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EQUATIONS FOR CURVES IN THEIR JACOBIANS

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In this paper we give equations of curves in their Jacobians. Some are global and others generalize Fay's trisecant identity.

1. Abelian varieties

Let X be an abelian variety with dual abelian variety X^v. Let P be a Poincaré sheaf on $X \times X^v$. Let $\mathcal{L}$ be an ample invertible sheaf on X. Then $W(\mathcal{L}) \equiv \pi_{X^v *}(\pi_X^* \mathcal{L} \otimes O_{X \times X^v} P)$ is a local free coherent sheaf on X^v whose formation commutes with base extension.

LEMMA (1). *Let $\varphi_{\mathcal{L}} : X \to X^v$ be the homomorphism defined by $\mathcal{L}$. Then $\varphi_{\mathcal{L}}^* W(\mathcal{L})$ is naturally isomorphic to $\Gamma(X, \mathcal{L}) \otimes_k \mathcal{L}^{\otimes -1}$.*

Proof. $(1_X \times \varphi_{\mathcal{L}})^* P \approx (\pi_1 + \pi_2)^* \mathcal{L} \otimes \pi_1^* \mathcal{L}^{\otimes -1} \otimes \pi_2^* \mathcal{L}^{\otimes -1}$. Thus $\varphi_{\mathcal{L}}^* W(\mathcal{L})$ is naturally isomorphic to

$$\pi_2 * \left((\pi_1 + \pi_2)^* \mathcal{L} \otimes_{O_{X \times X}} \pi_2^* \mathcal{L}^{\otimes -1} \right) \approx \pi_2 * ((\pi_1 + \pi_2)^* \mathcal{L}) \otimes_{O_X} \mathcal{L}^{\otimes -1}.$$

Now we have a π_2-isomorphism $\sigma : X \times X \to X \times X$ given by $(x_1, x_2) \to (x_1 + x_2, x_2)$ which induces an isomorphism

$$\Gamma(X, \mathcal{L}) \otimes_k O_X \approx \pi_2 * (\pi_1^* \mathcal{L}) \to \pi_2 * ((\pi_1 + \pi_2)^* \mathcal{L}).$$

Hence we get the result. Q.E.D.

Let H be a finite closed subscheme of X. We have an O_{X^v}-homomorphism

$$\alpha(\mathcal{L}, H) : W(\mathcal{L}) \to \pi_{X^v *}(\pi_X^* \mathcal{L} \otimes_{O_{X \times X^v}} P|_{H \times X^v})$$

of locally free coherent sheaves given by evaluation. For all non-negative integers we have a closed subscheme $Z^i(\mathcal{L}, H)$ of X^v defined by $\Lambda^i \alpha(\mathcal{L}, H) = 0$

Let $\beta(\mathcal{L}, H) : \Gamma(X, \mathcal{L}) \otimes_k O_X \to \pi_{2*} \left(\pi_1^* \mathcal{L}|_{(H,0)+\Delta} \right)$ be given by restriction.

LEMMA (2). $\beta(\mathcal{L}, H) \approx \varphi_{\mathcal{L}}^* (\alpha(\mathcal{L}, H)) \otimes_{O_X} \mathcal{L}$.

Proof. This follows from the proof of Lemma 1 as $\sigma(H \times X) = (H, 0) + \Delta$. Q.E.D.

We have closed subschemes $U^i(L, H)$ of X defined by $\Lambda^i \beta(\mathcal{L}, H) = 0$. Thus we get.

COROLLARY. $\varphi_{\mathcal{L}}^{-1} \left(Z^i(\mathcal{L}, H) \right) = U^i(\mathcal{L}, H)$.

2. Jacobians

Let J be the Jacobian of smooth complete curve C of genus $g \geq 1$. Then J is principal polarized by the theta divisor θ which is only determined up to translation. If an invertible sheaf $\mathcal{L}$ on J is algebraically equivalent to $O_X(n\theta)$ then deg $\mathcal{L} = n^g$. We will work only with this type of invertible sheaf.

Let $C \subset J$ be the usual embedding determined up to translation.

