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TWISTED TORI AND THE CHARACTERISTIC CLASSES OF A SELF-EQUIVALENCE

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In memory of José Adem.

The group of homotopy classes of self-equivalences from a space to itself has been studied for many years (see bibliography in [7]). It is a non-abelian invariant of spaces, which has proved to be very difficult to calculate. For simply-connected (or nilpotent) finite complexes it is known [9] to be finitely presented, although rarely finite. If the space is not simply- connected, it may be infinitely generated (see [2] or [5]).

The focus of this survey/research paper is somewhat different from earlier work. We wish to study a single homotopy equivalence from a space to itself, and to develop cohomology invariants which (it is hoped) might offer practical solutions to some key questions such as:

(a) Given a self-equivalence f representing a class of finite order in $\mathcal{E}(X)$, the group of homotopy classes of self-equivalences of X, can we find a homotopy equivalent space Y and an equivalent (conjugate) homeomorphism g on Y which has the same order as f. If we impose no conditions on Y, the late G. Cooke [1] has given a theoretical solution to this problem.

(b) What are the relations between the different subgroups of $\mathcal{E}(X)$ which are fixed by the usual functors? A. Legrand has called attention to the question of the relationship between these self-equivalences fixed by homology, and these fixed by homotopy groups (see problems in [7]).

The key space for the study of these characteristic classes is the twisted-torus of the self-equivalence. This is $T_f = X \times I/\sim$, where $(x, 1) \sim (f(x), 0)$. If f is the identity this is just $X \times S^1$. Our first section is devoted to T_f. We give various conditions for this to be a fibration over S^1, and calculate its homology in the basic cases. This space clearly deserves more study in its own right.

The second section studies characteristic classes which come from S^1 actions on T_f. These work best in the case where the self-equivalence is a homeomorphism of finite order. Here we have classes

$$w_i(f) \in \mathrm{Hom}(H_{i-1}(T_f), H_i(T_f))$$

This latter group is a subgroup of $H^{i-1}(T_f; H_i(T_f))$ (not naturally). For a general self-equivalence, we construct

$$\chi(f) \subseteq H^2(T_f)$$

We develop the basic properties of these classes and give some examples.

The third section looks at open questions and connections with other areas, such as cyclic cohomology.

We assume that all spaces are path connected and all maps and homotopies are continuous. In those cases, where we work in the pointed category, we shall mention it explicitly.

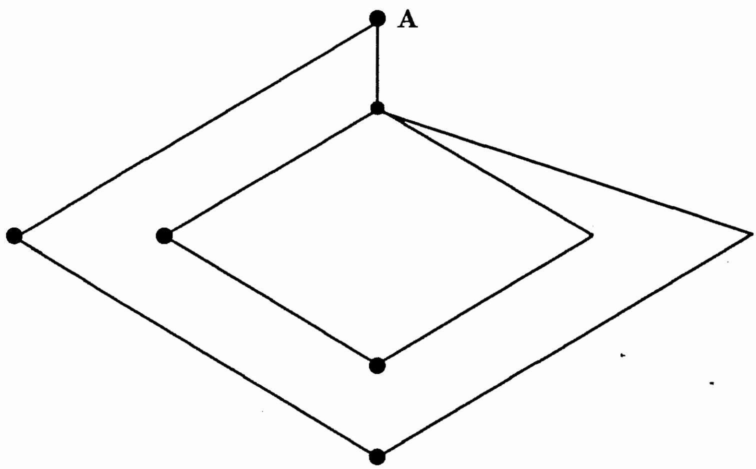
§1

Given a space X and a homotopy self-equivalence f, we write

$$T_f = X \times I/(x,1) \sim (f(x),0)$$

as the *twisted torus* of f. There are many easy examples.

In geneal, T_f is endowed with a projection $\pi : T_f \to S^1$ given by $\pi(x,t) = \{t\}$, where $\{t\}$ is the element in S^1, viewed as the quotient of $I = [0,1]$ where $0 \sim 1$. In general, this can be quite complicated. For example, π will usually not be a fibration, even in the sense of Serre. For example, let $f : I \to I$ be given by $f(t) = 0$. Then the twisted torus T_f looks like



Given a map $\alpha : pt \to T_f$ by $\alpha(pt) = A$, and a homotopy of $\pi \cdot \alpha$ which moves linearly away from $\pi(A)$ (in the negative direction), we see at once that $\pi : T_f \to S^1$ cannot have the covering homotopy property.

