its connective covers.

It is a pleasure to thank Joe Neisendorfer and Fred Cohen for helpful conversations, and Nick Kuhn for pointing out the relevance of Bousfield's paper and that Theorem (2) below is a consequence of Theorem (8).

Morava K-theories

For each prime p and each integer $n \geq 0$ there is a generalized homology theory $K(n)_*$ called the n^{th} Morava K-theory. These were invented by Jack Morava, but he never published an account of them. The definition is too complicated to give here; instead we will list some of their properties. Most of the following result is proved in [JW75]; a proof of (v) can be found in [Rav84].

PROPOSITION (1). *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geq 0$ with the following properties.* (We follow the standard practice of omitting p from the notation.)

(i) $K(0)_*(X) = H_*(X; \mathbf{Q})$ *and* $\overline{K(0)}_*(X) = 0$ *when* $\overline{H}_*(X; \mathbf{Z})$ *is all torsion.*

(ii) $K(1)_*$ *is one of $p-1$ isomorphic summands of mod p complex K-theory.*

(iii) $K(0)_*(\text{pt.}) = \mathbf{Q}$ *and for* $n > 0$, $K(n)_*(\text{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ *where the dimension of v_n is $2p^n - 2$. This ring is a graded field in the sense that every graded module over it is free.* $K(n)_*(X)$ *is a module over* $K(n)_*(\text{pt.})$.

(iv) *There is a Künneth isomorphism*

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*(\text{pt.})} K(n)_*(Y).$$

* Both authors partially supported by the National Science Foundation

(v) *Let X be a finite CW-complex. If $\overline{K(n)}_*(X)$ (the reduced Morava K-theory of X) vanishes, then so does $\overline{K(n-1)}_*(X)$.*

(vi) *There is an Atiyah-Hirzebruch spectral sequence converging to $K(n)_*(X)$ for any space X with*

$$E_2 = H_*(X; \mathbf{Z}/(p)) \otimes K(n)_*(\text{pt.}).$$

(vii) *If X is finite then the Atiyah-Hirzebruch spectral sequence collapses for large n, giving*

$$K(n)_*(X) = H_*(X; \mathbf{Z}/(p)) \otimes K(n)_*(\text{pt.}).$$

In particular if a finite complex has trivial Morava K-theory (for all n and p) then it is contractible.

(viii) *The Atiyah-Hirzebruch spectral sequence collapses if $H_*(X; \mathbf{Z})$ is torsion free.*

We will give a new proof of the following.

THEOREM (2). *Let $f : X \longrightarrow Y$ be a map between simply connected spaces with $K(n)_*(f)$ an isomorphism for all n and p. Then f is a weak homotopy equivalence, i.e., it induces an isomorphism of homotopy groups and is an equivalence if X and Y have the homotopy type of CW-complexes.*

An interesting corollary, which is actually equivalent to the theorem, is the following.

COROLLARY (3). *If X is a simply connected space with*

$$K(n)_*(X) = K(n)_*(\text{pt.})$$

for all n and p, then X is weakly contractible.

In Theorem (2) and Corollary (3) we could replace $K(n)_*$ by the related theories $E(n)_*$. (This assertion can be deduced from [Rav84, 2.1(d)].) These were introduced by Johnson-Wilson in [JW73]; $E(n)$ is a BP-module spectrum with

$$\pi_*(E(n)) = \mathbf{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}].$$

Generalizing the Whitehead theorem

The classical Whitehead theorem [Whi49] is the following.

THEOREM (4). *Let $f : X \longrightarrow Y$ be a map between simply connected spaces with $H_*(f)$ an isomorphism. Then f is a weak homotopy equivalence.*

One can ask to which homology theories E_* this can be generalized. Clearly the result will *not* hold if there is a noncontractible simply connected CW-complex X for which

$$E_*(X) = E_*(\text{pt.}),$$

i.e., an E_*-acyclic space X.