LEMMA (4). *If* $n \geq 2$, $i : \Gamma(J, \mathcal{L}) \to \Gamma(C, \mathcal{L}|_C)$ *is surjective.*

Proof. As the image of i is a closed vector subspace we need to show that its image is dense. Now, $O_J(\theta)|_C$ has degree g and $\mathcal{L}|_C$ has degree gn. Then $\dim_k \Gamma(C, \mathcal{L}|_C) = g(n-1)+1$. Let $c_1, \ldots, c_g$ be g general points of C, then $D = c_1 + \ldots + c_g = (\theta + j) \cdot C$ for some point $j \in J$. Let $\mathcal{L}' = \mathcal{L}(-(\theta + j))$, if $n \geq 2$ $D+$ (effective divisor in $|\mathcal{L}|_C)$ is in the image of $\Gamma(J, \mathcal{L}') \to \Gamma(C, \mathcal{L}'|_C)$. As $\mathrm{Pic}_0(J) \to \mathrm{Pic}_0(C)$ is an isomorphism, $\mathcal{L} \approx O_J\left(\sum_{i=1}^{n}(\theta + j_i)\right)$, then a general section α of $\mathcal{L}|_C$ has a divisor of the form $D_1 + \ldots + D_n$ (where the D_i are as above) which is in the image of $\Gamma(J, \mathcal{L}) \to \Gamma(C, \mathcal{L}|_C)$. Q.E.D.

Let deg $\mathcal{L} = n^g$ with $n \geq 2$. Let $V(\mathcal{L}) = \pi_{2*}\left(\pi_1^*\mathcal{L} \otimes P|_{C\times J}\right)$. Then $V(\mathcal{L})$ is a loclaly free coherent sheaf on J of rank $ng - g + 1$. We have

COROLLARY (5). *Restriction* $\rho : W(\mathcal{L}) \to V(\mathcal{L})$ *is surjective.*

Now let H be a closed finite subscheme of C. Then we have a restriction

$$\alpha'(\mathcal{L}, H) : V(\mathcal{L}) \to \pi_{2*}\left(\pi_1^*\mathcal{L} \otimes P|_{H\times J}\right).$$

Then $\alpha(\mathcal{L}, H)$ factors through the surjection ρ. Thus $Z^i(\mathcal{L}, H)$ is the closed subscheme $\Lambda^i\alpha'(\mathcal{L}, H) = 0$ and this subscheme has been studied extensively in [1] in dual form. Hence if

$$\begin{array}{c} 2g - 2 \geq ng - \deg H \geq g - 1 \\ ||| \\ 2g - 2 - i, \end{array}$$

then $Z^{\deg H}(\mathcal{L}, H)$ is a translative of $-W^i$ where $W^i = C + \ldots + C$ i times.

By the methods of the first section $n^{-1}Z^{\deg H}(\mathcal{L}, H) = U^{\deg H}(\mathcal{L}, H)$. This generalizes Fay identity with $n = 2$ and $\deg H = 3$.

3. Global equations

Let (X, θ) be a principally polarized abelian variety. Assume that the characteristic of the base field is not two. We will use the same notations as Jacobians. As in the case of Jacobians we will work only with invertible sheaves algebraically equivalent to $O_X(n\theta)$. Let us observe that the principal polarization defines an isomorphism $X^v \xrightarrow{\sim} X$; with this identification, given an invertible sheaf S of degree two on X, the morphism φ_S of §1 is a morphism $\varphi_S : X \to X$ such that $\phi_S(y) = 2y - 2\xi$ where $S \approx \tau_\xi^* O_X(2\theta)$.

THEOREM (6). *Let R and S be two invertible sheves on X of degree 2^g. Then $W(S) \otimes_{O_X} R$ is generated by its sections.*

Proof. Let x be a point on X. We need to see that $\Gamma(X, W(S) \otimes_{O_X} R) \to W(S) \otimes_{O_X} R|_x$ is surjective. Let $\mathcal{M} = \pi_1^* S \otimes_{O_{X\times X}} P \otimes_{O_{X\times X}} \pi_2^* R$. Then we need to see that

$$\Gamma(X \times X, \mathcal{M}) \to \Gamma\left(X \times \{x\}, \mathcal{M}|_{X\times\{x\}}\right)$$

is surjecture. Now $\varphi_S(y) = 2y - 2\xi$. By taking invariants under X_2 it is enough to see that

$$\Gamma(X \times x, (\mathrm{Id}_X, 2)^* \mathcal{M}) \to \Gamma\left(X \times 2^{-1}(x), (\mathrm{Id}_X, 2)^* \mathcal{M}|_{X\times 2^{-1}(x)}\right)$$

is surjective.

We may use our change of coordinates which now preserves $X \times 2^{-1}(x)$. Thus we need to see that

$$\Gamma\left(X \times X, \pi_1^* S \otimes_{O_{X\times X}} \pi_2^* \left(2^* R \otimes_{O_X} S^{\otimes -1}\right)\right) \to \Gamma\left(X \times 2^{-1}(x),\ \text{same}|_{X\times 2^{-1}(x)}\right)$$

is surjective.

