Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

THE SYMPLECTIC ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE FOR SPHERES

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1. Introduction

Every group in this paper is localized at the prime two. In [6] we developed the following efficient inductive method for calculating the stable homotopy groups of spheres based upon analyzing the Atiyah-Hirzebruch spectral sequence:

$$'E^2_{n,t} = H_nBP \otimes \pi^S_t \Longrightarrow \pi_{n+t}BP. \tag{1}$$

Since the Hurewicz homomorphism $h : \pi_*BP \to H_*BP$ is a monomorphism, $'E^\infty_{n,t} = 0$ if $t \neq 0$ and $'E^\infty_{n,0} = h(\pi_nBP)$. Moreover, π_*BP and H_*BP are known. Thus, if π^S_k is known for $k < t$ then, except for one step, it is algorithmic to deduce the composition series $d^{2r}('E^{2r}_{2r,t-2r+1})$ for $1 \leq r \leq (t+1)/2$ of π^S_t. The determination of π^S_t from this composition series is accomplished using Toda brackets. The algorithmic portions of the computation are done by computer. This procedure was used to compute the first 64 stable stems.

In this paper we carry out the analogous computation based upon analyzing the Atiyah-Hirzebruch spectral sequence:

$$E^2_{n,t} = H_nMSp \otimes \pi^S_t \Longrightarrow \pi_{n+t}MSp. \tag{2}$$

In this case, however, $h : \pi_*MSp \to H_*MSp$ has kernel Torsion π_*MSp and $h : \pi_*MSp/\text{Torsion} \to H_*MSp$ is a monomorphism. Thus, $E^\infty_{n,0} = h(\pi_nMSp)$ while $E^\infty_{n-s,s}$ for $1 \leq s \leq n$ is a composition series of Torsion π_nMSp. Thus, if π^S_k and π_kMSp are known for $k < t$ then the inductive method of [6] can be used to deduce the composition series $d^{4r}(E^{4r}_{4r,t-4r+1})$ for $1 \leq r \leq (t+1)/4$ of π^S_t. In this context the method of [6] is substantially easier because H_*MSp is concentrated in degrees congruent to zero modulo four while H_*BP is concentrated in even degrees. Thus, the same computational effort can compute twice as many stable stems. Moreover, we avail ourselves of the straightforward computer computations from [6, Chapter 4, Section 4] of the cokernel of the differentials in (1) which originate in the 0 row and have image in $\operatorname{Im} J \otimes H_*BP$. In this way, we avoid having to make the analogous computations in our spectral sequence (2). (Note that we do not make use of any of the

1991 Mathematics Subject Classification: Primary 55Q4; Secondary 55N22, 55T25.

This research was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

difficult and subtle computations of [6, Chapters 5, 6 and 7].) *Using no additional computer computations we use these methods to analyze* (2) *through degree 50 and compute the first 49 stable stems.*

There are two reasons for developing new methods for computing stable stems. First, every method for computing stable stems can be analyzed routinely except for occasional very difficult technical problems. However, a difficult problem in one method often corresponds to a simple problem in another method. Thus, a new method of computation will have significant impact in carrying out further computations. Second, each method of computation of stable stems has led to substantial new insights into homotopy theory. (See Ravenel [10] for a summary of the research resulting from the study of the BP Adams-Novikov spectral sequence.) It is hoped that the study of the symplectic Atiyah–Hirzebruch spectral sequence will also lead to new directions in homotopy theory.

In Section 2, we determine all differentials which originate on the 0 row in our spectral sequence. In Section 3, we describe the d^4, d^8 and d^{12}-differentials in our range of computations. In Section 4, we give tables of leaders which follow from the results of Sections 2, 3 and describe the structure of our spectral sequence through degree 50. In Section 5, we determine two relations and twenty Toda brackets which follow from our computations.

We assume that the reader is familiar with the methods of [6] for analyzing the BP Atiyah-Hirzebruch spectral sequence as developed in [6, Chapters 1,2,4] as well as the details of this analysis through degree 50 as summarized in [6, Appendices 1,2,4]. In addition, we will use the structure of $\pi_n MSp$ for $n \leq 50$ which was determined in [7, Section 8] and is summarized in [8, Theorem (2.4)].

Let F be a ring spectrum. We will need the analogue of [6, Theorem (1.2.6)] which defines an action of the Landweber-Novikov operations [1], [9] on the Atiyah-Hirzebruch spectral sequence for F_*MSp:

$$E^2_{n,t} = H_n MSp \otimes F_t \Longrightarrow F_{n+t} MSp. \tag{3}$$

Each Landweber-Novikov operation $s_\omega \in \mathrm{MSp}^k\mathrm{MSp}$ can be represented by a map of spectra $s_\omega : \Sigma^k MSp \to MSp$ which induces a natural map of spectral sequences:

$$s_\omega : E^r_{n,t} \to E^r_{n-k,t}$$

for $2 \leq r \leq \infty$. These s_ω satsify the Cartan formula, are given by $s_\omega \otimes 1$ on E^2 and are induced on E^∞ by the usual s_ω on F_*MSp.

2. Differentials originating on the 0 row

In this section, we study the differentials $d^{4r} : E^{4r}_{4n,0} \to E^{4r}_{4n-4r,4r-1}$. Recall [6, Chapter 4] that all differentials which originate on the 0 row of the BP spectral sequence (1) land in $\mathrm{Im}\, J \otimes H_*BP$ where

$$\mathrm{Im}\, J = \oplus_{n\geq 0}[Z_2\alpha_n \oplus Z_2\eta\alpha_n \oplus Z_8\beta_n \oplus Z_{2^{C(n)}}\gamma_n \oplus Z_2\eta\gamma_n \oplus Z_2\eta^2\gamma_n].$$

Here α_n, β_n, γ_n has degree $8n+1$, $8n+3$, $8n+7$, respectively. In Theorem (2.3) we prove that, in our range of computations, the differentials originating on the 0 row in our spectral sequence land in $I \otimes H_*MSp$ where

$$I = \oplus_{n\geq 0}[Z_8\beta_n \oplus Z_{2^{C(n)}}\gamma_n] \subset \operatorname{Im} J.$$