Nevertheless, the twisted torus has many agreeable properties under suitable hypotheses.

PROPOSITION (1.1). *Let $f : X \to X$ and T_f be as above.*

1) If f is a homeomorphism, T_f is a locally trivial fibration, with fibre X, over S^1. If $G \subseteq Homeo(X)$ is a subgroup containing f then π is a fibre bundle with fibre X and group G.

2) If f is a homeomorphism, T_f is foliated with 1-dimensional leaves given by the action of R on T_f. (Here we assme X is a manifold.)

3) If f is a homeomorphism of finite order there is an S^1 action which is compatible via the map π, with the S^1 action on S^1 given by $(e^{i\theta}, e^{i\tau}) \mapsto (e^{i(\theta+n\tau)})$, where n is the order of f.

Proof. 1) Given a small open interval I about $0 \sim 1$, $\pi^{-1}(I) \equiv (-e, 0] \times X \cup [0, e) \times X$ where $0 \times X$ is identified with $0 \times X$ by $(0,x) \sim (0,f(x))$. This is clearly homeomorphic to $(-e, e) \times X$.

2) The action of R on T_f is as follows: take the standard action of R on S^1 by $(e^{i\theta}, t) \mapsto e^{i(\theta+t)})$. Lift this to T_f locally, since π is locally trivial with a single identification, via the homeomorphism f, as one crosses $0 \sim 1$.

3) If f has order n, it's easy to check that the action of n or any multiple, via 2) above, is the identity. Thus the action of 2) passes to quotient, giving an action of

$$\mathrm{R}/nZ \equiv S^1.$$

The relation between a homeomorphism and a homotopy equivalence has been examined by the late G. Cooke [7]. Specifically, Cooke asked if given a homotopy equivalence $f : X \to X$, if there is a space Y, and a homotopy equivalence $g : X \to Y$ so that there is a homeomorphism $\bar{f} : Y \to Y$ and a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ f\downarrow & & \downarrow \bar{f} \\ X & \xrightarrow{g} & Y \end{array}$$

In fact, Cooke addressed the more general question: given a group G and a homomorphism $\alpha : G \to \mathcal{E}(X)$, the group of homotopy classes of homotopy equivalences, can X be replaced by a homotopy equivalent Y so that the equivalent map $\tilde{\alpha} : G \to \mathcal{E}(Y)$ actually factors through a homomorphism of G to the group of homeomorphisms of Y. He expressed the solution in terms of the lifting problem

$$\begin{tikzcd} & B_{G(X)} \arrow[d] \\ K(G,1) \arrow[ur, dashed] \arrow[r] & K(\mathcal{E}(X),1) \end{tikzcd}$$

where $B_{G(X)}$ is the classifying space for the associative H- space of all self-homotopy equivalences. S. Smith [8] has now made considerable progress in analyzing the rational homotopy of $G(X)$, $B_{G(X)}$, etc.

Specializing to the case of a single homotopy equivalence f, which generates, of course, a cyclic subgroup of $\mathcal{E}(X)$ we can always find an equivalent Y on which an equivalent $\bar{f}$ acts as homeomorphism. This can be shown from Cooke's result, using the fact that $K(Z,1) = S^1$, or more directly by constructing an infinite telescope $L = \cup X \times I$, identifying end points by $(x,0) \sim (f(x),1)$, etc. The telescope has the homotopy-type of X and the map $\bar{f}$ is the shift clearly a homeomorphism of infinite order.

However, if one seeks an equivalent homeomorphism $\bar{f}$ which has precisely the same order as the class of f, $\{f\} \in \mathcal{E}(X)$, when this is in fact finite, then there are obstructions to finding such a homeomorphisms $\bar{f}$. In fact, one may find a counter-example in case $\{f\}$ has order 2, in Cooke's paper [1].

The infinite telescope or infinite mapping cylinder

$$L_f = \cup X \times I/\sim$$

where $(x,n) \sim (f(x), n+1)$, serves two purposes here. First, it is a space generally homotopically equivalent to X, on which a single homotopy equivalence may be realized via a shift map. (However, this homotopy equivalence will have infinite order under composition, while the class of the original equivalence could have finite order.) The shift map is clearly a homeomorphism. Secondly, the quotient of L_f by the shift map is T_f. This will permit a definition of our $\chi(f)$ in full generality.

