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UNSTABLE ELEMENTS RELATED TO THE STABLE HOMOTOPY OF PROJECTIVE SPACES

BY MITSUNORI IMAOKA

1. Introduction

Calculations of the Atiyah-Hirzebruch spectral sequences for the stable homotopy groups of the complex or quaternionic projective spaces are found in [12],[7] and so on. Differentials in such spectral sequences represent much information on the stable homotopy of the projective spaces, and they are closely related to the transfer maps developed largely by Knapp [6]. In this paper, I will show that the differentials have a meaning also in the unstable homotopy groups of spheres.

Let $H_*(CP^\infty;Z) = Z\{b_1, b_2, \ldots\}$ be the homology of the complex projective space, where $b_i \in H_{2i}(CP^\infty;Z)$ is the generator. Then, in the Atiyah-Hirzebruch spectral sequence $E^2_{i,j} = H_i(CP^\infty;Z) \otimes \pi^s_j(S^0) \Rightarrow \pi^s_{i+j}(CP^\infty)$, a differential $d^r : E^{2r}_{2n,j} \to E^{2r}_{2(n-r),j+2r-1}$ defined on the $2r$–term has the form

$$(A) \qquad d^r[b_n \otimes \alpha] = [b_{n-r} \otimes \beta] \quad \text{for } \alpha \in \pi^s_j(S^0) \text{ and } \beta \in \pi^s_{j+2r-1}(S^0).$$

Our result can be stated as follows, in which $H : \pi_i(S^{2k}) \to \pi_i(S^{4k-1})$ and $P : \pi_{i+2}(S^{4k-1}) \to \pi_i(S^{2k-1})$ are the homomorphisms representing the Hopf invariant and the Whitehead product respectively in the *EHP*-sequence:

THEOREM (I) *Whenever the relation* (A) *occurs,*

(1) *we have an element* $\gamma \in \pi_{j+4n-2r+2}(S^{2(n-r+1)})$, *for* $0 \le j \le 4n - 6r + 1$, *satisfying* $H(\gamma) = \beta$ *and* $\Sigma^{2r-1}\gamma = P(\alpha_{4n+3})$, *and*

(2) *the Whitehead product* $[\iota_s, \beta_s]$ *is equal to* 0 *for* $0 \le j \le 2n - 4r$ *and* $s = 2(n-r) + 1$,

where $\alpha_t \in \pi_{j+t}(S^t)$, $\beta_s \in \pi_{j+2r+s-1}(S^s)$ *and* $\iota_s \in \pi_s(S^s)$ *are the desuspensions of* α, β *in* (A) *and a generator* $\iota \in \pi^s_0(S^0)$ *respectively.*

We will show a more general result in Theorem (2) and Corollary (3), which include results for the quaternionic projective space HP^∞ and the quaternionic quasi-projective space Q_∞. The basic idea is to use desuspensions of the transfer maps concerning the projective spaces, which generalizes the use of the J–homomorphisms. We treated some related results in [3].

For the real projective spaces, this sort of study has already been done extensively by Mahowald [8] and Toda [14], and it has given some important parts in the James exact sequence. In a sense, our discussion is a complex or quaternionic version of their methods, and has some aspect, as is seen in the excellent work of Milgram [10], of the unstable homotopy from the stable point of view.

In §2 we treat desuspensions of the transfer maps for projective spaces, and prove Theorem (1) which is the key to our procedure. We show our main results Theorem (2) and Corollary (3) in §3, and give some examples in §4.

2. Desuspensions of transfer maps

First, we prepare some notation about projective spaces. Let C or H be the field of the complex or quaternionic numbers, and put $(F,d) = (C,2)$ or $(H,4)$ respectively. By the definition of the F-projective space FP^m, we have a principal S^{d-1} bundle $S^{d(m+1)-1} \to FP^m$ and denote it by $\tilde{\xi}$. Associated to $\tilde{\xi}$, we have a canonical F-line bundle ξ, and a $(d-1)$ dimensional real vector bundle ζ defined by the adjoint representation of S^{d-1}, which satisfies $\xi \otimes_F \xi^* \cong \zeta \oplus 1$. For the Thom space $(FP^{n-1})^\alpha$ of $\alpha = 1_{d-1}$ or ζ, we put $QP^n = (FP^{n-1})^\alpha$ for $n \geq 1$. Thus, by QP^n we mean $\Sigma^{d-1}FP^{n-1}_+$ or the quasi-projective space Q_n over F respectively. In both cases, $\dim QP^n = dn - 1$. Let $QP^n_1 = QP^n$ and $QP^n_m = QP^n/QP^{m-1}$ for $n \geq m \geq 2$. Then, by [1], we have a homeomorphism