In Theorem (2.4) we use the canonical map $\lambda : MSp \rightarrow BP$ to show how elements of $(\operatorname{Im} J \otimes H_*BP)_{4*-1}$ which do not bound in the BP spectral sequence determine elements of $I \otimes H_*$MSp which do not bound in our spectral sequence. Recall that H_*MSp is a polynomial algebra with one generator in each degree $4n$ for $n \geq 1$. We begin by introducing notation for polynomial generators of H_*MSp which reduce to canonical elements of $H_*(MSp; Z_2)$ and determine monomials which give a minimal generating set for the cokernel of the Hurewicz homomorphism. Let $\Phi_a \in E_2^{\epsilon(a)+8a-3,1}$ for $a \geq 0$ denote the Ray elements [11] where $\epsilon(a) = 4$ if $a = 0$ and $\epsilon(a) = 0$ otherwise. Recall [4] that for $0 \leq a < b$, the Massey product $P(a,b) = \langle \Phi_a, h_0, \Phi_b\rangle$ is defined with zero indeterminacy in $E_2^{\epsilon(a)+8a+8b-5,2}$ of the classical Adams spectral sequence (ASS):

$$E_2^{n,k} = \operatorname{Ext}^k_{\mathfrak{A}}(Z_2, Z_2)_n \Longrightarrow \pi_n MSp \tag{4}$$

where $\mathfrak{A}$ is the mod two Steenrod algebra. For $0 \leq a < b$, let $v_{a,b} \in E_2^{\epsilon(a)+8a+8b-4,0}$ be an element such that $d_2(v_{a,b}) = P(a,b)$ in the ASS. If $0 < a < b$ then $v_{a,b}$ is uniquely determined. For $n \neq 2^t$, let v_{2n} denote any choice of an indecomposable element of $E_2^{8n,0}$ which survives to E_3 in the ASS. For $a \geq 0$, let $q_a \in E_2^{4\epsilon(a)+32a-8,4}$ denote the unique element such that $P(a,b)^4 = \Phi_a^4 q_b + q_a\Phi_b^4$ in E_2 of the ASS. Let $q(a) \in E_2^{2\epsilon(a)+16a-4,3}$ denote the unique element such that $q(a)^2 = h_0^2 q_a$ in E_2 of the ASS. The Φ_n, q_a and $q(a)$ are all infinite cycles.

THEOREM (2.1) [7, Section 4]. *There are elements $V_{a,b}$, V_{2n}, Q_a of H_*MSp such that:*

(a) *the $4V_{a,b}$, $2V_{2n}$ and $8Q_n$ are in the image of the Hurewicz homomorphism;*

(b) *$V_{a,b}$, V_{2n} reduces to $v_{a,b}$, v_{2n}, respectively, in $H_*(MSp; Z_2)$;*

(c) *$h^{-1}(8Q_a)$ contains an element which projects to $q(a)$ in E_∞ of the ASS;*

(d) *each of these elements can be used as a polynomial generator of H_*MSp.*

We show next that λ_* has a simple description in terms of these polynomial generators of H_*MSp and the canonical polynomial generators $M_n \in H_{2^{n+1}-2}BP$ which are denoted as m_{p^n-1} in [1, page 111]. Recall [6, Theorem

(3.2.2)] that the $\overline{M}_2 = 3M_2 - M_1^3$ and $\overline{M}_n = M_n - M_1M_{n-1}^2$ for $n \geq 3$ are d^2-cycles in the BP spectral sequence.

LEMMA (2.2). *$\lambda_* : H_*MSp \to H_*BP$ is given by:*

(a) $\lambda_*(Q_n) \equiv M_{n+1}^2$ *modulo* (2);

(b) $\lambda_*(V_{0,2^t}) \equiv 2M_1M_{t+2}$ *modulo* (4);

(c) $\lambda_*\left(V_{2^s,2^t}\right) \equiv 2M_{s+2}M_{t+2}$ *modulo* (4).

Proof. (a) Note that

$$\text{Image}\,[\lambda_* : H_*(MSp; Z_2) \to H_*(BP; Z_2)] = Z_2[M_n^2 \mid n \geq 1].$$

Thus, $\lambda_*(Q_n) = U_n^2 + 2A_n$ where $8U_n^2 + 16A_n \in \text{Image}\, h$. Recall [2], [3] that

$$\pi_*BP = Z_{(2)}[V_n \mid n \geq 1]$$

and $h\,(V_n) = 2W_n$ where the $V_n \in \pi_{2^{n+1}-2}BP$ are the Hazewinkel generators. Write $U_n = \alpha W_{n+1} + P_n$ where α is odd and P_n is a decomposable polynomial in the W_k. Then

$$8U_n^2 + 16A_n - 8\alpha^2 W_{n+1}^2 = 16\alpha W_{n+1}P_n + 8P_n^2 + 16A_n \in \text{Image}\, h$$

and there are no common monomial summands of $W_{n+1}P_n$ and P_n^2. Since the square of a decomposable element of Image h must be divisible by 16, it follows that P_n is divisible by two. Thus, $\lambda_*(Q_n) \equiv W_{n+1}^2 \equiv M_{n+1}^2$ modulo two.

(b), (c) Since $d^4(V_{0,1}) = 2\nu Q_0$, it follows that $d^4\lambda_*(V_{0,1}) = 2\nu M_1^2 = d^4(2M_1M_2)$. Since $\lambda_*(Q_0^2) = M_1^4$, we can define $V_{0,1}$ so that $\lambda_*(V_{0,1}) = 2M_1M_2$. Let $\Psi_0 = 1$, and for $k \geq 1$ define $\Psi_k \in H_{8k-4}MSp$ by $\Phi_k = \eta\Psi_k \in E_{8k-3,1}^2 = Z_2\eta \otimes H_*MSp$. Then Landweber-Novikov operations imply that $d^4(V_{m,n}) \equiv 2\nu\Psi_m\Psi_n$ modulo (4ν). Thus, $d^4\lambda_*\left(V_{m,n}\right) \equiv 2\nu\lambda_*\left(\Psi_m\right)\lambda_*\left(\Psi_n\right)$ modulo (4ν). Write $\Psi_{2^k} = Q_{k+1} + D_{k+1}$ where D_{k+1} is a sum of $Q_{k(1)}\cdots Q_{k(r)}$ for $r \geq 3$. Since $[\text{Image}\ d^4] \cap [Z_4(2\nu) \otimes H_*BP] = d^4\left(2H_*BP\right)$, it follows that $d^4\lambda_*\left(V_{[2^s],2^t}\right) \equiv d^4\left(2M_{s+2}M_{t+2} + 2A_{s,t}\right)$ modulo (4ν) for $-1 \leq s < t$ where $A_{s,t}$ is a linear combination of $M_{k(1)}\cdots M_{k(r)}$ for $r \geq 6$. Observe that $\lambda_*\left(v_{[2^s],2^t}\right) \in H_*(BP; Z_2)$ is annihilated by all dual Steenrod operations and hence equals zero. Thus, $\lambda_*\left(V_{[2^s],2^t}\right) \equiv 2M_{s+2}M_{t+2} + 2A_{s,t} + 2K$ modulo (4) where $2K$ is a d^4-cycle. By [6, Corollary (3.3.12)], $K \in Z\{1, M_1, M_2\} \otimes Z\left[\langle M_1^4\rangle,\ \langle M_2^2\rangle,\ \langle M_n\rangle\ \mid\ n \geq 3\right]$. Since $4\lambda_*\left(V_{[2^s],2^t}\right) \in$ Image h, it follows that $8A_{s,t} + 8K \in$ Image h and $A_{s,t} + K$ is divisible by two. Thus, $\lambda_*\left(V_{[2^s],2^t}\right) \equiv 2M_{s+2}M_{t+2}$ modulo (4). □

We prove next that all the leading differentials originating on the 0 row in our range of computation are $d^{8n+4}\left(2^{4n}Q_0^{2n+1}\right) = \beta_n$ and $d^{8n+8}(2^{4n-C(n)+4} Q_0^{2n+2}) = \gamma_n$. The proof verifies that these differentials determine the correct value of $E^\infty_{*,0}$ and that no hidden differentials occur. Recall that a leader $L \in E^r_{n,0}$ in the 0 row is an element of least degree such that $d^r(L) \neq 0$.