Now, we wish to study the homology (and cohomology) of a twisted torus. This clearly depends on the effect of f on homology. As the examples of the torus and Klein bottle show, even a simple change of sign will give significant changes in results. We will work this out in the case where the coefficients are in a field using the Mayer-Vietoris sequence as the method.

PROPOSITION (1.2). *Let $f : X \to X$ be a homotopy equivalence of a path connected space, $H_i(X), i > 0$, the homology with coefficients in a field. Then*

$$H_i(T_f) = \left\{ \frac{H_i(X) \oplus H_i(X)}{Im \begin{pmatrix} 11 \\ 1f \end{pmatrix}_{i*}} \right\} \oplus \operatorname{Ker} \begin{pmatrix} 11 \\ 1f \end{pmatrix}_{(i-1)*}$$

where

$$\begin{pmatrix} 11 \\ 1f \end{pmatrix}_{i*} : H_i(X) \oplus H_i(X) \to H_i(X) \oplus H_i(X)$$

is the endomorphism given as the indicated matrix

$$\begin{pmatrix} 11 \\ 1f \end{pmatrix}_{i*} = \begin{pmatrix} 1 & 1 \\ 1 & f_{*i} \end{pmatrix},$$

with $f_{*i} : H_i(X) \to H_i(X)$ induced by f.

Proof. Write $T_f = A \cup B$ where $A = X \times I$, $B = X \times I$ and we identify A to B via $(x,0) \sim (x,0)$, $(x,1) \sim (f(x),1)$. Of course, A, B, and X all have the same homotopy-type.

It is easy to see that we may assume that $X \cup X = A \cap B$ (disjoint) and $T_f = A \cup B$. Consider the Mayer-Vietoris sequence

$$\ldots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(A \cup B) \to$$
$$\to H_{i-1}(A \cap B) \to H_{i-1}(A) \oplus H_{i-1}(B) \to \ldots$$

By our earlier remarks, we can rewrite this as

$$- \ldots \to H_i(X) \oplus H_i(X) \xrightarrow{\phi_{*i}} H_i(X) \oplus H_i(X) \to H_i(T_f) \longrightarrow$$

$$\longrightarrow H_{i-1}(X) \oplus H_{i-1}(X) \xrightarrow{\phi_{*i-1}} H_{i-1}(X) \oplus H_{i-1}(X)$$

By exactness (as our coefficients lie in a field),

$$H_i(T_f) \approx \frac{H_i(X) \oplus H_i(X)}{Im\phi_{*i}} \oplus \mathrm{Ker}\,\phi_{*(i-1)}$$

It remains to identify the image and kernel of ϕ_*. Because of the identifications of A and B, to make T_f, the inclusion of the first summand $H_i(X)$ in $H_i(A) = H_i(X)$ and $H_i(B) = H_i(X)$ is the identity. The second summand maps according to the identity to $H_i(A)$ and f_* to $H_i(B)$, as desired. The description of ϕ_* follows immediately.

Remark. It is easy to express this result in terms of dimension. We also note that, when the coefficients lie in a field, $H_*(T_f)$ depends entirely on $H_*(X)$ and f_* and not on any finer information about the self-equivalence. The determination of $\pi_n(T_f), n \geq 0$, is more complicated. However, we have a basic, partial result.

PROPOSITION (1.3). *Let X be path connected with base point x_o, $f : X \to X$ a self-equivalence, and T_f as above; $\bar{x}_o = (x_o, 0) \in T_f$ is chosen as the base point. Then there is a group G and a semi-direct product decomposition*

$$\pi_1(T_f, \bar{x}_o) \approx G \bowtie Z,$$

where Z is the integers.

Proof. There is the obvious quotient map, which we now write

$$\rho : T_f \to S^1$$

$\rho(\bar{x}_o) = y_o$. We simply need to show that

$$\rho_\# : \pi_1(T_f, \bar{x}_o) \to \pi_1(S^1, y_o) = Z$$

is a split surjection.

Choose a path $\alpha : I \to X \subseteq T_f$ thought of as $X \equiv \{(x, 0)\}$, from $f(x_o)$ to x_o. Let $\gamma : I \to T_f$ be the path $\gamma(t) = (x_o, t)$.