$$QP^n_m \approx (FP^{n-m})^{\alpha \oplus (m-1)\xi}. \tag{2.1}$$

Let $M_n(F)$ be the order of $J(\xi)$ in the J-group $\tilde{J}(FP^{n-1})$. Then, by (1.1) we have a stable homotopy equivalence $QP^{n+M}_{i+M} \simeq_s \Sigma^{dM}(FP^{n-i})^{\alpha\oplus(i-1)\xi}$ for any positive integer M satisfying $M \equiv 0 \bmod M_{n-i+1}(F)$. Thus, the attaching map ∂ in the cofiber sequence $QP^{n+M}_M \to QP^{n+M}_{1+M} \xrightarrow{\partial} S^{dM}$ is considered as a stable map $\tau_n : QP^n \to S^0$, and the cofiber $C(\tau_n)$ of τ_n is identified as $C(\tau_n) \simeq_s \Sigma(FP^n)^{\alpha-\xi}$. τ_n is natural for n, that is, for the inclusion $i : QP^{n-1} \to QP^n$, we have $\tau_n \circ i \simeq_s \tau_{n-1}$. By Knapp [6], it is shown that τ_n is stably homotopic to the transfer map defined on the principal bundle $\tilde{\xi}$. Thus we call τ_n the transfer map.

By the suspension theorem, there is a desuspension

$$\tilde{\tau}_n : QP^n \to \Omega^{dn}S^{dn}$$

of τ_n. We also denote by $\tilde{\tau}_n$ its adjoint map $\Sigma^i QP^n \to \Omega^{dn-i}S^{dn}$ for $0 \leq i \leq dn$. Then the homotopy class of $\Sigma\tilde{\tau}_n : \Sigma^{dn+1}QP^n \to S^{dn+1}$ is uniquely determined by the stable homotopy class of τ_n.

Let $E^k : \Omega^i S^j \to \Omega^{i+k}S^{j+k}$ be the canonical inclusion, and

$$\Omega^{i+2}S^{4j+3} \xrightarrow{P} \Omega^i S^{2j+1} \xrightarrow{E} \Omega^{i+1}S^{2j+2} \xrightarrow{H} \Omega^{i+1}S^{4j+3} \quad \text{for } i,j \geq 0 \tag{2.2}$$

be the fiber sequence called the *EHP*-sequence, which we will comment on later. Then the purpose of this section is to show the following:

THEOREM (1)

(1) *For the transfer map* τ_n, *there is a desuspension* $\tilde{\tau}_n : QP^n \to \Omega^{dn}S^{dn}$ *which satisfies* $H \circ \tilde{\tau}_n \simeq E^{dn} \circ p : QP^n \to \Omega^{dn}S^{2dn-1}$, *where* $p : QP^n \to S^{dn-1}$ *is the collapsing map to the top cell.*

(2) *For any desuspension* $\tilde{\tau}_n$ *of* τ_n, *we have* $P \circ E^{d(n+1)+1} \simeq E^{d-1} \circ \tilde{\tau}_n \circ \phi_n : S^{d(n+1)-2} \to \Omega^{d(n+1)-1}S^{d(n+1)-1}$, *where* $\phi_n : S^{d(n+1)-2} \to QP^n$ *is the attaching map of the top cell of* QP^{n+1}.