THEOREM (2.3). *The following results are valid through degree 48.*

(a) *The leaders on the 0 row are* $2^{4n}Q_0^{2n+1}$ *and* $2^{4n-C(n)+4}Q_0^{2n+2}$.

(b) $d^{8n+4}\left(2^{4n}Q_0^{2n+1}\right) = \beta_n$ *and* $d^{8n+8}\left(2^{4n-C(n)+4}Q_0^{2n+2}\right) = \gamma_n$.

(c) $d^{4r}\left(E^{4r}_{*,0}\right) \subset I \otimes H_*MSp$ *for* $r \geq 1$.

Proof. Let $\mu : S \to MSp$ denote the unit of the spectrum MSp. Observe that each of the β_n or γ_n is zero in π_*MSp because the image of its representative in $E_2^{4k+3,p}$ under the map of Adams spectral sequences induced by μ is in such high filtration degree p that the ASS of MSp is zero there. Hence all of the β_n and γ_n must be boundaries in our spectral sequence. Since they bound from the 0 row in the BP spectral sequence, they must bound from the 0 row in our spectral sequence. Recall [6, Corollary (4.3.6)] that in the BP spectral sequence, $d^{8n+4}(2^{4n}M_1^{4n+2}) = \beta_n$ and $d^{8n+8}\left(2^{4n-C(n)+4}M_1^{4n+4}\right) = \gamma_n$. Since $\lambda_*(Q_0) = M_1^2$, the only possibility for the β_n and γ_n to bound in our spectral sequence is given by (b). It remains to show that there are no other leaders on the 0-row. Assume that $L \in E^{4r}_{4m,0}$ is a leader of least degree with $d^{4r}(L) = \xi X \neq 0$ where $\xi \in \pi^S_{4r-1}$ such that $\xi \notin I_{4r-1}$ and $X \in H_{4m-4r}MSp$. Under the assumption that this theorem is true we make the following three observations.

(1) A routine tedious computation, summarized in Table 1, shows that the kernel of all the differentials determined by these leaders equals $E^\infty_{*,0}$.

(2) Landweber-Novikov operations show that ξX is a leader.

(3) Either ξX is a nonbounding infinite cycle representing an element of π_*MSp or ξX supports a nonzero differential $d^{4s}(\xi X) = \zeta Y$ with $\zeta \in \pi^S_*$ but $\zeta \notin I$ and $Y \in H_*MSp$. Moreover, there is a $u > 0$ such that L survives to $E^{4r+4u}_{4n,0}$ with $d^{4r+4u}(L) = \mu W$ where $\mu \in I$ and $W \in H_*MSp$.

In the latter case of (3), the correct result is that μW is a nonzero leader with $d^{4s-4u}(\mu W) = \zeta Y$. Since $\nu\beta_n = 0$ for $n \geq 1$ and $\nu\gamma_n = 0$ for $n \geq 0$, we must have $4s \geq 12$. Since this theorem is true in degrees less than $4r - 1$, the computations of the remainder of this section and of Section 3 can proceed in these degrees to determine the structure of this spectral sequence as depicted in the tables of Section 4. From those tables we see that the only possibilities for ξX are: $2\nu C[20]Q_0V_2$, $A[19]Q_0^2\overline{Q}_1$ and $\eta A[30]Q_0\overline{Q}_1$. Since $\lambda_*(2\nu C[20]Q_0V_2) = 4\nu C[20]M_1^3M_2$ and $d^{12}\left(4\nu C[20]M_1^3M_2\right) = A[14]C[20]$, it follows that $2\nu C[20]Q_0V_2$ can not bound and $\xi X \neq 2\nu C[20]Q_0V_2$. If $\xi X = A[19]Q_0^2\overline{Q}_1$ then

$$W \in \text{Span}\,\{Q_0Q_1,\ Q_0^4,\ Q_0^2V_2,\ V_2^2,\ V_4, Q_1,\ Q_0V_2,\ Q_0^3\}.$$

Let $s_{k\Delta_n}$ denote the Landweber–Novikov operation s_I defined in [9] where I is the sequence whose only nonzero entry is k in the n^{th} position. Thus, $W \in \{s_{\Delta_1}(W_1),\ s_{2\Delta_1}(W_2),\ s_{\Delta_3}(W_3),\ s_{\Delta_1}(W_4)\}$. Let $C_{A[19]}$ denote the mapping cone of a representative map of $A[19] \in \pi_{19}^S$. In the Atiyah-Hirzebruch spectral sequence for $C_{A[19]*}(MSp)$, $\mu W_1 = d^{20+4u}(L_1)$, $\mu W_2 = d^{20+4u}(L_2)$, $\mu W_3 = d^{20+4u}(L_3)$ or $\mu W_4 = d^{20+4u}(L_4)$. Thus, in our spectral sequence $s_{\Delta_1}d^{20}(L_1)$, $s_{2\Delta_1}d^{20}(L_2)$, $s_{\Delta_3}d^{20}(L_3)$ or $s_{\Delta_1}d^{20}(L_4)$ equals $d^{20}(L)$ which equals $A[19]Q_0^2\overline{Q}_1$, an impossibility. Thus $\xi X \neq A[19]Q_0^2\overline{Q}_1$. If $\xi X = \eta A[30]Q_0\overline{Q}_1$ then $W \in \text{Span}\,\{1,\ Q_0,\ Q_0^2,\ V_{0,1},\ Q_1,\ Q_0^3,\ Q_0V_{0,1}\}$. An argument analogous to the previous one using s_{Δ_1}, s_{Δ_2} and $C_{\eta A[30]}$ produces an L_1 or L_2 with $s_{\Delta_1}d^{32}(L_1)$ or $s_{\Delta_2}d^{32}(L_2)$ equal to $d^{32}(L)$ which equals $\eta A[30]Q_0\overline{Q}_1$, an impossibility. Thus, $\xi X \neq \eta A[30]Q_0\overline{Q}_1$. Thus there is no possibility for ξX, and $(a), (c)$ must be true. □

In Table 1, $G^N = \log_2[\text{order } E^2_{4N,0}/E^\infty_{4N,0}]$. The entry in the row with $N = 4k$ and the column labeled by $\xi \in \pi^S_{4r-1}$ equals $\log_2[\text{order } d^{4r}\left(E^{4r}_{4k,0}\right)]$.