The composition path $\gamma * \alpha : I \to T_f$ begins and ends at x_o. Define a homomorphism

$$\sigma : \pi_1(S^1, y_o) \to \pi_1(T_f, \bar{x}_o) \text{ by } \sigma(1) = \{\gamma * \alpha\}.$$

We need to check that $\rho_\# \cdot \sigma = Id$. But $\rho_\# \cdot \sigma$ on the identity map is represented by

$$g(t) = \begin{cases} 2t, 0 \le t \le \frac{1}{2} \\ y_o, \frac{1}{2} \le t \le 1 \end{cases}$$

where S^1 is parameterized to have circumference 1. It is easy to verify that this is homotopic to the identity.

We need one more result for the definition of $\chi(f)$.

PROPOSITION (1.4). *If f is a properly discontinuous homeomorphism (see [5]), and X/f is the quotient by the action of f, then there is a fibration $S^1 \to \mathcal{E} \to X/f$, where $\mathcal{E}$ is the quotient of T_f by the map which identifies each segment $(x,t)(0 \le t \le 1)$ with $(f(x),t)$ $(0 \le t \le 1)$ and $(x,0)$ with $(f(x),0)$.*

Proof. Choose an open neighborhood of x, say U, with all the $f^n(U)$ disjoint. The (x,t), $x \in U$ and $0 \le t \le 1$, yields a trivial S^1 fibration over $\{U\} \subseteq X/f$.

§2

We now come to the definition and basic properties of the characteristic classes of a homotopy self-equivalence. We begin with the classes $w_i(f)$, which are defined when f is a homeomorphism $f : X \to X$ of finite order.

Definition (1). Let $f : X \to X$ be a homeomorphism of finite order, i.e. $f^n = id$ for some $n \ge 1$. Let T_f be the twisted torus and let

$$\mu : T_f \times S^1 \to T_f$$

be the action given in Proposition 1.1, part 3 (which covers the "standard" map of S^1 of the same degree as the order of f). Because the homology of S^1 is so simple. $H_{k-1}(T_f) \approx H_{k-1}(T_f) \otimes H_1(S^1)$ is a summand of $H_k(T_f \times S^1)$. The composition

$$H_{k-1}(T_f) \subseteq H_k(T_f \times S^1) \xrightarrow{\mu_*} H_k(T_f)$$

defines the class

$$\begin{aligned} w_k(f) &\in \operatorname{Hom}(H_{k-1}(T_f); H_k(T_f)) \\ &\subseteq H^{k-1}(T_f; H_k(T_f)) \end{aligned}$$

where the last inclusion is not natural, in general.

To define our second type of charateristic class, let $f : X \to X$ be a homotopy-equivalence. By earlier remarks, we may replace f by an equivalent shift homeomorphism $\bar{f}$ of the telescope L_f. $\bar{f}$ acts in a properly discontinuous fashion on L_f. Proposition 1.4 yields an S^1 fibration $S^1 \to \mathcal{E} \to L_f/\bar{f}$. It is clear that the quotient space $L_f/\bar{f}$ is homeomorphic to T_f.

Definition (2). The "characteristic class" $\chi(f)$ is defined as the image of a generator of $H^1(S^1)$ under the transgression τ, which is the correspondence

$$H^1(S^1) \xrightarrow{d^*} H^2(\mathcal{E}, S^1) \xleftarrow{p^*} H^2(T_f, *) \approx H^2(T_f).$$

(To make this precise and well-defined, one might take $Z/2$ coefficients so that $\pi_1(T_f)$ acts trivially on the cohomology of the fibre; or alternatively consider, instead of τ above, the operator $d_2^{0,1}$ in the cohomology Serre spectral sequence with local coefficients).

We now look into some properties of $w_i(f)$

PROPOSITION (2.1).