If E is connective (i.e., if $E_{-i}(\text{pt.}) = 0$ for $i \ll 0$) then the class of E_*-acyclic spaces depends only on certain *arithmetic* properties of $E_*(\text{pt.})$, namely for which primes p multiplication by p in $E_*(\text{pt.})$ is either an isomorphism or has a nontrivial kernel.

For nonconnective theories such as classical complex K-theory, the situation is more delicate. For example it is known that

$$K_*(K(\mathbf{Z}/(p), 2)) = K_*(\text{pt.}),$$

so there is no Whitehead theorem for K-theory.

The theory of interest in Theorem (2) is E_* where

$$E = \bigvee_{p} \bigvee_{n \geq 0} K(n), \tag{5}$$

the wedge of the Morava K-theories for all n and p. It has suitable arithmetic properties, but it is not connective, so Theorem (2) is not obvious.

A stable question related to generalizing the Whitehead theorem is the following. *For which spectra is it true that a map f is a weak equivalence whenever $E_*(f)$ is an isomorphism?* It is true for *E_*-local* spectra, which are defined as follows.

Definition (6). A spectrum X is **E_*-local** if for each E_*-acyclic spectrum Y, there are no essential maps from Y to X. A space X is E_*-local if each map $f : Y_1 \to Y_2$ with $E_*(f)$ an isomorphism induces an isomorphism

$$[Y_2, X] \xrightarrow{f^*} [Y_1, X].$$

The main theorem in the subject, which is due to Bousfield ([Bou75] and [Bou79]), says that for any theory E_*, any space or spectrum X is E_*-equivalent to one which is local. For more discussion, see [Rav84].

Definition (7). A spectrum is **harmonic** if it is E_*-local for E as in (5). If $E_*(X) = 0$ then X is **dissonant**. A space is harmonic or dissonant if its suspension spectrum is.

Then Theorem (2) is a consequence of the following result, which answers a question posed in [Rav84].

THEOREM (8). *Every suspension spectrum is harmonic.*

COROLLARY (9). *A simply connected dissonant space is weakly contractible.*

Proof that Theorem (8) *implies Theorem* (2). Suppose that

$$f : X \longrightarrow Y$$

is a map between simply connected spaces with $E_*(f)$ an isomorphism. Then it follows that $E_*(C_f)$ is trivial, where C_f denotes the cofibre of f, and C_f is therefore weakly contractible by (9), so f is a weak equivalence as claimed. □

Properties of harmonic spectra

The following two results can be found in [Rav84].

PROPOSITION (10). *For any homology theory E_*,*
(i) *Any homotopy inverse limit of E_*-local spectra is E_*-local.*
(ii) *If*

$$W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$$

is a cofibre sequence and any two of W, X and Y are E_*-local, then so is the third.
(iii) *If $X \vee Y$ is E_*-local, then so are X and Y.*

On the other hand, a homotopy direct limit of local spectra need not be local. We will use three different inverse limit constructions in this paper: an Adams resolution in the proof of Theorem (14), a Postnikov tower in the proof of Lemma (8), and a geometric construction of the Eilenberg-Moore spectral sequence in the proof of Lemma (17).

PROPOSITION (11).*The mod p Eilenberg-MacLane spectrum H/p is dissonant. More generally, if $\pi_*(X)$ is bounded above and torsion, then X is dissonant.*

Given this result, Theorem (8) has the following consequence.

THEOREM (12). *Let X be a suspension spectrum with $\pi_*(X) \otimes \mathbf{Q} = 0$. Let $X\langle m\rangle$ denote the $(m-1)$-connected cover of X (as a spectrum), i.e., there is a map $X\langle m\rangle \rightarrow X$ inducing an isomorphism in stable homotopy in dimensions m and above. Then the homotopy type of X is uniquely determined by that of $X\langle m\rangle$.*

Proof. Assume for simplicity that X is p-local. The general case will follow by an arithmetic square argument which we leave to the reader.