In view of the extra factors of two which arise when $\lambda_*$is applied to a monomial in the $V_{a,b}$, it follows that $\lambda^2 : I \otimes H_*MSp \to I \otimes H_{4*}BP$ is neither one-to-one nor onto. Nevertheless, elements of $(\text{Im}\,J \otimes H_*BP)_{4*-1}$ which do not bound in the BP spectral sequence (even those not in $I \otimes H_*BP$) determine elements of $I \otimes H_*MSp$ which do not bound in our spectral sequence. For $I = (i(1), \ldots, i(t))$ let $M_I = M_1^{i(1)} \cdots M_t^{i(t)}$ and $Q_I = Q_0^{i(1)} \cdots Q_{t-1}^{i(t)}$.

$4N$	G^N	ν	σ	β_1	γ_1	β_2	γ_2	β_3	γ_3	β_4	γ_4	β_5	γ_5
4	3	3											
8	6	2	4										
12	13	6	4	3									
16	21	6	7	3	5								
20	36	14	8	6	5	3							
24	58	16	16	9	10	3	4						
28	91	29	19	15	15	6	4	3					
32	134	33	30	20	25	9	8	3	6				
36	204	57	39	31	35	15	12	6	6	3			
40	291	69	60	40	53	21	20	9	12	3	4		
44	419	105	75	62	70	33	28	15	18	6	4	3	
48	585	129	106	80	105	45	44	21	30	9	8	3	5

TABLE 1: Image of the $d^{4r}\left(E^{4r}_{*,0}\right)$

THEOREM (2.4). *Let $1 \le n$, $2 \le k$, $1 \le s < t$ and $0 \le e$ below.*

(a) *If $\alpha_n \overline{M}_k M_{2I}$ does not bound then $2^{C(n)-1}\gamma_{n-1}V_{0,2^k-2}Q_I$ does not bound.*

(b) *If $\eta^2\gamma_{n-1}M_1M_{2I}$ does not bound and is not homologous to an element of $Z_2\alpha_n \otimes H_*BP$ then either:*

(i) *$4\beta_{n-1}V_{0,1}Q_I$ does not bound or*

(ii) *$d^{8n-4}(X) = 4\beta_{n-1}V_{0,1}Q_I$ where*
$\lambda^{8n-4}(X)$ survives to $E^{8n}_{*,0}$ and $d^{8n}\lambda^{8n-4}(X) \ne 0$.

(c) (i) *If $2^e\beta_n M_{2I}$ does not bound then $2^e\beta_n Q_I$ does not bound.*

(ii) *If $2^{e+1}\beta_n\overline{M}_s\overline{M}_t M_{2I}$ does not bound then $2^e\beta_n Q_I V_{[2^s-2],2^t-2}$ does not bound.*

(iii) *If $\beta_n\overline{M}_s\overline{M}_t M_{2I}$ does not bound and i_1 is even then either:*

(α) *$2^{C(n)-1}\gamma_{n-1}Q_0Q_IV_{[2^s-2],2^t-2}$ does not bound or*

(β) *$d^{8n}(X) = 2^{C(n)-1}\gamma_{n-1}Q_0Q_IV_{[2^s-2],2^t-2}$ where*
$\lambda^{8n}(X)$ survives to $E^{8n+2}_{*,0}$ and $d^{8n+2}\lambda^{8n}(X) \ne 0$.

(d) (i) *If $2^e\gamma_{n-1}M_{2I}$ does not bound then $2^e\gamma_{n-1}Q_I$ does not bound.*

(ii) *If $2^{e+1}\gamma_{n-1}\overline{M}_s\overline{M}_t M_{2I}$ does not bound then $2^e\gamma_{n-1}Q_IV_{[2^s-2],2^t-2}$ does not bound.*

Proof. For $I = (i(1), \ldots, i(t))$ let $\tau(I) = \Delta_{2^{i(1)}-1} + \cdots + \Delta_{2^{i(t)}-1}$.
(a) Observe that $d^{8n+2}\lambda_{8n+2}\left(2^{4n-1}Q_0^{2n}V_{0,2^k-2}\right) = \alpha_n\overline{M}_k$ and $d^{8n}\left(2^{4n-1}Q_0^{2n}V_{0,2^k-2}\right) = 2^{C(n)-1}\gamma_{n-1}V_{0,2^k-2}$. If

