Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

ON THE COEFFICIENTS OF THE DOUBLE TOTAL SQUARING OPERATION

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1. Introduction and statement of the results.

As it is well known, the combinatorics involved in the study of the iterated total squaring operation is closely related to the stable homotopy theory, via the Adams spectral sequence. For this reason we try to analyse the coefficients of such cohomology operation and their relations with modular invariant theory. We will use here the standard notation of modular invariant theory, which can be found, for example, in [2] or [4]. All the symbols used here will be explained later anyway, in the attempt to make this paper reasonably self-contained.

Let us consider the iterated total squaring operation

$$(1) \qquad S_m : H^*(X) \longrightarrow \Delta_m \otimes H^*(X) \qquad (X \text{ a } CW\text{-complex})$$

$$x \longmapsto \sum_I v^{-I} \otimes \mathrm{Sq}^I(x)$$

where $I = (i_1, \dots, i_m)$ and $i_s \geq 0$ for each $s = 1, \dots, m$. Here $-I$ indicates the string $(-i_1, \dots, -i_m)$ and Sq^I stands for the monomial $\mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_m}$ in the mod 2 Steenrod algebra $\mathcal{A}$. As it has been pointed out by H. Múi [3] we have that

$$(2) \qquad S_m(x) = \sum_R Q_{m,0}^{-\sum r_j} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k} \otimes \xi_R^*(x)$$

(see also [1] for the present normalized version of this result , originally due to H. Múi) where $R = (r_1, \dots, r_k)$, with $r_s \geq 0$ for each $s = 1, \dots, k$, and ξ_R^* is the element of the Milnor basis of $\mathcal{A}$ dual to the monomial

$$\xi_R = \xi_1^{r_1} \dots \xi_k^{r_k} \in \mathcal{A}_*$$

with respect to the basis $\mathcal{B}$ of admissible monomials in $\mathcal{A}$.

Here we would like to carry out a similar procedure using the basis $\mathcal{B}$ of $\mathcal{A}$. In other words, if we write each summand of (1) expressing the monomials Sq^I in terms of admissible monomials, we find an expression of the form

$$S_m(x) = \sum_{\alpha \in \mathcal{B}} \mathrm{coeff}(\alpha) \otimes \alpha(x) .$$

As in (2), for invariant theoretical reasons we know that

$$\mathrm{coeff}(\alpha) \in \Gamma_m .$$

We would like to compute it.

In the present paper we only succeed in doing it in the case $m = 2$, where we employ the Adem relations as a tool for our calculation. We find that

$$S_2(x) = \sum_{h,k} \sum_i \binom{k-2i}{i} Q_{2,0}^{2i-h-k} Q_{2,1}^{k-3i} \otimes \mathrm{Sq}^{2h+k}\mathrm{Sq}^h(x)$$

i.e.

$$\mathrm{coeff}(\mathrm{Sq}^{2h+k}\mathrm{Sq}^h) = \sum_i \binom{k-2i}{i} Q_{2,0}^{2i-h-k} Q_{2,1}^{k-3i} .$$

2. Invariant theory.

We need to recall some notation from modular invariant theory. Full detail can be found in [4] (as well as in many other papers).

Let $\mathbb{F}_2$ be the field of order 2 and let us consider, for each $m \geq 1$, the polynomial ring

$$P_m = \mathbb{F}_2[t_1, \ldots, t_m] .$$

We fix an element

$$e_m = \prod_\lambda (\lambda_1 t_1 + \cdots + \lambda_m t_m) \qquad (\lambda = (\lambda_1, \ldots, \lambda_m) \neq (0, \ldots, 0))$$

the so called Euler class of P_m, and invert it. We get a ring

$$\Phi_m = P_m[e_m^{-1}] .$$

Let us write GL_m for the general linear group $GL_m(\mathbb{F}_2)$. There is an action of GL_m on P_m which extends the obvious action of GL_m on the $\mathbb{F}_2$-vector space spanned by $\{t_1, \ldots, t_m\}$. Φ_m too is acted upon by GL_m as e_m is fixed under the action of GL_m on P_m. Therefore it makes sense to consider the rings of invariants

$$\Gamma_m = \Phi_m^{GL_m} \quad ; \quad \Delta_m = \Phi_m^{T_m} \qquad (m \geq 2) .$$