The fibre F of the map $X\langle m\rangle \rightarrow X$ has homotopy that is all torsion and is bounded above. Thus F is dissonant by (11). It follows that $X\langle m\rangle$ and X have the same E_*-localization (for E as in (5)), namely X, so X is the E_*-localization of $X\langle m\rangle$. □

THEOREM (13). *If X is a connective spectrum with free abelian (or $\mathbf{Z}_{(p)}$-free) homology then it is harmonic.*

If we add a finiteness hypothesis, then this becomes a special case of a more general result ([Rav84, 4.4]) which implies that all finite complexes are harmonic.

We will sketch the proof of Theorem (13) here. It suffices to do it in the p-local case. A critical lemma is the following.

LEMMA (14)*.A connective p-local spectrum X is harmonic if $BP \wedge X$ is harmonic, where BP denotes the Brown-Peterson spectrum.*

This lemma in turn depends on the following.

LEMMA (15). *Let W be a (possibly infinite) wedge of spheres. Then $BP \wedge W$ is harmonic.*

If X is as in Theorem (13), then $BP \wedge X_{(p)}$ is a wedge of suspensions of BP, and hence harmonic by Lemma (15). Theorem (13) then follows from Lemma (14).

Proof of Lemma (15). In [Rav84, Theorem 4.2] it is shown by direct calculation that BP is harmonic. We need to know the same for an arbitrary wedge of suspensions of BP, but the proof given there does not obviously generalize to this case. Instead, we argue as follows.

By [Rav84, Theorem 2.1] and [Rav87, Lemma 3] a spectrum is L_nBP_*-local if and only if it is local with respect to $\vee_{0\le i\le n}K(i)$. In particular such a spectrum is harmonic. Hence by [Rav84, Prop. 1.24] (which says that an E-module spectrum is E_*-local for a ring spectrum E), the spectrum

$$X_n = W \wedge L_nBP$$

is harmonic for each n. It follows that $\lim_{\leftarrow} X_n$ is also harmonic. We claim that this inverse limit is equivalent to $W \wedge BP$, so the latter is harmonic as desired. We have

$$\pi_*(X_n) = H_*(W) \otimes \pi_*(L_nBP)$$

and $\pi_*(L_nBP)$ was computed in [Rav84]. The maps in the inverse system respect this direct sum decomposition, so we have

$$\begin{aligned}\pi_*(\varprojlim X_n) &= \varprojlim \pi_*(X_n)\\ &= H_*(W) \otimes \varprojlim \pi_*(L_nBP)\\ &= H_*(W) \otimes \pi_*(BP)\\ &= \pi_*(W \wedge BP). \quad \square\end{aligned}$$

Proof of Lemma (14). We use the canonical BP-based Adams resolution for X. This is a diagram which displays X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_s$$

with X_0 contractible and cofibre sequences

$$X \wedge \overline{BP}^{(s)} \wedge BP \longrightarrow X_{s+1} \longrightarrow X_s.$$

where $\overline{BP}$ denotes the cofibre of the unit map $S^0 \to BP$ and $\overline{BP}^{(s)}$ is its s-fold smash product. The smash product $\overline{BP}^{(s)} \wedge BP$ is a wedge of suspensions of BP, so its smash product with X is harmonic by hypothesis. It follows by induction on s that each X_s is harmonic, so X itself is. $\square$

The proof of Theorem (8)

We need the following definition.

Definition (16). A space has **depth** 0 if its integral homology groups are free abelian or (in the p-local case) if its homology groups are free $\mathbf{Z}_{(p)}$-modules. It has depth $\leq k+1$ if it is the fibre of a map between spaces (with simply connected target) each having depth $\leq k$.

Note that the category of spaces having depth $\leq k$ is closed under finite products and weak infinite products.

Now we need two lemmas.