$$d^{8n}(X) = 2^{C(n)-1}\gamma_{n-1}V_{0,2^k-2}Q_I$$

then $d^{8n}\lambda^{8n}(X) = 0$ and $\lambda^{8n}(X)$ survives to $'E^{8n+2}_{*,0}$. Since $s_{\tau(I)}(X) = 2^{4n-1}Q_0^{2n}V_{0,2^k-2}$ modulo $E^{8n+4}_{*,0}$ it follows that $r_{2\tau(I)}\lambda^{8n}(X) = 2^{4n}M_1^{4n+1}M_k$ modulo $'E^{8n+4}_{*,0}$ and $d^{8n+2}\lambda^{8n}(X) = \alpha_n\overline{M}_kM_{2I}$, a contradiction.
(b) Observe that $d^{8n+2}\lambda^{8n+2}\left(2^{4n-2}\left(Q_0^{2n-1}V_{0,1} + 2Q_0^{2n+1}\right)\right) = \eta^2\gamma_{n-1}M_1$ and $d^{8n-4}\left(2^{4n-2}\left(Q_0^{2n-1}V_{0,1} + 2Q_0^{2n+1}\right)\right) = 4\beta_{n-1}V_{0,1}$. If $d^{8n-4}(X) = 4\beta_{n-1}V_{0,1}Q_I$ then $d^{8n-4}\lambda^{8n-4}(X) = 0$. Assume that X can be chosen so that $\lambda^{8n}(X)$ is a d^{8n}-cycle. Since $s_{\tau(I)}(X) = 2^{4n-2}Q_0^{2n-1}V_{0,1}$ modulo $E^{8n}_{*,0}$ it follows that $d^{8n+2}\lambda^{8n-4}(X) = \eta^2\gamma_{n-1}M_1M_{2I}$, a contradiction.
(c) (i) Since $\lambda^{8n+4}\left(2^e\beta_nQ_I\right) = 2^e\beta_nM_{2I}$ it follows that $2^e\beta_nQ_I$ can not bound.
(ii) Since $\lambda^{8n+4}\left(2^e\beta_nQ_IV_{[2^s-2],2^t-2}\right) = 2^{e+1}\beta_n\overline{M}_s\overline{M}_t\overline{M}_{2I}$ it follows that $2^e\beta_nQ_IV_{[2^s-2],2t-2}$ can not bound.
(iii) Observe that $d^{8n+4}\lambda^{8n}\left(2^{4n-1}Q_0^{2n+1}V_{[2^s-2],2t-2}\right) = \beta_n\overline{M}_s\overline{M}_t$ and $d^{8n}\left(2^{4n-1}Q_0^{2n+1}V_{[2^s-2],2^t-2}\right) = 2^{C(n)-1}\gamma_{n-1}Q_0V_{[2^s-2],2t-2}$. If $d^{8n}(X) = 2^{C(n)-1}\gamma_{n-1}Q_0Q_IV_{[2^s-2],2^t-2}$ then $d^{8n}\lambda^{8n}(X) = 0$. Assume that $d^{8n+2}\lambda^{8n}(X) = 0$. Since $s_{\tau(I)}(X) = 2^{4n-1}Q_0^{2n+1}V_{[2^s-2],2^t-2}$ modulo $E^{8n}_{*,0}$ it follows that $r_{2\tau(I)}\left(\lambda^{8n}(X)\right) = 2^{4n}M_1^{4n+2}\overline{M}_s\overline{M}_t + \cdots$ and $d^{8n+4}\lambda^{8n}(X) = \beta_n\overline{M}_s\overline{M}_tM_{2I}$, a contradiction.
(d) (i) Since $\lambda^{8n}\left(2^e\gamma_{n-1}Q_I\right) = 2^e\gamma_{n-1}M_{2I}$ it follows that $2^e\gamma_{n-1}Q_I$ can not bound.
(ii) Since $\lambda^{8n}\left(2^e\gamma_{n-1}Q_IV_{[2^s-2],2^t-2}\right) = 2^{e+1}\gamma_{n-1}\overline{M}_s\overline{M}_tM_{2I}$ it follows that $2^e\gamma_{n-1}Q_IV_{[2^s-2],2^t-2}$ can not bound. □

Notes: (1) This theorem covers all cases which arise through degree 50.
(2) In the BP spectral sequence, $\alpha_1M_1^6\overline{M}_2$ is a leader which by this theorem implies that $8\sigma Q_0^3V_{0,1}$ does not bound. However, $8\sigma Q_0^3V_{0,1}$ is not a leader as $d^{12}(8\sigma Q_0^3V_{0,1}) = 4C[18]V_{0,1}$. In fact, the leader in this bidegree is $\sigma Q_0V_{0,1}^2$.

3. Differentials originating on higher rows

We determine E^8 of our spectral sequence in Theorem (3.3) by showing that d^4 is multiplication by ν. For elements of order two we determine Kernel d^8 and Image d^8 in Theorem (3.5) as well as Kernel d^{12} and Image d^{12} in

Theorem (3.6). The key to these computations is the determination of polynomial generators of H_*MSp which are d^4, d^8 or d^{12} cycles modulo two. We begin by showing that H_*MSp has polynomial generators in degrees greater than four which are d^4-cycles.

LEMMA (3.1). (a) *For $n \geq 1$, there are choices $\overline{Q}_n$ of Q_n modulo decomposable elements which are d^4-cycles. In addition, $\overline{V}_{0,1} = \langle Q_0^2 \rangle = Q_0^2 - V_{0,1}$ is a d^4-cycle.*
(b) *There are choices $\overline{V}_{a,b}$ of $V_{a,b}$ and $\overline{V}_{2n}$ of V_{2n} for $n \neq 2^t$ which are d^4-cycles.*

Proof. Since the cell of MSp of degree 4 is attached to the bottom cell of MSp of degree 0 by ν, it follows that $d^4(Q_0) = \nu$. Since $\lambda^4 d^4(V_{0,1}) = d^4(2M_1M_2) = 2\nu M_1^2 = \lambda^4(2\nu Q_0)$ it follows that $d^4(V_{0,1}) = 2\nu Q_0$ and $\langle Q_0^2\rangle$ is a d^4-cycle. Let $I = (0, e_1, \ldots, e_s)$ and $V_J = V_{0,2}^{f(0,2)} V_{1,2}^{f(1,2)} V_6^{f(6)} \cdots$. Observe that:

(i) $d^4\left(Q_0\langle Q_0^2\rangle^e Q_I V_{0,1}^f V_J\right) = \nu\langle Q_0^2\rangle^e Q_I V_{0,1}^f V_J + \cdots$;

(ii) $d^4\left(V_{0,1}\langle Q_0^2\rangle^e Q_I V_{0,1}^{2f} V_J\right) = 2\nu Q_0\langle Q_0^2\rangle^e Q_I V_{0,1}^{2f} V_J + \cdots$;

(iii) $d^4\left(V_{0,1}^2\langle Q_0^2\rangle^e Q_I V_{0,1}^{4f} V_J\right) = 4\nu Q_0 V_2\langle Q_0^2\rangle^e Q_I V_{0,1}^{4f} V_J + \cdots$;

(iv) $d^{12}\left(2\nu Q_0 V_{0,1}\langle Q_0^2\rangle^e Q_I V_{0,1}^{2f} V_J\right) = A[14]\langle Q_0^2\rangle^e Q_I V_{0,1}^{2f} V_J + \cdots$;

(v) $4\nu Q_0 V_{0,1}^3 = \Phi_1^2\Phi_3 \in MSp_{31}$.

It follows from (v) that $Z_2\left(4\nu Q_0 V_{0,1}^3\right) \otimes Z_2[Q_0^2,\ Q_1, \ldots] \otimes Z_2[V_{0,1}^4,\ V_{0,2}, \ldots]$ can contain no d^4-boundaries. Thus, all nonzero d^4-boundaries are given by (i)–(iii). Now (a) and (b) follow from the observation that $d^4(V_m)$ for $m \geq 4$ and $d^4(Q_n)$ for $n \geq 1$ are sums of boundaries given in (i)–(iii). □

In describing the d^4, d^8 and d^{12}-differentials we will use the following subalgebras of H_*MSp as well as the algebra B defined in Lemma (3.4).