Here T_m is the upper triangular subgroup of GL_m. Clearly

$$\Gamma_1 = \Delta_1 = \Phi_1 = \mathbb{F}_2[t_1^{\pm 1}] .$$

It is well known that Γ_m, Δ_m are rings of Laurent series; more precisely we have that

$$\Delta_m = \mathbb{F}_2[v_1^{\pm 1}, \ldots, v_m^{\pm 1}] \quad ; \quad \Gamma_m = \mathbb{F}_2[Q_{m,0}^{\pm 1}, Q_{m,1}, \ldots, Q_{m,m-1}]$$

where the v_j's and the $Q_{m,i}$'s are rational expressions of the indeterminates

$$t_1, \ldots, t_m .$$

For more detail, see [4], [1]. Obviously

$$\Gamma_m \subseteq \Delta_m$$

and, in particular, the $Q_{m,i}$'s can be expressed in terms of the v_j's. The following formulas clarify such a dependence.

$$\text{(3)} \qquad \begin{cases} (i) & Q_{1,0} = v_1 = t_1 \quad ; \quad Q_{m,m} = 1 \quad \forall\, m \geq 1 \\ (ii) & Q_{m,i} = 0 \quad \text{if } i < 0 \text{ or } i > m \\ (iii) & Q_{m,i} = Q_{m-1,0} Q_{m-1,i} v_m + Q_{m-1,i-1}^2 \, . \end{cases}$$

In particular, for $m = 2$, we have

$$\Gamma_2 = \mathbb{F}_2[Q_{2,0}^{\pm 1}, Q_{2,1}] \quad ; \quad \Delta_2 = \mathbb{F}_2[v_1^{\pm 1}, v_2^{\pm 1}]$$

and, using (3)(i),(ii),(iii), we get that

$$Q_{2,0} = Q_{1,0} Q_{1,0} v_2 = v_1^2 v_2 \tag{4}$$

$$Q_{2,1} = Q_{1,1} Q_{1,0} v_2 + Q_{1,0}^2 = v_1 v_2 + v_1^2 = v_1(v_1 + v_2) \, . \tag{5}$$

3. The iterated total squaring operation.

Let us start by explaining what we mean by *total squaring operation*.

Definition (1). Let M be an unstable graded left algebra over $\mathcal{A}$. The total squaring operation (over M) is the ring homomorphism

$$\mathrm{Sq} : M \longrightarrow \mathbb{F}_2[t_1^{\pm 1}] \otimes M = \Phi_1 \otimes M$$

defined by setting

$$\mathrm{Sq}(x) = \sum_{j \geq 0} t_1^{-j} \otimes \mathrm{Sq}^j(x) \, .$$

We make $\Phi_1 \otimes M$ into an $\mathcal{A}$-module by setting

$$\begin{aligned} \mathrm{Sq}^k(t_1^a \otimes x) &= \sum_{h=0}^{k} \mathrm{Sq}^h t_1^a \otimes \mathrm{Sq}^{k-h}(x) \\ &= \sum_{h=0}^{k} \binom{a}{h} t_1^{a+h} \otimes \mathrm{Sq}^{k-h}(x) \, . \end{aligned}$$

Definition (2). We set $S_1 = \mathrm{Sq}$ and define

$$S_m : M \longrightarrow \Phi_m \otimes M \qquad (m \geq 2)$$

by setting

$$S_m(x) = \sum_{i_1,\dots,i_m \geq 0} (t_1^{-i_1}\mathrm{Sq}^{i_1})\dots(t_m^{-i_m}\mathrm{Sq}^{i_m})(x) . \tag{6}$$

The fact that the RHS of (6) is actually an element of $\Phi_m \otimes M$ is not immediately clear, and it is shown in [2], [3].

In [1] it has been proved that

$$\mathrm{im}\, S_m \subseteq \Delta_m \otimes M$$

and we actually have that

$$S_m(x) = \sum_I v^{-I} \otimes \mathrm{Sq}^I(x) \tag{7}$$

where the sum runs over the multi-indices

$$I = (i_1, \dots, i_m)$$

with $i_s \geq 0$, $s = 1, \dots, m$. In (7) we write $-I$ for the multi-index

$$-I = (-i_1, \dots, -i_m)$$

and v^{-I}, Sq^I for the monomials

$$v^{-I} = v_1^{-i_1} \dots v_m^{-i_m} \quad ; \quad \mathrm{Sq}^I = \mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_m} .$$

