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REAL ANALYTIC Γ-STRUCTURES WITH ONE LEAF ON THE TORUS

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Let $B\Gamma_1^\omega$ be Haefliger's classifying space for codimension one real analytic foliations, [2]. $B\Gamma_1^\omega$ is equivalently the classifying space of the groupoid, with the sheaf topology, of germs of local, orientation preserving, real analytic diffeomorphisms of $\mathbf{R}$. If we give the topological groupoid $\Gamma = \Gamma_1^\omega$ the discrete topology we obtain a space $B\Gamma_\delta$ which classifies codimension one Γ-structures with one leaf, [8].

We consider the homomorphism $H_2(B\Gamma_\delta) \to H_2(B\Gamma)$ induced by the identity $\Gamma_\delta \to \Gamma$.

THEOREM. *Any map from the torus T to $B\Gamma_\delta$ represents a homology class in $H_2(B\Gamma_\delta)$ whose image in $H_2(B\Gamma)$ is zero.*

Remarks. (1): The theorem says that any two Γ_δ-structures on the torus are cobordant as Γ-structures.
(2):The space $B\Gamma_\delta$ is homotopy equivalent to $BG_0 = K(G_0, 1)$ where G_0 is the group of orientation preserving real analytic germs at the origin of $\mathbf{R}$. This is an easy fact.
(3): The space $B\Gamma$ is a $K(G, 1)$, [2], and G_0 is a subgroup of G, [3]. These are deep facts.
(4): By means of the homotopy equivalences in (2) and (3) the homomorphism $H_2(B\Gamma_\delta) \to H_2(B\Gamma)$ can be viewed as the homomorphism $H_2(G_0) \to H_2(G)$ induced by the inclusion $G_0 \to G$. When $\mathcal{G}$ is a discrete group we use $H_*(\mathcal{G})$ to denote Eilenberg-MacLane homology. This is the same thing as $H_*(B\mathcal{G})$.
(5): It is not difficult to see, [5], that the group G_0 splits off its subgroup of linear germs $\mathbf{R}^+$ so that $H_2(R^+)$ is a direct summand of $H_2(G_0)$. As an abelian group $H_2(\mathbf{R}^+)$ is isomorphic to $\mathbf{R}$ so that there are uncountably many homology classes represented by tori in G_0.
(6): As we remarked above each non-trivial element of $\pi_1(B\Gamma_\delta) = G_0$ is non-trivial in $\pi_1(B\Gamma)$ = G. Hence, a map $T \to B\Gamma_\delta$ which represents a non-zero homology class in G_0 *is not homotopic to a constant* as a map into $B\Gamma$.
(7): By way of contrast, in the C^∞ case, the natural map $BG_0 \to B\Gamma$ is in fact homotopic to a constant,[8].
(8): Recent work on Γ-structures, [4], and on the homology of groups of jets and germs, [5], has led us to believe that, in the real analytic case, $H_2(G_0)$ has an uncountable summand which does not map to zero in the homology of $B\Gamma$. Such classes would necessarily be represented by structures on surfaces of genus > 1. Indeed it can be shown that any homology class can be represented by a structure on a disjoint union of two-holed tori.

The Group $G = \pi_1(B\Gamma)$. We recall now the description of G given in [3], for it will be used in the proof of the theorem. Let $S = \pi_0(\Gamma)$ be the set of connected components of the topological space Γ. The elements of S may be viewed as a maximally extended, orientation preserving, real analytic diffeomorphisms between open intervals of $\mathbf{R}$. There is a partially defined multiplication among elements of S given by the composite fg whenever the range of g intersects the domain of f.

It is proved in [3] that $G = \pi_1(B\Gamma)$ is the free group on the elements of S modulo the relations $f \cdot g = fg$ whenever the right hand side is defined. Furthermore it is shown that that no element of S, other than the identity diffeomorphism, is trivial in the group G.