Definition (3.2)

$$
\begin{aligned}
A &= Z\left[\langle Q_0^2\rangle,\ \overline{Q}_n \mid n \geq 1\right].\\
C &= Z[Q_n \mid n \geq 0].\\
S &= Z[V_{a,b},\ V_{2n} \mid 0 \leq\ a < b,\ n \neq 2^t].\\
S_1 &= Z[V_{0,1}^2,\ \overline{V}_{a,b},\ \overline{V}_{2n} \mid 0 \leq a < b,\ (a,b) \neq (0,1),\ n \neq 2^t].\\
S_2 &= Z[V_{0,1}^4,\ \overline{V}_{a,b},\ \overline{V}_n \mid 0 \leq a < b,\ (a,b) \neq (0,1),\ n \neq 2^t].\\
T &= Z[V_{0,2^k}^2,\ V_{2n},\ [V_{2n-1}] \mid k \geq 0,\ n \neq 2^t]
\end{aligned}
$$

$$\textit{where } \Phi_n = \eta[V_{2n-1}] \textit{ in } E^{\infty}_{8n-4,1} \textit{ for } n \neq 2^t.$$

The following theorem describes Kernel d^4 and Image d^4 in all cases thereby determining E^8. Let $Z_\infty = Z$.

THEOREM (3.3). *Assume that* $d^4\ (\xi Q_0) = \zeta$ *where* ξ *has order* M *with* $2 \leq M \leq \infty$.

(a) *Then* $\zeta = \nu\xi$.

(b) *If* ζ *has order two then*

(i) $[\mathit{Kernel}\ d^4] \cap [Z_M\ \xi \otimes H_*MSp] = Z_M\ \xi \otimes A \otimes S$ *and*

(ii) $[\mathit{Image}\ d^4] \cap [Z_2\ \zeta \otimes H_*MSp] = Z_2\ \zeta \otimes A \otimes S.$

(c) *If* ζ *has order four then*

(i) $[\mathit{Kernel}\ d^4] \cap [Z_M\ \xi \otimes H_*MSp] = [Z_M\ \xi \otimes A \otimes S_1]$

$$\oplus[Z_{M/2}\ (2\xi V_2) \otimes A \otimes S_1]$$

$$\oplus[Z_{M/4}\ (4\xi Q_0) \otimes A \otimes S] \textit{ and}$$

(ii) $[\mathit{Image}\ d^4] \cap [Z_2\ \zeta \otimes H_*MSp] = [Z_4\ \zeta \otimes A \otimes S] \oplus [Z_2\ (\zeta Q_0) \otimes A \otimes S_1].$

(d) *If* ζ *has order eight then*

(i) $[\mathit{Kernel}\ d^4] \cap [Z_M\ \xi \otimes H_*MSp] = [Z_M\ \xi \otimes A \otimes S_2]$

$$\oplus[Z_{M/2}\ \left(2\xi V_{0,1}^2\right) \otimes A \otimes S_2]$$

$$\oplus[Z_{M/4}\ (4\xi V_{0,1}) \otimes A \otimes S_1] \oplus [Z_{M/8}\ (8\xi Q_0) \otimes A \otimes S] \textit{ and}$$

(ii) $[\mathit{Image}\ d^4] \cap [Z_2\ \zeta \otimes H_*MSp] = [Z_8\ \zeta \otimes A \otimes S] \oplus [Z_4\ (2\zeta Q_0) \otimes A \otimes S_1]$

$$\oplus[Z_2\ (4\zeta Q_0 V_{0,1}) \otimes A \otimes S_2].$$

Proof. These computations follow from Lemma (3.1) and its proof. □

The next lemma determines polynomial generators $\langle Q_n \rangle$ for $n \geq 2$ of H_*MSp which will be used to describe the d^8 and d^{12}-differentials.

LEMMA (3.4). *For each* $n \geq 2$, *one can choose an element* $\langle Q_n \rangle \in H_*MSp$ *differing from* Q_n *by a decomposable element such that for any* b *of degree* k *in the ring*

$$B = Z\left[\langle Q_0^2 \rangle^2,\ \overline{Q}_1^2,\ \langle Q_n \rangle \mid n \geq 2\right]$$

and any operation s_ω *of degree* $k-4$ *or* $k-8$, *the class* $s_\omega(b)$ *is divisible by 2.*

Proof. Observe that

$$\begin{aligned}
d^4\left(A[32,3]Q_0\right) &= \nu A[32,3] \neq 0,\\
d^8\left(A[32,3]V_{0,1}\right) &= 0,\\
d^8\left(A[32,3]Q_0^2\right) &= \sigma A[32,3] \neq 0,\\
d^4\left(\sigma A[32,3]Q_0\right) &= 0,\\
d^8\left(\sigma A[32,3]V_{0,1}\right) &= 0 \text{ and}\\
d^8\left(\sigma A[32,3]Q_0^2\right) &= \sigma^2 A[32,3] = \eta^2 C[44] \neq 0.
\end{aligned}$$

We use induction on $n \geq 2$ to show that the $\langle Q_n\rangle$ exist. Assume that $k \geq 2$ and $\langle Q_k\rangle$ exists for $k < n$. By Lemma (3.1),

$$\text{Kernel } d^4 \mid Z_2A[32,3] \otimes H_*MSp = Z_2A[32,3] \otimes A \otimes S.$$

Then through degree $2^{k+3}+28$: Kernel $d^8 \mid Z_2\sigma A[32,3] \otimes H_*MSp$

$$\begin{aligned}
&= Z_2\sigma A[32,3]\{1,\ Q_0,\ Q_1\} \otimes Z_2\left[\langle Q_0^2\rangle,\ \overline{Q}_1^2,\ \langle Q_2\rangle,\ldots,\langle Q_{n-1}\rangle\right] \otimes S\\
&= \text{Image } d^8 \mid Z_2A[32,3] \otimes D_n
\end{aligned}$$

where $D_n = Z_2\{1, Q_0^2,\ \overline{Q}_1,\ Q_0^2\overline{Q}_1\} \otimes Z_2\left[\langle Q_0^2\rangle,\ \overline{Q}_1^2,\ \langle Q_2\rangle,\ldots,\langle Q_{n-1}\rangle\right] \otimes S$. Thus, $d^8\left(A[32,3]\overline{Q}_n\right) = d^8\left(A[32,3]\delta_n\right)$ with $\delta_n \in D_n$. Define $\langle Q_n\rangle = \overline{Q}_n + \delta_n$. Let degree s_ω equal $2^{n+3}-8$ or $2^{n+3}-12$. Since $A[32,3]\langle Q_n\rangle$ survives to E^{12}, it follows that $s_\omega\left(\langle Q_n\rangle\right)$ can not contain an odd multiple of Q_0 or Q_0^2 as a summand. Since $\langle Q_n\rangle$ is a polynomial of d^4-cycles, it follows that $s_\omega\left(\langle Q_n\rangle\right)$ can not contain an odd multiple of $V_{0,1}$ as a summand. □

We now determine Kernel d^8 and Image d^8 for elements of order two.