For each multi-index

$$R = (r_1, \dots, r_k) \qquad (k \in \mathbb{N}, r_j \geq 0)$$

we write ξ_R for the monomial

$$\xi_R = \xi_1^{r_1} \dots \xi_k^{r_k} \in \mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

where $\mathcal{A}_*$ is the dual of $\mathcal{A}$. We indicate with ξ_R^* the element of $\mathcal{A}$ which is dual to ξ_R with respect to the basis of admissible monomials of $\mathcal{A}$. We know that the elements of the form ξ_R^* form the so-called Milnor basis of $\mathcal{A}$. Hence, for each summand which appears in the RHS of (7), the monomial Sq^I can be uniquely expressed as a sum of Milnor elements ξ_R^*. After suitable groupings, we write formula (7) as follows:

$$S_m(x) = \sum_R \mathrm{coeff}(\xi_R^*) \otimes \xi_R^*(x)$$

where

$$\operatorname{coeff}(\xi_R^*) \in \Delta_m \, .$$

In [1], [3] it has been proved that in fact

$$\operatorname{coeff}(\xi_R^*) \in \Gamma_m \, .$$

More precisely, we have

$$S_m(x) = \sum_R Q_{m,0}^{-\sum r_j} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k} \otimes \xi_R^*(x) \, .$$

This procedure can be carried out using any linear basis $\mathcal{B}$ of $\mathcal{A}$, as any linear combination of invariants is again invariant. Therefore we find that

$$S_m(x) = \sum_{\alpha \in \mathcal{B}} \operatorname{coeff}(\alpha) \otimes \alpha(x)$$

with

$$\operatorname{coeff}(\alpha) \in \Gamma_m \qquad \forall\, \alpha \in \mathcal{B} \, .$$

4. The coefficients of the double total squaring operation.

It appears quite natural to ask ourselves what happens when we choose the basis of admissible monomials in $\mathcal{A}$. We fix our attention on the double total squaring operation

$$S_2 : M \longrightarrow \Delta_2 \otimes M$$

$$x \longmapsto \sum_{i,j \geq 0} v_1^{-i} v_2^{-j} \otimes \operatorname{Sq}^i \operatorname{Sq}^j (x) \, .$$

An admissible monomial of length 2 is of the form

$$\operatorname{Sq}^{2h+k} \operatorname{Sq}^k \qquad (h, k \geq 0) \, .$$

If we write each monomial $\operatorname{Sq}^i \operatorname{Sq}^j$ as a sum of admissible monomials, we can write $S_2(x)$ as follows:

$$S_2(x) = \sum_{h,k} \sum_{i+j=3h+k} \alpha(i,j,h,k) v_1^{-i} v_2^{-j} \otimes \operatorname{Sq}^{2h+k} \operatorname{Sq}^k (x)$$

where

$$\alpha(i,j,h,k) = \begin{cases} 0 \\ 1 \end{cases}$$

according on whether $\operatorname{Sq}^{2h+k} \operatorname{Sq}^k$ appears or does not appear in the admissible expression of $\operatorname{Sq}^i \operatorname{Sq}^j$.

THEOREM. *We have*

$$S_2(x) = \sum_{h,k}\sum_i \binom{k-2i}{i} Q_{2,0}^{2i-h-k} Q_{2,1}^{k-3i} \otimes \mathrm{Sq}^{2h+k}\mathrm{Sq}^k(x)\,. \tag{8}$$

Proof. Let us write $R_{h,k}$ for the coefficient of $\mathrm{Sq}^{2h+k}\mathrm{Sq}^k(x)$ in (8). We have

$$R_{h,k} = \sum_{i+j=3h+k} \alpha(i,j,h,k) v_1^{-i} v_2^{-j}$$

i.e. $R_{h,k}$ is the sum of those monomials $v_1^{-i}v_2^{-j}$ such that $i+j=3h+k$ and $\mathrm{Sq}^{2h+k}\mathrm{Sq}^k$ appears in the admissible expression of $\mathrm{Sq}^i\mathrm{Sq}^j$. Let us compute $R_{h,k}$. We have

$$j = 3h+k-i\,.$$

If we write the Adem relation

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_c \binom{3h+k-i-1-c}{i-2c} \mathrm{Sq}^{3h+k-c}\mathrm{Sq}^c \tag{9}$$

we see that the monomial $\mathrm{Sq}^{2h+k}\mathrm{Sq}^k$ appears in the RHS of (9) when $c=h$ and its coefficient is

$$\binom{2h+k-i-1}{i-2h}\,.$$

Hence $v_1^{-i}v_2^{-j}$ is a summand of $R_{h,k}$ (or, equivalently $\alpha(i,j,h,k)=1$) if and only if

$$\binom{2h+k-i-1}{i-2h} = 1$$

i.e.