A. *Let $f_1, f_2 : X \to X$ be conjugate homeomorphism of finite order. That is to say, we have a homeomorphism g so that $f_2 = g^{-1} \cdot f_1 g$. Then $w_i(f_1)$ and $w_i(f_2)$ agree up to automorphism of the domain and range.*

B. *Let $f = Id : X \to X$. Let $x \in H_{i-1}(T_f), i > 0$, lie in the image of the map induced by the inclusion $X \subseteq T_f$ $(x \mapsto (x, 0))$. Here $T_f \equiv X \times S^1$. Then $w_i(f)(x) = x \otimes \iota$, where $\iota \in H_1(S^1)$ is a generator. Furthermore, $w_i(f)(y \otimes \iota) = 0$ where $y \in H_{i-2}(x)$.*

C. *Let f be a homeomorphism of finite order, $g = f^m$. Let $u \in H_{i-1}(X), i : X \subseteq T_f$. Then $\rho_* w_i(f)(i_*(x)) = m w_i(g)(i_*(x))$ where ρ is the natural map $T_f \to T_g$.*

Proof: A. Given conjugate homeomorphisms, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X \\ \downarrow h & & \downarrow h \\ X & \xrightarrow{f_2} & X \end{array}$$

This situation defines a map $\tilde{h} : T_{f_1} \longrightarrow T_{f_2}$ by $\tilde{h}(x, t) = (h(x), t), 0 \leq t < 1$ and $\tilde{h}(x, 1) = \tilde{h}(f_1(x), 0) = (hf_1(x), 0) = (f_2 h(x), 0) = (h(x), 1)$.

Clearly, $\tilde{h}$ is compatible with the S^1 action $\mu : T_{f_1} \times S^1 \to T_{f_1}$. The claim in part A follows immediately.

B. If $f = Id$, $T_f = X \times S^1$ and the map

$$\mu : T_f \times S^1 \to T_f$$

or

$$X \times S^1 \times S^1 \to X \times S^1$$

identifies with the map $1_x \times \bar{\mu}$, where $\bar{\mu}$ is the multiplication map in S^1.

The first claim is obvious. For the second, suppose $y \otimes \iota \in H_{k-1}(T_f) = H_{k-1}(X \times S^1)$. Then

$$\mu_*(y \otimes \iota \otimes \iota) = y \otimes \bar{\mu}_*(\iota \otimes \iota) = 0$$

because $\bar{\mu}_*(\iota \otimes \iota) \in H_2(S^1) = 0$.

C. To prove C, we observe that there is a map $\rho : T_f \to T_g \equiv T_f m$ which covers - via the projection maps $T_f \to S^1$ - the standard map which maps S^1 around itself m times, which we write ρ_m. This yields a commutative diagram

$$\begin{array}{ccc} T_f & \times S^1 \xrightarrow{\mu_f} & T_f \\ \rho \times \rho_m & \downarrow & \downarrow \rho \\ T_g & \times S^1 \xrightarrow{\mu_g} & T_g \end{array}$$

Calculating on homology,

$$\begin{aligned} \rho_* \mu_{f*}(i_*(x) \otimes \iota) &= \mu_{g*}(\rho \times \rho_m) * (i_*(x) \otimes \iota) \\ &= \mu_{g*}((\rho_* i_*(x)) \otimes \rho_*(\iota)) = \mu_{g*}(i_*(x) \otimes m \cdot \iota) \end{aligned}$$

Here, we have used the fact that ρ leaves the time zero values fixed, and ρ_m is the map described above.

We get finally $m \cdot \mu_{g*}(i_*(x) \otimes \iota) = m w_i(g)(i_*(x))$. But this is $\rho_* w_i(f)(i_*(x))$

We now look at some examples of these characteristic classes. To begin, with the most simple cases, let $X = S^1$. If $f = Id$, then part B of the previous proposition gives us the result. Note T_f is the standard torus.

$$w_1(f)(1) = Id : Z \to Z \subseteq H_1(T_f).$$

if $x \in H_1(S^1)$, $w_2(f)(i_*(x)) = x \otimes \iota$, where $i : S^1 \subseteq T_f$;

$$w_2(f)(1 \otimes \iota) = 0.$$

If $j > 2 \quad w_j(f) \equiv 0$.

On the other hand, if $f : S^1 \to S^1$ is the standard map of degree -1 then T_f is the Klein bottle. $w_1(f)$ is still the inclusion. But T_f is non-orientable, $H_2(T_f) = 0$. Thus, in this case $w_j(f) = 0$ for all $j > 1$. Note that $w_2(f)$ is like the identity for $f = Id$, and like 0 for its negative, on the image of i_*.