We recall the definition of the *EHP*–sequence. Let $(S^{2j+1})_\infty$ be the reduced product complex of S^{2j+1}. Then $(S^{2j+1})_\infty$ is homotopy equivalent to ΩS^{2j+2}, and there is a map $H : ((S^{2j+1})_\infty, S^{2j+1}) \to (S^{4j+2})_\infty$ called the James-Hopf invariant, which gives an isomorphism $H_* : \pi_i(\Omega S^{2j+2}, S^{2j+1}) \to \pi_{i+1}(S^{4j+3})$ (cf.[14]). Converting H into a fiber map, we get the *EHP*–sequence as in (1.2). $E^k : \Omega^i S^j \to \Omega^{i+k}S^{j+k}$ corresponds to a suspension, and $P : \Omega^{i+2}S^{4j+3} \to \Omega^i S^{2j+1}$ is related to the Whitehead product under some conditions.

Theorem (1) is a special case of the next situation. Let K be a $(2n-1)$–dimensional *CW*-complex with one top cell, $p : K \to S^{2n-1}$ the collapsing map to the top cell, and $f : K \to S^0$ a stable map. We further assume that

(2.3) $\quad H_{2n-1}(K;Z) \cong Z \quad$ and $\quad f_* = 0 : \tilde{H}_0(K;Z/2) \to \tilde{H}_0(S^0;Z/2).$

In Theorem (1), K means QP^n and f means τ_n. By the suspension theorem, we have a desuspension $\tilde{f} : K \to \Omega^{2n}S^{2n}$ of f, which corresponds to $\tilde{\tau}_n$ in Theorem (1). We denote the homotopy class of a map g by $[g]$ and the set of homotopy classes of maps from X to Y by $[X, Y]$.

Let $\tilde{f}_0 : K \to \Omega^{2n}S^{2n}$ be one of the desuspensions of f. Then, concerning the homomorphism $H_* : [K, \Omega^{2n}S^{2n}] \to [K, \Omega^{2n}S^{4n-1}]$, the following lemma is easily shown by the fact that $H_* \circ P_*(\iota_{4n+1}) \simeq \pm 2\iota_{4n-1}$ for the generator $\iota_i \in \pi_i(S^i)$ (see [14;Prop.2.7]).

LEMMA (2.4) *The set of* $H_*([\tilde{f}])$ *for all desuspensions* $\tilde{f} : K \to \Omega^{2n}S^{2n}$ *of* f *is equal to* $H_*([\tilde{f}_0]) + 2[K, \Omega^{2n}S^{4n-1}]$, *where the group structure of* $[K, \Omega^{2n}S^{4n-1}]$ *is the usual one.*

Since $p^* : [S^{2n-1}, \Omega^{2n}S^{4n-1}] \to [K, \Omega^{2n}S^{4n-1}]$ is an isomorphism by the assumption (2.3), there is a unique integer $d(\tilde{f})$, for each desuspension $\tilde{f}$, satisfying

(2.5) $$p^*(d(\tilde{f})E^{2n}) = H_*([\tilde{f}]).$$

Let $H_2[f] \in Z/2$ be the mod 2 integer of $d(\tilde{f})$. Then, by Lemma (2.4), $H_2[f]$ depends only on the stable homotopy class of f, and the following is clear:

LEMMA (2.6) *If* $H_2[f] = 1$, *then there is a desuspension* $\tilde{f} : K \to \Omega^{2n}S^{2n}$ *which satisfies* $H \circ \tilde{f} \simeq E^{2n} \circ p$.

Let $C(f)$ be the cofiber of f, and $\iota \in \tilde{H}^0(C(f);Z/2) \cong Z/2$ the generator. Then we have the following:

PROPOSITION (2.7) *If* $\mathrm{Sq}^{2n}\iota \neq 0$ *in* $H^{2n}(C(f);Z/2)$, *then* $H_2[f] = 1$.

Proof. Suppose that $H_2[f] = 0$. Then there is a desuspension $\tilde{f}$ with $H_*[\tilde{f}] = 0$ by Lemma (2.4). Then, by the *EHP*–sequence we have a desuspension $g : \Sigma^{2n-1}K \to S^{2n-1}$ of f, and a canonical isomorphism $\tilde{H}^i(C(f);Z/2) \cong \tilde{H}^{i+2n-1}(C(g);Z/2)$, where $C(g)$ denotes the cofiber of g. To the generator $\iota_g \in \tilde{H}^{2n-1}(C(g);Z/2) \cong Z/2$, we have $\mathrm{Sq}^{2n}\iota_g = 0$, and it implies $\mathrm{Sq}^{2n}\iota = 0$ via the above isomorphism. This contradicts the assumption, and we have the desired result.