Proof of theorem. Recall if $\mathcal{G}$ is a discrete group and α is a $\mathcal{G}$ structure on a space X then the homotopy class of $\alpha, [\alpha]$, can be considered, via the usual equivalences, to be an element of $Hom(\pi_1(X), \mathcal{G})_{\mathcal{G}}$ where the lower $\mathcal{G}$ denotes conjugacy class. Let α be a G_0-structure on T. Since $\pi_1(T) = Z \times Z$, $[\alpha]$ identifies with an equivalence class of pairs of commuting elements: $[\alpha] = (A, B)_{G_0}$ where A and B commute in G_0. The following lemma shows that we can simultaneously linearize a and b.

LEMMA. *Let A and B be commuting elements of G_0. Let $f \in G$ be a local C^0 homeomorphism at $0 \in \mathbf{R}$ which is C^ω to the right of the origen. Suppose f conjugates A to a linear germ at the origen $L_r : x \to rx, r > 0$. Then f also conjugates B to a linear germ.*

Proof of the Lemma. We are assuming $fAf^{-1} = L_r$ and let us suppose r > 1. Let $g \in$ G be the element corresponding to $x \to \frac{\ln x}{\ln r}$. Then

$$gfAf^{-1}g^{-1} = T_1 = unit\ \ translation$$

Furthermore, if we set $A^q = f^{-1}L_{r^q}f \in G_0$ then

$$gfA^q f^{-1}g^{-1} = T_q = translation\ \ by\ \ q$$

Now consider $gfBf^{-1}g^{-1}$. Since B commutes with A this must be a periodic real analytic diffeomorphism of $\mathbf{R}$, call it P. Here periodic means $P(x + 1) = P(x) + 1$. There is a T_s so that T_sP is a periodic homeomorphism whose graph intersects that of T_1 an infinite number of times. We have $T_s = gfA^sBf^{-1}g^{-1}$, so that the graph of a^sB intersects that of A arbitrarily close to the origin on the right hand side. But A^sB and A are real analytic *at the origin*. Therefore $A^sB = A$, and B is conjugate to a linear germ, as claimed.

Let us state the result of the lemma somewhat more formally. Consider the sequence of maps

$$B\mathbf{R}^+ \to BG_0 \to BG$$

induced by inclusions of groups. The lemma says that $[\alpha] \in [T, BG_0]$ maps to an element $[\beta] \in [T, BG]$ which is in the image of an element $[\gamma] \in [T, B\mathbf{R}^+]$

We will observe below that an f satisfying the conditions of the lemma exists. Meanwhile, we complete the proof of the theorem.

Proof of Theorem continued. We see from the lemma that α is homotopic *in*BG to a structure represented by a pair of positive multiplicative reals

(r_1, r_2), that is α is homotopic to a $\mathbf{R}^+$ structure on T. Now homology classes of A-structures on T are known when A is abelian, [7]. They correspond to elements of the group $A \otimes_Z A$. Let us conjugate $r_1 \otimes r_2$ by the element of G coming from the map $g : x \to x^2$. Then up to homology in BG we obtain

$$[\alpha] = r_1 \otimes r_2 = g r_1 g^{-1} \otimes g r_2 g^{-1} = r_1^2 \otimes r_2^2 = 4(r_1 \otimes r_2) = 4[\alpha]$$

So $3[\alpha] = 0$, and the same argument using x^3 gives $8[\alpha] = 0$. It follows that $[\alpha] = 0$ as claimed.

Existence of f. The following is proved in [6]: Let F_p and G_q be germs of real analytic diffeomorphisms at p and q with $F(p) > p$ and $G(q) > q$. Then there exists a real analytic diffeomorphism H so that $H^{-1} F_p H = G_q$, the equation holding for the appropriate germs of H.

This result follows from the uniqueness of real analytic manifold structures on the circle, S^1,[1]. To summarize the argument in [6]: F_p, and G_q each can be used to determine C^ω structures on S^1 in such a way that a C^ω equivalence between the two manifolds gives rise to the conjugating diffeomorphism H.

The existence of the f required in the lemma is an immediate consequence of this fact.

This completes the proof of the theorem.

As a final comment, it seems somewhat unsatisfactory to fall back upon the deep result of [1] in order to come up with the conjugating diffeomorphism H. We discuss some alternatives in [6], for example the use of the uniformization theorem to induce a real analytic metric on S^1. However, we know of no significantly easier or more direct way to redo the proof.

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