By Künneth formula, we need to prove the surjectivity of

$$\Gamma\left(X, 2^* R \otimes_{O_X} S^{\otimes -1}\right) \to \Gamma\left(2^{-1}(x), 2^* R \otimes_{O_X} S^{\otimes -1}|_{2^{-1}(x)}\right)$$

Now this follows from [2]. Q.E.D.

Now if X is the Jacobian of C and $H \subset C$ satisfied $\deg H = 3$ then taking global sections in the homomorphism $\alpha(S,H) \otimes 1 : W(S) \otimes_{O_J} R \to \pi_2 * \left(\pi_1^* S \otimes_{O_{J\times J}} P|_{H\times J}\right) \otimes_{O_J} R$ we have the homomorphism

$$\Gamma\left(J, W(S) \otimes_{O_J} R\right) \xrightarrow{\delta} \Gamma\left(J \times J, \pi_1^* S \otimes_{O_{J\times J}} P \otimes_{O_{J\times J}} \pi_2^* R|_{J\times H}\right)$$

which satisfies $\Lambda^3 \delta = 0$ equals a translate of $-C$. Thus $-C$ is the zeroes of some sections of $\mathcal{T}$ where $\deg \mathcal{T} = 6$. We will describe these equations explicitly.

In the general case, given H a finite subscheme of X and S and R as in the theorem, the homomorphism $\alpha(S,H)$ of §1 induces a homomorphism

$$\alpha(S,H) \otimes 1 : W(S) \otimes_{O_X} R \to \pi_{2*}\left(\pi_1^* S \otimes_{O_{X\times X}} P|_{H\times X}\right) \otimes_{O_X} R$$

and taking global sections:

$$\delta(S,R,H) : \Gamma(X, W(S)\otimes_{O_X} R) \to \Gamma\left(X \times X, \pi_1^* S \otimes_{O_{X\times X}} P \otimes_{O_{X\times X}} \pi_2^* R|_{H\times X}\right).$$

COROLLARY. *The closed subscheme $Z^i(S,H)$ of X coincides with the subscheme defined by the global equations $\Lambda^i \delta(S,R,H) = 0$.*

Using the results of §1 we can compute explicitly the global equations of the subschemes $U^i(S,H)$.

Let us assume that $H = \{c_1, \dots, c_{n+2}\}$ is a finite subscheme of X of length $n+2$ given by $n+2$ distinct points of X and $\xi \in X$ is a point such that $2\xi = c_1 + \dots + c_{n+2}$. We will assume that $S \approx R \approx \tau_{-\xi}^* O_X(2\theta) = O_X(2\theta_\xi)$. With these notations:

$$\delta(S,S,H) : \Gamma(X \times X, \pi_1^* S \otimes P \otimes \pi_2^* S) \to \bigoplus_{i=1}^{n+2} \Gamma\left(X, \tau_{\widetilde{c}_i}^* S\right)$$

(where $\widetilde{c}_i \in X$ is such that $2\widetilde{c}_i = c_i$). Taking inverse images with respect to φ_s we obtain a homomorphism

$$\begin{aligned}\beta(S,H) \ : \ & \pi_2 * (\pi_1^* S) \otimes_{O_X} \left(S^{\otimes -1} \otimes 2^* S\right) \to \\ & \bigoplus_{i=1}^{n+2} \left(\pi_2 * \left(\pi_1^* S|_{(c_i,0)+\Delta}\right) \otimes_{O_X} \left(S^{\otimes -1} \otimes 2^* S\right)\right)\end{aligned}$$

For each c_i the homomorphism:

$$\begin{aligned}\pi_2 * (\pi_1^* S) \ &\approx \ \Gamma(X,S) \otimes_k O_X \xrightarrow{\beta} \pi_2 * \left(\pi_1^* S|_{(c_i,0)+\Delta}\right) \\ &\approx \ \pi_2 * \left(\pi_1^* \tau_{c_i}^* S\right) \approx \Gamma\left(X, \tau_{c_i}^* S\right) \otimes_k O_X\end{aligned}$$

is given by $\beta(s(z)) = s(z + c_i)$, $s(z)$ being a global section of S.