Proof of Theorem (1) Let $\alpha = 1_{d-1}$ or ζ over FP^n as before. To prove (1), it is sufficient to show that $\mathrm{Sq}^{dn}U \neq 0$ for the Thom class $U \in \tilde{H}^{-1}((FP^n)^{\alpha-\xi}; Z/2)$ of $\alpha-\xi$, by Lemma 1.6 and Proposition 1.7, because $C(\tau_n) \simeq_s \Sigma(FP^n)^{\alpha-\xi}$ as mentioned above. But $\mathrm{Sq}^{dn}U = Uw_{dn}(\alpha-\xi) = Uw_{dn}(-\xi) = Ux^n \neq 0$, where w_{dn} and x are the dn–th Stiefel–Whitney class and the mod 2 Euler class of ξ respectively. Thus we have (1), and the integer $d(\tilde{\tau}_k)$ as in (2.5) is always odd for any desuspension $\tilde{\tau}_k : QP^k \to \Omega^{dk}S^{dk}$ of τ_k. Then we have the following diagram which is homotopy commutative up to sign:

$$\begin{array}{ccccccc}
QP^n & \xrightarrow{\ i\ } & QP^{n+1} & \xrightarrow{\ p\ } & S^{m-1} & \xrightarrow{\ \phi_n\ } & \Sigma QP^n \\
\Big\downarrow g & & \Big\downarrow \tilde{\tau}_{n+1} & & \Big\downarrow d(\tilde{\tau}_{n+1})E^m & & \Big\downarrow g \\
\Omega^{m-1}S^{m-1} & \xrightarrow{\ E\ } & \Omega^m S^m & \xrightarrow{\ H\ } & \Omega^m S^{2m-1} & \xrightarrow{\ P\ } & \Omega^{m-2}S^{m-1},
\end{array}$$

where $g = E^{d-1} \circ \tilde{\tau}_n$ and $m = d(n+1)$. Thus, from the right square in the diagram, we have $\pm d(\tilde{\tau}_{n+1})P \circ E^{m+1} \simeq E^{d-1} \circ \tilde{\tau}_n \circ \phi_n : S^{m-2} \to \Omega^{m-1}S^{m-1}$. But $[P \circ E^{m+1}] = [\iota_{m-1}, \iota_{m-1}]$ is of order 2 and $d(\tilde{\tau}_{n+1})$ is odd. Hence we have (2).

3. Main theorem

Let QP^n_m be the stunted projective space as in (1.1), and $\partial_s : QP^t_{s+1} \to \Sigma QP^s$ the attaching map. For the collapsing map $p : QP^s \to S^{ds-1}$ to the top cell, we put $\partial = \Sigma p \circ \partial_s : QP^t_{s+1} \to S^{ds}$.

Now we consider the composition $t_n = \tilde{\tau}_n \circ \partial_n : \Sigma^{dn-2}QP^l_{n+1} \to \Omega S^{dn}$ and its induced homomorphism $\Gamma = (t_n)_* : \pi_k(\Sigma^{dn-2}QP^l_{n+1}) \to \pi_{k+1}(S^{dn})$, where $\tilde{\tau}_n : \Sigma^{dn-1}QP^n \to \Omega S^{dn}$ is the desuspension of τ_n given in Theorem 1 (1). For a class $\tilde{\omega} \in \pi_k(\Sigma^{dn-2}QP^l_{n+1})$, we put $\tilde{\alpha} = p_*(\tilde{\omega}) \in \pi_k(S^{d(n+l)-3})$ and $\tilde{\beta} = \partial_*(\tilde{\omega}) \in \pi_k(S^{2dn-2})$. Let H and P also denote the homomorphisms, between unstable homotopy groups, induced from the maps H and P in (2.2) respectively. Then, by Theorem (1), we have the following:

PROPOSITION (3.1) *For any* $\tilde{\omega} \in \pi_k(\Sigma^{dn-2}QP^l_{n+1})$, *we have* $H(\Gamma(\tilde{\omega})) = \Sigma\tilde{\beta}$ *and* $\Sigma^{m-3}\Gamma(\tilde{\omega}) = P(\Sigma^m\tilde{\alpha})$, *where* $m = d(l-n)+2$.