Finally, let us look at the case of the shift homeomorphism

$$\sigma : X \times \ldots \times X \to X \times \ldots \times X,$$

m factors, defined by $\sigma(x_1, \ldots, x_m) = (x_{m_1}, x_1, \ldots, x_{m-1})$; assume that X is connected and has some no zero homology, in a positive dimension. σ is clearly a homeomorphism of order m. The action $\mu_\sigma : T_\sigma \times S^1 \to T_\sigma$ covers the "standard" map of degree m. In general, there are many possible classes of dimension $i-1$ in the homology of T_σ, yielding different values under the homomorphism

$$w_i(\sigma) : H_{i-1}(T_\sigma) \to H_i(T_\sigma);$$

suppose $u \in H_{i-1}(T_\sigma)$ is in the image of the inclusion $j : X \times \ldots \times X \to T_\sigma$ at time 0 and $u = j(v)$ with v invariant under σ_*. Then $w_i(\sigma)(u)$ is easily calculated to be a cross product of u and $m \cdot i$, where i is the 1-dimensional homology class represented by a single circle through a base point $* \times \ldots \times * \in X \times \ldots \times X$.

§3

One might criticize the present paper as raising more questions than it solves. Certainly, there is a need to study the characteristic classes $w_i(f)$ in a great many examples, especially some examples which involve homotopy self-equivalences which (unlike the above examples) are themselves poorly understood. Only then will we really know the value of the $w_i(f)$.

The twisted torus T_f is clearly of basic importance in itself, and it has figured in (unpublished) work of Andre Legrand. Endowed with an S^1 action, in case f is a homeomorphism of finite order, there are many possible domains of study. An example which we have not pursued but probably merits study, is to look at T_f as a fibre bundle over S^1, with fibre X and group a finite subgroup of the group of homeomorphisms of X. Another thought would be to study the "equivariant cohomology" of X (or the related idea of the homotopy fixed points). Let E_{S^1} be the total space of the universal S^1-bundle and form the product over S^1

$$E_{S^1} \times_{S^1} T_f.$$

The (co)homology of this space is clearly of importance in understanding f.

Goodwillie [3] has interpreted the cohomology of the full, free loop space in terms of cyclic cohomology and we can only expect that such an interpretation of $H^*(T_f)$ is possible. See also the work of J.D.S. Jones [4].

Finally, we come to the more elusive "class" $\chi(f) \subseteq H^2(T_f)$. While this makes sense for any homotopy equivalence f, rather than the restrictive homomorphisms of finite order, it seems very difficult to control. One problem concerns the fibre space $S^1 \to \mathcal{E} \to T_f$. Our task would be easy, if this was known to be a principle $SO(2)$ bundle.

Unfortunately, there exist bundles of this sort which are far from principal (for example, consider a short exact sequence $0 \to Z \to G \to A \to 0$ where A is a group and the sequence splits, that is G is a semi-direct product. This easily yields a fibre space $S^1 \to K(G,1) \to K(A,1)$. If it were principal, it would be a cartesian product, because the splitting would give a section to the bundle. This would force $G \cong Z \times A$). To understand the class $\chi(f)$, we need better understand the bundle $S^1 \to \mathcal{E} \to T_f$.

Finally, we ask whether the classes $w_i(f)$ could perhaps be defined for more general homotopy-equivalences f.

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REFERENCES

[1] G. COOKE, *Replacing homotopy actions by topological actions* Trans. A.M.S. **237**, (1978) 391-406.
[2] D. FRANK AND D. KAHN, *Finite complexes with infinitely-generated groups of self-equivalences*, Topology **16** (1977), 189-192.
[3] T. GOODWILLIE, *Cyclic homology, derivations and the free loop space*, Topology **24** (1985), 187-215.
[4] J.D.S. JONES, *Cyclic homology and equivariant homology*, Invent. Math. **87** (1987), 403-423.
[5] W.S. MASSEY , Algebraic Topology; an Introduction, Harcourt, Brace, and World, 1967.
[6] D. MCCULLOUGH, *Connected 3-manifolds with infinitely generated groups of self-homotopy-equivalences*, Proc. A.M.S., **91** (1984), 625-629.
[7] R. PICCININI, *Groups of self-equivalences and related topics*, Springer Lecture Notes in Math. **1425**, Springer Verlag 1990.
[8] SAMUEL SMITH, On Rational Homotopy of Function Spaces, Ph. D. University of Minnesota, 1993.
[9] D. SULLIVAN, *Infinitesimal computations in topology*, IHES Publ. Math. **47** (1977) Paris.