Let $\{\theta_\sigma(z), \sigma \in (\mathbb{Z}/2\mathbb{Z})^g\}$ be a basis for the vector space $\Gamma(X, O_X(2\theta))$ (for example, $\{\theta_\sigma\}$ could be the classical basis of second order theta functions of (X,θ)). A basis for the vector space $\Gamma(X,S)$ is given by $\{\theta_\sigma(z-\xi)\}$ and the homomorphism $\beta(S,H) : \pi_2 * (\pi_1^* S) \to \bigoplus_{i=1}^{n+2} \pi_2 * \left(\pi_1^* \tau_{c_i}^* S\right)$ is given in this basis by:

$$\beta\left(\theta_\sigma(z-\xi)\right) = \left(\theta_\sigma(z-\xi+c_1), \dots, \theta_\sigma(z-\xi+c_{n+2})\right).$$

From this discussion and §1 we obtain

THEOREM (7). *$U^{n+2}(S,H) = \varphi_s^{-1}(Z^{n+2}(S,H))$ is scheme-theoretically defined by the system of global equations:*

$$S_{\overline{\sigma}\lambda} = \det\left(\theta_{\sigma_{\lambda_i}}(z-\xi+c_j)\right) = 0$$

for every $\overline{\sigma} = \left(\sigma_{\lambda_1}, \ldots, \sigma_{\lambda_{n+2}}\right) \in [(\mathbb{Z}/2\mathbb{Z})^g]^{(n+2)}$.

Let us assume now that $H = \{(n+2)c\}$ is a subscheme concentrated at the closed point $c \in X$ of the form Spec $k[\epsilon]/\epsilon^{n+2} \hookrightarrow X$; that is, H is given by a ring homomorphism $O_{X,c} \xrightarrow{p_H} k[\epsilon]/\epsilon^{n+2}$:

$$p_H(f(z)) = \sum_{k=0}^{n+1}(\Delta_k f)(c)\epsilon^k$$

where $\Delta_k = \sum_{h_1+2h_2+\ldots+kh_k=k} \frac{1}{h_1!\ldots h_k!}D_1^{h_1}\ldots D_k^{h_k}$, D_i being constant vector fields on X. (We are assuming now that the base field has characteristic 0).

THEOREM (8). *If $H = \{(n+2)c\}$ is the subscheme given above, the subscheme $U^{n+2}(S,H)$ is scheme-theoretically defined by the system of global equations:*

$$\det\left(\Delta_j\theta_{\sigma_{\lambda_i}}(z+c)\right) = 0$$

for every $(\sigma_{\lambda_1}, \ldots, \sigma_{\lambda_{n+2}}) \in [(\mathbb{Z}/2\mathbb{Z})^g]^{(n+2)}$.

Proof. Let us observe that in this case

$$\pi_{2*}\left(\pi_1^*S|_{(H;0)+\Delta}\right) \approx \left[\Gamma(X,\tau_c^*S) \oplus \varepsilon\Gamma(X,\tau_c^*S) \oplus \ldots \oplus \varepsilon^{n+1}\Gamma(X,\tau_c^*S)\right] \otimes_k O_X$$

and the homomorphism $\beta(S,H)$ is given by:

$$\beta(\theta_\sigma(z)) = \theta_\sigma(z+c) + \varepsilon\Delta_1\theta_\sigma(z+c) + \ldots + \varepsilon^{n+1}\Delta_{n+1}\theta_\sigma(z+c)$$

Q.E.D.

In general, if $H = H_1 \amalg \ldots \amalg H_r$ and each H_i is a subscheme of X of the form Spec $k[\epsilon]/\epsilon^{n_i} \hookrightarrow X$, we obtain in the same way a system of global equations for $U^{n+2}(S,H)$.

The global equations obtained here generalize Fay's trisecant identity and Gunning's relations (see [4]). To obtain the classical results of Fay and Gunning in the Jacobian case, we can proceed as follows:

Let C be a smooth complete curve of genus $g \geq 1$, $J = \text{Pic}^0(C)$ its Jacobian variety, $p_0 \in C$ a closed point and $i : C \hookrightarrow J$ the immersion defined by p_0. We fix a half canonical divisor Δ on C (that is, $O_C(2\Delta) \approx \omega_C$ the canonical sheaf). These data allow us to determine a canonical polarization $\Theta \hookrightarrow J$ with the condition: $\Theta|_C = \Delta + p_0$.

Let $p_1, \ldots, p_{n+2}$ $n+2$ distinct points of C and $c_1, \ldots, c_{n+2}$ their images in J, we select a point $\xi \in J$ such that $2\xi = c_1 + \ldots + c_{n+2}$ and define $S = O_J(2\Theta_\xi)$ and $H = \{c_1, \ldots, c_{n+2}\} \subset J$. Applying the above results to (J, S, H) we obtain the Fay trisecant identity for $n = 1$ and the Gunning relations for arbitrary n.

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