Proof. Since $(H \circ \tilde{\tau}_n)_* = (E \circ p)_*$ by Theorem 1 (1), we have $H(\Gamma(\tilde{\omega})) = (E \circ \partial)_*(\tilde{\omega}) = \Sigma\tilde{\beta}$, which gives the first equality. Next, consider the following diagram:

$$\begin{array}{ccccccc}
S^k & \xrightarrow{\tilde{\omega}} & \Sigma^{dn-2}QP^l_{n+1} & \xrightarrow{\partial_n} & \Sigma^{dn-1}QP^n & \xrightarrow{\tilde{\tau}_n} & \Omega S^{dn} \\
\Big\downarrow = & & \Big\downarrow p & & \Big\downarrow i & & \Big\downarrow E^{d(l-n-1)} \\
S^k & \xrightarrow{\tilde{\alpha}} & S^{d(n+l)-3} & \xrightarrow{\phi_{l-1}} & \Sigma^{dn-1}QP^{l-1} & \xrightarrow{\tilde{\tau}_{l-1}} & \Omega^{m-d-1}S^{d(l-1)}.
\end{array}$$

The right square is homotopy commutative, since we have a stable equivalence $\tau_{l-1} \circ i \simeq_s \tau_n$ as remarked in §2. The middle square is homotopy commutative by the naturality of the cofiberings, and so is the left square by the definition of $\tilde{\alpha}$. By Theorem 1 (2), we have $E^{d-1} \circ \tilde{\tau}_{l-1} \circ \phi_{l-1} \simeq P \circ E^m$: $S^{d(n+l)-3} \to \Omega^{m-2}S^{dl-1}$. Hence we have $\Sigma^{m-3}\Gamma(\tilde{\omega}) = P(\Sigma^m\tilde{\alpha})$, which is the second required equality.

Now we consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{i,j} = \tilde{H}_i(QP^\infty; R) \otimes \pi^s_j(S^0) \Longrightarrow \pi^s_{i+j}(QP^\infty) \otimes R, \tag{3.2}$$

where R is Z or $Z_{(p)}$ for a prime p. We denote the differential in dr–terms by

$$d^r : E^{dr}_{i,j} \to E^{dr}_{i-dr,j+dr-1}.$$

The homology group $\tilde{H}_*(QP^\infty; R)$ is a free abelian group with basis $\{b_i \in \tilde{H}_{d(i+1)-1}(QP^\infty; R) \mid i \geq 0\}$. We denote by $[b_i \otimes \alpha]$ an element of $E^{dr}_{d(i+1)-1,j}$ given by $b_i \otimes \alpha \in E^2_{d(i+1)-1,j}$ which persists to the dr–term.

As in §2, $M_r(F)$ denotes the order of $J(\xi)$ in the J–group $\tilde{J}(FP^{r-1})$, where ξ is the canonical F–line bundle over FP^{r-1}. Thus we have a stable homotopy equivalence $QP^{k+n}_{k+m} \simeq_s \Sigma^{dk}QP^n_m$ if $n - m \leq r-1$ and $k \equiv 0 \bmod M_r(F)$, by (2.1).

Now we consider a relation

$$d^r[b_n \otimes \alpha] = [b_{n-r} \otimes \beta] \quad \text{for} \quad \alpha \in \pi^s_j(S^0) \quad \textit{and} \quad \beta \in \pi^s_{j+dr-1}(S^0) \tag{3.3}$$

in the spectral sequence (3.2). We remark that, by the suspension theorem, α (resp. β) has a unique desuspension $\alpha_i \in \pi_{i+j}(S^i)$ (resp. $\beta_i \in \pi_{i+j+dr-1}(S^i)$)

if $j \leq i-2$ (resp. $j \leq i-dr-1$). Then by Proposition (3.1) we have the following main theorem.