THEOREM (3.5). *Assume that ξ and ζ both have order two.*

(a) *If $d^8\left(\xi Q_0^2\right) = \zeta$ and $d^8\left(\xi V_{0,1}\right) = 0$ then $\zeta = \sigma\xi$.*

(i) *If $\nu\xi = 0$ then Kernel $d^8 = Z_2\ \xi\{1,\ Q_0,\ Q_1\} \otimes B \otimes S$ and Image $d^8 = Z_2\ \zeta\{1,\ Q_0,\ Q_0^2,\ Q_1,\ Q_0\overline{Q}_1\} \otimes B \otimes S$.*

(ii) *If $\nu\xi \neq 0$ then Kernel $d^8 = Z_2\ \xi \otimes B \otimes S$ and Image $d^8 = Z_2\ \zeta\{1,\ Q_0,\ Q_1\} \otimes B \otimes S$.*

(b) *If $d^8\left(\xi V_{0,1}\right) = \zeta$ and $d^8\left(\xi Q_0^2\right) = 0$ then $\zeta = \sigma\xi$.*

(i) *If $\nu\xi = 0$ then Kernel $d^8 = Z_2\ \xi \otimes C \otimes T$ and Image $d^8 = Z_2\ \zeta \otimes H_*MSp$.*

(ii) *If* $\nu\xi \neq 0$ *then Kernel* $d^8 = Z_2\, \xi \otimes A \otimes T$ *and Image* $d^8 = Z_2\, \zeta \otimes A \otimes S$.

(c) *If* $d^8\left(\xi V_{0,1}\right) = \zeta$ *and* $d^8\left(\xi Q_0^2\right) = \zeta$ *then one can choose an element* $\langle V_{a,b}\rangle$, $\langle V_{2n}\rangle$ *in* H_*MSp *differing by a decomposable element from* $V_{a,b}$, V_{2n}, *respectively, defining*

$$\langle S\rangle = Z\left[\langle V_{a,b}\rangle,\ \langle V_{2n}\rangle \mid 0 \le a < b,\ n \neq 2^t\right]$$

such that:

(i) *if* $\nu\xi = 0$ *then Kernel* $d^8 = Z_2\, \xi\{1,\ Q_0,\ Q_1,\ Q_0\overline{Q}_1\} \otimes B \otimes \langle S\rangle$ *and Image* $d^8 = Z_2\, \zeta\{1,\ Q_0,\ Q_1,\ Q_0\overline{Q}_1\} \otimes B \otimes \langle S\rangle$;

(ii) *if* $\nu\xi \neq 0$ *then Kernel* $d^8 = Z_2\, \xi\{1,\ \overline{Q}_1\} \otimes B \otimes \langle S\rangle$ *and Image* $d^8 = Z_2\, \zeta\{1,\ \overline{Q}_1\} \otimes B \otimes \langle S\rangle$.

Proof. By Lemma (3.4), d^8 restricted to those elements with representatives in $Z_2\xi \otimes H_*MSp$ is a homomorphism of B-modules. Since the $\overline{Q}_n$, $n \geq 1$, are d^4-cycles, it is impossible for $s_\omega(\overline{Q}_n)$ to be an odd multiple of $V_{0,1}$. Thus in (b), d^8 on $Z_2\xi \otimes H_*MSp$ is a map of A-modules. Note that $Q_0^2 - V_{0,1}$ survives to E^8 and $d^8\left(Q_0^2 - V_{0,1}\right) = \sigma$. Thus, $\zeta = \sigma\xi$ in cases (a) and (b). In (c), let $\langle V_k\rangle$ denote $\langle V_{a,b}\rangle$ or $\langle V_{2n}\rangle$. We establish the existence of the $\langle V_n\rangle$ by induction on $4n = \deg\langle V_n\rangle$ such that if degree s_ω equals $4n-4$, $4n-8$ then $s_\omega\left(\langle V_n\rangle\right) \equiv 0 \bmod (2)$, $s_\omega\left(\langle V_n\rangle\right) \equiv k\left(Q_0^2 + V_{0,1}\right) \bmod (2)$, respectively. Define $\langle V_{0,1}\rangle = V_{0,1} + Q_0^2$. Observe that

$$\begin{aligned}
d^4\left(A[14]Q_0\right) &= \nu A[14] \neq 0,\\
d^8\left(A[14]Q_0^2\right) &= d^8\left(A[14]V_{0,1}\right) = \eta C[20] \neq 0,\\
d^4\left(\eta C[20]Q_0\right) &= 0 \text{ and}\\
d^8\left(\eta C[20]Q_0^2\right) &= d^8\left(\eta C[20]V_{0,1}\right) = A[8]C[20] \neq 0.
\end{aligned}$$

Assume that the $\langle V_k\rangle$ have been chosen for $2 \leq \deg\langle V_k\rangle < \deg\langle V_n\rangle$. Let $n' = n-2$ if n is a power of two, and let $n' = n-1$ otherwise. Let $D_n = Z_2\left[\langle V_k\rangle \mid \deg\langle V_k\rangle < \deg\langle V_{n'}\rangle\right]$. Then through degree $4n+13$,

$$\text{Kernel } d^8 \mid Z_2\eta C[20] \otimes H_*MSp = Z_2\, \eta C[20]\{1,\ Q_0,\ Q_1,\ Q_0\overline{Q}_1\} \otimes B \otimes D_n.$$

In addition,

$$\text{Image } d^8 \mid Z_2 A[14] \otimes B \otimes D_n = Z_2\, \eta C[20]\{1,\ Q_1\} \otimes B \otimes D_n.$$

Since $\eta C[20]Q_0$ is not a d^8-boundary, zero is the only d^8-boundary in $Z_2\eta C[20]\{Q_0,\ Q_0\overline{Q}_1\} \otimes B \otimes D_n$. Thus, we can find $\delta_n \in D_n$ such that

$d^8(A[14]\delta_n) = d^8\left(A[14]\overline{V}_n\right)$. Define $\langle V_n\rangle = \overline{V}_n + \delta_n$. Since $\langle V_n\rangle$ is a d^4-cycle and a d^8-cycle modulo two, the $s_\omega(\langle V_n\rangle)$ have the desired property. The computation of Kernel d^8 and Image d^8 in (a), (b), (c) are straightforward. □

We now determine Kernel d^{12} and Image d^{12} for the elements of order two in our range of computation.

THEOREM (3.6).

(a) *Assume that* $\nu\xi = 0$ *with* $d^8\left(\xi Q_0^2\right) = d^8(\xi V_{0,1}) = 0$, $d^{12}\left(\xi Q_0^3\right) = \zeta$ *and* $d^{12}(\xi Q_1) = d^{12}(\xi Q_0 V_{0,1}) = 0$. *Then* *Kernel* $d^{12} = Z_2\,\xi\{1,\ Q_0,\ Q_0^2, Q_1,\ Q_0Q_1\} \otimes B \otimes S$ *and* *Image* $d^{12} = Z_2\,\xi\{1,\ \langle Q_0^2\rangle,\ \overline{Q}_1\} \otimes B \otimes S$.