LEMMA (17). *If a space has finite depth, then it is harmonic.*

LEMMA (18). *Every simply connected Eilenberg-MacLane space has finite depth.*

Now consider the Postnikov tower for a simply connected space X. It displays X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_i$$

(not to be confused with the Adams resolution used earlier) where $X_2 = K(\pi_2(X), 2)$ and there are fibre sequences

$$X_{i+1} \longrightarrow X_i \longrightarrow K(\pi_{i+1}(X), i+2).$$

It follows from Lemma (18) that each X_i has finite depth, so each of them is harmonic by Lemma (17). Hence X is harmonic by Proposition (10).

This completes the proof of Theorem (8) and hence our Whitehead theorem, modulo the proofs of Lemmas (17) and (18).

The proof of Lemma (17)

We will of course argue by induction on the depth, using Theorem (13) to start. Let

$$X \longrightarrow E \longrightarrow B$$

be a fibre sequence where E and B each have depth $\leq k$.

The geometric construction for the Eilenberg-Moore spectral sequence (due originally to L. Smith [Smi69, Smi70] and Rector [Rec70]) displays X as the total space of the cosimplicial space

$$[i] \mapsto E \times \overbrace{B \times B \times \cdots B}^{i \text{ factors}}.$$

Applying the functor "suspension spectrum" to everything in side gives a cosimplicial spectrum. It follows from the strong convergence of the Eilenberg-Moore spectral sequence that the total spectrum of this cosimplicial spectrum is the suspension spectrum of X. This writes the suspension spectrum of X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_i$$

(not to be confused with the Postnikov tower or the Adams resolution used earlier) with the following properties: the spectrum X_1 is the suspension spectrum of E, and there are cofibre sequences

$$X_{i+1} \longrightarrow X_i \longrightarrow K_i$$

where $\Sigma^{i-1}K_i$ is a stable summand of

$$E \times \overbrace{B \times B \times \cdots B}^{i \text{ factors}}.$$

(Note that this is a stable construction in that there is a space called $\Sigma^{i-1}X_i$ which need not desuspend.)

Now there is no theorem saying *a priori* that the product of harmonic spaces is harmonic, but it is clear from the definition that any finite product of spaces of depth $\leq k$ has depth $\leq k$. Hence each $E \times B^i$ is harmonic by the inductive hypothesis, so each K_i is by (10)(iii). It follows that each X_i is harmonic by induction on i (using (10)(ii)), so X itself is harmonic by (10)(i).

The proof of Lemma (18)

It suffices to show that $K(\mathbf{Z}, i+2)$ has finite depth for each $i \geq 2$, since any abelian group admits a free resolution. We claim that it suffices to prove it for $K(\mathbf{Z}_{(p)}, i+2)$ for each i and p. There are short exact sequences

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \textstyle\prod_p \mathbf{Q}/\mathbf{Z}_{(p)} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z}_{(p)} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z}_{(p)} \longrightarrow 0$$

which leads to corresponding fibre sequences of Eilenberg-Mac Lane spaces. We know that $K(\mathbf{Q}, i)$ has depth 0 since its homology is torsion free. If $K(\mathbf{Z}_{(p)}, i)$ has finite depth, then so does $K(\mathbf{Q}/\mathbf{Z}_{(p)}, i)$ since $\mathbf{Q}/\mathbf{Z}_{(p)}$ is a p-torsion group. The first short exact sequence would then imply that $K(\mathbf{Z}, i)$ has finite depth.

We will show that $K(\mathbf{Z}_{(p)}, i+2)$ has depth $\leq i$. We need some results from Steve Wilson's thesis [Wil73] and [Wil75]. There he studies the spaces BP_i in the Ω-spectrum for BP. The main result of [Wil73] is that each of them has torsion free homology and hence depth 0.

Recall that the Brown-Peterson spectrum BP has homotopy groups

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$$

with $|v_i| = 2p^i - 2$. There are BP-module spectra $BP\langle n\rangle$ (known as Johnson-Wilson spectra) with

$$\pi_*(BP\langle n\rangle) = \mathbf{Z}_{(p)}[v_1, v_2, \cdots v_n].$$

In particular, $BP\langle 0\rangle = H_{(p)}$, the Eilenberg-Mac Lane spectrum for $\mathbf{Z}_{(p)}$.