THEOREM (2) *Whenever a relation* (3.3) *occurs, there exists an element* $\gamma \in \pi_{j+d(2t-r)-2}(S^{d(t-r)})$ *which satisfies*

$$H(\gamma)=\beta \quad \text{and} \quad \Sigma^{dr-1}\gamma = P(\alpha_{2dt-1}).$$

Here, $t=k+n+1$ *for any non negative integer* k *with* $k \equiv 0 \bmod M_r(F)$, $0 \leq j \leq d(2t-3r)-3$, *and* $H(\gamma)$ *is regarded as an element of* $\pi^s_{j+dr-1}(S^0)$.

Proof. By the definition of the differential d^r, the relation (3.3) implies that there is an element $\omega \in \pi^s_{j+dt-1}(QP^t_{t-r+1})$ satisfying $p_*(\omega)=\alpha$ and $\partial_*(\omega)=\beta$. By the suspension theorem, if $j \leq d(2t-3r)-3$, then each of the elements ω, α and β has a unique desuspension $\tilde{\omega} \in \pi_{j+d(2t-r)-3}(\Sigma^{d(t-r)-2}QP^t_{t-r+1})$, $\alpha_{d(2t-r)-3}$ and $\beta_{2d(t-r)-2}$ respectively. Thus, $\gamma=\Gamma(\tilde{\omega})$ satisfies the required properties by Proposition (3.1), and we have the desired result.

COROLLARY (3) *Under the same assumptions and notations as in Theorem* (2),

(1) *if* $j \leq dt-3$ *moreover, then* $\Sigma^{dr-1}\gamma=[\iota_{dt-1},\alpha_{dt-1}]$;

(2) *if* $j \leq d(t-2r)-2$ *moreover, then* $[\iota_{d(t-r)-1},\beta_{d(t-r)-1}]=0$,

where [,] *denotes the Whitehead product.*

Proof. It is known that the formula $P(\Sigma^{i+2}\theta)=[\iota_i,\Sigma\theta]$ holds for $\theta \in \pi_m(S^{i-1})$ if $m \leq 2i-3$ (cf.[15;XII Cor 2.5]). Thus (1) follows from the second equality in Theorem (2) under the additional condition. By the first equality in Theorem (2) and the *EHP*–sequence, we have $P(\beta_{2d(t-r)-1})=0$. Then, applying the above formula, we also have (2).

Theorem (I) in the introduction follows from Theorem (2) and Corollary 3 (2) by taking the case of $(F,d)=(C,2)$ and $k=0$.

4. Some examples

We will illustrate some concrete cases derived from known differentials in the Atiyah-Hirzebruch spectral sequences.

Let $\nu \in \pi^s_3(S^0) \cong Z/24$ and $\sigma \in \pi^s_7(S^0) \cong Z/240$ be the respective generators. Then the cell structure of the quaternionic projective space HP^n is represented as $S^4 \bigcup_\nu e^8 \bigcup_{2\nu} \cdots \bigcup_{(n-1)\nu} e^{4n}$. Thus, in the Atiyah-Hirzebruch spectral sequence $E^2_{r,s}=H_r(HP^\infty;Z)\otimes\pi^s_s(S^0) \Rightarrow \pi^s_*(HP^\infty)$, we have the following, just by the definition of the Toda bracket $<\ ,\ ,\ >$ [14].

LEMMA (4.1) *Let* $\alpha \in \pi^s_j(S^0)$.

(1) $[b_n \otimes \alpha] \in E^8_{4n,j}$ *if and only if* $(n-1)\nu\alpha=0$.

(2) *If an equality* $d^2[b_n \otimes \alpha] = [b_{n-2} \otimes \beta]$ *holds, then* $\beta \in < (n-2)\nu, (n-1)\nu, \alpha >$.

(3) *If* $(n-1)\nu\alpha = 0$ *and* $\sigma\alpha = 0$, *then* $d^2[b_n \otimes \alpha] = [b_{n-2} \otimes \beta]$ *for any* $\beta \in < (n-2)\nu, (n-1)\nu, \alpha >$.

Thus we apply Theorem (2) to these differentials, and have the following.