$$R_{h,k} = \sum_\ell \binom{2h+k-i-1}{i-2h} v_1^{-i} v_2^{i-3h-k}\,.$$

We set

$$\ell = i-2h$$

and get

$$\begin{aligned} R_{h,k} &= \sum_\ell \binom{k-\ell-1}{\ell} v_1^{-2h-\ell} v_2^{\ell-h-k} \\ &= \sum_\ell \binom{k-\ell-1}{\ell} (v_1^{-2h-2k} v_2^{-h-k})(v_1^{2k-\ell} v_2^{\ell}) \\ &= Q_{2,0}^{-h-k} \cdot \sum_\ell \binom{k-\ell-1}{\ell} v_1^{2k-\ell} v_2^{\ell} \end{aligned}$$

(using formula (4)). Therefore we want to prove that

$$\sum_{\ell} \binom{k-\ell-1}{\ell} v_1^{2k-\ell} v_2^{\ell} = \sum_{i} \binom{k-2i}{i} Q_{2,0}^{2i} Q_{2,1}^{k-3i} . \tag{10}$$

We have

$$\begin{aligned}\sum_{i} \binom{k-2i}{i} Q_{2,0}^{2i} Q_{2,1}^{k-3i} &= \sum_{i} \binom{k-2i}{i} v_1^{4i} v_2^{2i} \cdot v_1^{k-3i} \cdot \sum_{s} \binom{k-3i}{s} v_1^{s} v_2^{k-3i-s} \\ &= \sum_{i,s} \binom{k-2i}{i} \binom{k-3i}{s} v_1^{k+i+s} v_2^{k-i-s} .\end{aligned}$$

In the expression above, we look for the summand $v_1^{2k-\ell} v_2^{\ell}$, for a fixed ℓ, and find that it occurs when $\ell = k-i-s$ i.e.

$$s = k - i - \ell$$

and its coefficient is

$$\sum_{i} \binom{k-2i}{i} \binom{k-3i}{k-i-\ell} . \tag{11}$$

We want to show that the expression (11) coincides with the coefficient of $v_1^{2k-\ell} v_2^{\ell}$ in the LHS of (10), which is

$$\binom{k-1-\ell}{\ell}$$

Hence we want to show that

$$\sum_{i} \binom{k-2i}{i} \binom{k-3i}{k-i-\ell} = \binom{k-1-\ell}{\ell} . \tag{12}$$

We are now left with a purely combinatorial computation. We are working on the field $\mathbb{F}_2$, hence the following identities hold:

$$1+x = 1-x \qquad ; \qquad 1+x^2 = (1+x)^2 .$$

We have that

$$\begin{aligned}\binom{k-2i}{i} \binom{k-3i}{k-i-\ell} &= \frac{(k-2i)!(k-3i)!}{i!(k-3i)!(k-i-\ell)!(\ell-2i)!} \\ &= \frac{(k-\ell)!}{i!(k-i-\ell)!} \cdot \frac{(k-2i)!}{(k-\ell)!(\ell-2i)!} \\ &= \binom{k-\ell}{i} \binom{k-2i}{\ell-2i}\end{aligned}$$

We observe now that $\binom{k-\ell}{i}$ is the coefficient of x^i in the expansion of $(1+x)^{k-\ell}$, or, equivalently, the coefficient of x^{2i} in the expansion of $(1+x^2)^{k-\ell}$. Moreover

$$(1-x)^{-n} = \sum_t \binom{n+t-1}{t} x^t .$$

Therefore $\binom{k-2i}{\ell-2i}$ is the coefficient of $x^{\ell-2i}$ in the expansion of $(1-x)^{-(k-\ell+1)}$. Thus

$$\sum_i \binom{k-\ell}{i}\binom{k-2i}{\ell-2i}$$

is the coefficient of

$$x^{\ell-2i} \cdot x^{2i} = x^{\ell}$$

in the expansion of

$$\begin{aligned}(1+x^2)^{k-\ell} \cdot (1-x)^{-(k-\ell+1)} &= (1+x)^{2k-2\ell} \cdot (1+x)^{-k+\ell-1} \\ &= (1+x)^{k-\ell-1}\end{aligned}$$

i.e.

$$\sum_i \binom{k-\ell}{i}\binom{k-2i}{\ell-2i}$$

is the coefficient of x^{ℓ} in the expansion of $(1+x)^{k-\ell-1}$. But such coefficient is exactly $\binom{k-\ell-1}{\ell}$.

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