There are cofibre sequences of spectra

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle$$

which yield fibre sequences of spaces

$$BP\langle n-1\rangle_i \longrightarrow BP\langle n\rangle_{i+1+|v_n|} \longrightarrow BP\langle n\rangle_{i+1}. \tag{19}$$

In [Wil75] Wilson showed that for $i \leq 2 + 2p + 2p^2 + \cdots 2p^n$, $BP\langle n\rangle_i$ is a factor of BP_i and therefore has depth 0.

Lemma (18) follows from the $n = 0$ case of the following.

LEMMA (20). *Let $f(n) = 2 + 2p + 2p^2 + \cdots 2p^n$. Then for each $n \geq 0$ and $i \geq 0$, the space $BP\langle n\rangle_{i+f(n)}$ has depth $\leq i$.*

Proof. Wilson proved this for $i = 0$ and we will argue by induction on i. From (19) we get a fibre sequence

$$BP\langle n\rangle_{i+f(n)} \longrightarrow BP\langle n+1\rangle_{i+f(n)+1+|v_{n+1}|} \longrightarrow BP\langle n\rangle_{i+f(n)+1}. \tag{21}$$

Since

$$f(n) + |v_{n+1}| = f(n+1) - 2,$$

we have

$$i + f(n) + 1 + |v_{n+1}| = f(n+1) + i - 1.$$

This means that the base and total space of (21) each have depth $\leq i - 1$ by the inductive hypothesis, and the result follows. □

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA 02139

UNIVERSITY OF ROCHESTER
ROCHESTER, NY 14627

REFERENCES

[Bou75] A. K. BOUSFIELD, *The localization of spaces with respect to homology*. Topology, **14** 133-150, 1975.

[Bou79] ———, *The localization of spectra with respect to homology*. Topology, **18** 257-281,.1979.

[Bou82] ———, *On homology equivalences and homological localizations of spaces*. American Journal of Mathematics, **104** 1025-1042, 1982.

[JW73] D. C. JOHNSON AND W. S. WILSON, *Projective dimension and Brown-Peterson homology*. Topology, **12** 327-353, 1973.

[JW75] ———, *BP-operations and Morava's extraordinary K-theories*. Mathematische Zeitschrift, **144** 55-75, 1975.

[Rav84] D. C. RAVENEL, *Localization with respect to certain periodic homology theories*. American Journal of Mathematics, **106** 351-414, 1984.

[Rav87] ———, *The geometric realization of the chromatic resolution*. In W. Browder, editor, Algebraic topology and algebraic K-theory, pages 168-179, 1987.

[Rec70] D. L. RECTOR. *Steenrod operations in the Eilenberg-Moore spectral sequence*. Commentarii Mathematici Helvetici, **45** 540-552, 1970.

[RW80] D. C. RAVENEL AND W. S. WILSON, *The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture*. American Journal of Mathematics, **102** 691-748, 1980.

[Smi69] L. SMITH, *On the construction of the Eilenberg-Moore spectral sequence*. Bulletin of the American Mathematical Society, **75** 873-878, 1969.

[Smi70] ———, *Lectures on the Eilenberg-Moore spectral sequence*. Volume 134 of Lecture Notes in Mathematics, Springer-Verlag, 1970.

[Whi49] J. H. C. WHITEHEAD. *Combinatorial homotopy I*, Bulletin of the American Mathematical Society, **55** 213-249, 1949.

[Wil73] W. S. WILSON. *The Ω-spectrum for Brown-Peterson cohomology, Part I*. Commentarii Mathematici Helvetici, **48** 45-55, 1973.

[Wil75] ———, *The Ω-spectrum for Brown-Peterson cohomology, Part II*. American Journal of Mathematics, **97** 101-123, 1975.