COROLLARY (4.2) *Let* $0 \leq j \leq 8n - 19$ *and* $n \geq 3$. *Then, for any* $\alpha \in \pi_j^s(S^0)$ *with* $(n-1)\nu\alpha = 0$, *there is an element* $\gamma \in \pi_{j+8n-2}^s(S^{4(n-1)})$ *which satisfies*

$$H(\gamma) \in < (n-2)\nu, (n-1)\nu, \alpha > \subset \pi_{j+7}^s(S^0) \quad and \quad \Sigma^7\gamma = P(\alpha_{8n+7}).$$

Of course, analogous results hold with respect to the quaternionic quasi-projective space Q_∞, in which case n must be replaced by $n+1$ in the above result. As for the complex projective space CP^∞, the calculations by Mosher ([12] etc.) give many corresponding results.

Morisugi [11] proved that Mahowald's elements [9] $\eta_j \in \pi_{2^j}^s(S^0)$ for $j \geq 3$ are all in the image of the S^3–transfer map $\tau_n : Q_n \to S^0$. His result means that, in the Atiyah-Hirzebruch spectral sequence $E_{r,s}^2 = H_r(Q_\infty; Z) \otimes \pi_s^s(S^0) \Rightarrow \pi_*^s(Q_\infty)$, a relation $d^r[b_{k+r-1} \otimes \eta] = [b_{k-1} \otimes \eta_j]$ holds for $r = 2^{j-2}$ and $k \equiv 0 \bmod M_r(H)$, where $\eta \in \pi_1^s(S^0)$ is the generator. Thus applying Theorem (2) to this, we have the following.

COROLLARY (4.3) *Let* $j \geq 3$ *and put* $r = 2^{j-2}$. *Then, for any positive integer k with* $k \equiv 0$ *mod* $M_r(H)$, *we have an element* $\gamma \in \pi_{8k+4r-1}(S^{4k})$ *which satisfies*

$$H(\gamma) = \eta_j \quad and \quad \Sigma^{4r-1}\gamma = P(\eta)$$

where $\eta \in \pi_{8(k+r)}(S^{8(k+r)-1})$ *is the generator.*

In [4;Theorem 5], we defined an integer t_n and an element $\alpha(n) \in \pi_{4n-5}^s(S^0)$ for $n \geq 2$, and proved that $d^{n-1}[b_n \otimes t_n] = [b_1 \otimes \alpha(n)]$ in the spectral sequence for $\pi_*^s(HP^\infty)$ as above. Thus, applying Theorem (2) and Corollary (3) to this relation, we have the following.

COROLLARY (4.4) *Assume that* $2k \geq n - 4 \geq 0$ *and* $k \equiv 0$ *mod* $M_{n-1}(H)$. *Then we have an element* $\gamma_k \in \pi_{8k+4n+10}(S^{4k+8})$ *such that* $H(\gamma_k) = \alpha(n)$, $\Sigma^{4n-5}\gamma_k = 0$, *and thus* $[\iota_{4k+7}, \alpha(n)_{4k+7}] = 0$.

There is an analogous result from the quaternionic quasi-projective space by [4;Theorem I]. We can also get some results of this sort from Crabb-Knapp's work ([2] etc.) about the stable homotopy of the complex projective spaces.

Lastly, we remark that Theorem (1) (2) is a generalization of the well known formula of James–Whitehead [5]. Let $J : G(n) \to \Omega^{dn}S^{dn}$ be the

J–map, and $r : Q_n \to G(n)$ the reflection map for $(G(n), d) = (U(n), 2)$ or $(Sp(n), 4)$. By Stolz [13], it is shown that $\tau = J \circ r : Q_n \to \Omega^{dn} S^{dn}$ is stably homotopic to the S^{d-1}–transfer map $\tau_n : Q_n \to S^0$. Thus, by Theorem (1), (2), we have the formula of James–Whitehead $\Sigma^{d-1}[\tau \circ \phi_n] = [\iota_{d(n+1)-1}, \iota_{d(n+1)-1}]$.

WAKAYAMA UNIVERSITY
930 SAKAEDANI,
WAKAYAMA 640, JAPAN.

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