(b) *Assume that* $\nu\xi = 0$ *with* $d^8\left(\xi Q_0^2\right) = d^8(\xi V_{0,1}) = 0$, $d^{12}\left(\xi Q_0^3\right) = d^{12}(\xi Q_0 V_{0,1}) = \zeta$ *and* $d^{12}(\xi Q_1) = 0$. *Then* *Kernel* $d^{12} = Z_2\,\xi\{1,\ Q_0,\ Q_0^2,\ Q_1,\ Q_0Q_1,\ Q_0^2\overline{Q}_1\} \otimes B \otimes \langle S\rangle$ *and* *Image* $d^{12} = Z_2\,\zeta\{1,\ \langle Q_0^2\rangle,\ \overline{Q}_1,\ \langle Q_0^2\rangle\overline{Q}_1\} \otimes B \otimes S_1$.

(c) *Assume that* $\nu\xi \neq 0$ *with* $d^8\left(\xi Q_0^2\right) = d^8(\xi V_{0,1}) = 0$ *and* $d^{12}\left(\xi\overline{Q}_1\right) = \zeta$. *Then Kernel* $d^{12} = Z_2\,\xi\{1,\ Q_0^2\} \otimes B \otimes S$ *and* *Image* $d^{12} = Z_2\,\zeta\{1,\ Q_0^2\} \otimes B \otimes S$.

(d) *Assume that* $\nu\xi = 0$ *with* $d^8(\xi V_{0,1}) = d^8\left(\xi Q_0^2\right) = 0$, $d^{12}(\xi Q_0 V_{0,1}) = 0$ *and* $d^{12}(\xi Q_1) = d^{12}\left(\xi Q_0^3\right) = \zeta$. *Then* *Kernel* $d^{12} = Z_2\,\xi\{1,\ Q_0,\ Q_0^2,\ \overline{Q}_1,\ \langle Q_0^2\rangle Q_1\} \otimes B \otimes S$ *and* *Image* $d^{12} = Z_2\,\zeta\{1,\ Q_0,\ \overline{Q}_1\} \otimes B \otimes S$.

Proof. These computations are straightforward. □

4. Leaders

Routine computations, using the theorems of Sections 2 and 3, determine the structure of our spectral sequence through dimension 50. The tables below summarize these computations. They display all the leading differentials exept:

$$d^8(\alpha_n V_{0,1}) = \eta\gamma_n \text{ and } d^8(\eta\alpha_n V_{0,1}) = \eta^2\gamma_n$$

for $n \geq 1$. They also include all the leaders except $\alpha_n = \eta q_0^n$ and $\eta\alpha_n = \eta^2 q_0^n$ for $n \geq 1$. Each of the leaders $d^r(XM) = YN$ with $X, Y \in \pi_*^S$ and $M, N \in H_*MSp$ given below determines differentials $d^r(Xm) = Yn$ in higher degrees using Landweber–Novikov operations. The elements of $\pi_*MSp = E^\infty$ are represented by their leaders, those elements of π_*MSp which are annihilated by all Landweber-Novikov operations. In our range of computations, these leaders are:

$$\begin{array}{rcllrcl} 2^n & \in & MSp_0 \text{ for } n \geq 0, & & \eta = \Phi_0 & \in & MSp_1, \\ \eta^2 = \Phi_0^2 & \in & MSp_2, & & \Phi_1^2\Phi_3 & \in & MSp_{31}, \\ R(0,1;0,2) & \in & MSp_{47} \text{ and} & & \Phi_1\Phi_2[\Phi_1 V_6 + \Phi_2 V_{0,1}^2] & \in & MSp_{47}. \end{array}$$

Note that the first of these leaders represents all elements which project to a nonzero element of $MSp_*/\text{Torsion}$ while the other five leaders represent Torsion MSp_* in our range of computations. The first 49 stable stems are depicted by leading differentials $d^r(XM) = \xi[n,k]$ where $X \in \pi_*^S$, $M \in H_*MSp$ and $\xi[n,k]$ denotes the k^{th} element of degree n of order 2, 4, 8, 16 if ξ equals A, B, C, D, respectively. If there is only one such element then we denote $\xi[n,k]$ by $\xi[n]$.

$$\begin{array}{cccccccccc} \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} & \underline{10} \\ \eta & \eta^2 & \nu \longleftarrow & Q_0 & & \nu^2 \longleftarrow & \nu Q_0 & \eta\sigma + A[8] \longleftarrow & \eta V_{0,1} & \\ & & 2\nu \longleftarrow & 2Q_0 & & & & A[8] \longleftarrow & \eta Q_0^2 & \\ & & 4\nu \longleftarrow & 4Q_0 & & & \sigma \longleftarrow & Q_0^2 & \eta^2\sigma \longleftarrow & \eta^2 V_{0,1} \\ & & & & & & 2\sigma \longleftarrow & 2Q_0^2 & \eta A[8] \longleftarrow & \nu^2 Q_0 \\ & & & & & & 4\sigma \longleftarrow & 4Q_0^2 & & \\ & & & & & & 8\sigma \longleftarrow & 8Q_0^2 & & \end{array}$$

$$\begin{array}{ccccccc} \underline{13} & \underline{14} & \underline{15} & \underline{16} & \underline{17} & \underline{18} & \underline{19} \\ \eta A[8]Q_0 \longleftarrow & \eta^2 Q_0^3 & \eta A[14] \longleftarrow & A[8]V_{0,1} & \nu A[14] \longleftarrow & A[14]Q_0 & \\ & \sigma^2 \longleftarrow & \sigma Q_0^2 & A[16] \longleftarrow & \eta Q_0^4 & C[18] \longleftarrow & 2\sigma Q_0^3 \\ & A[14] \longleftarrow & 2\nu Q_0 V_{0,1} & & \eta A[16] \longleftarrow & \eta^2 Q_0^4 & A[19] \longleftarrow \\ & & & & & 2C[18] \longleftarrow & 4\sigma Q_0^3 \\ & & & & & 4C[18] \longleftarrow & 8\sigma Q_0^3 \end{array}$$

20	21	22	23	24	25
$C[20] \longleftarrow$	$\eta Q_0 V_{0,1}^2$	$\eta^2 C[20] \longleftarrow$	$\eta A[14] V_{0,1}$		$\nu C[18] Q_0 \longleftarrow$
$2C[20] \longleftarrow$	$\eta A[8] Q_0 V_{0,1}$	$\nu A[19] \longleftarrow$	$A[19] Q_0$	$\eta\sigma A[16] \longleftarrow$	$\eta A[16] V_{0,1}$
$\longleftarrow A[8] Q_0 \langle V_{0,1} \rangle$			$\sigma A[16] \longleftarrow$	$A[16] V_{0,1}$	$\eta C[20] Q_0 \longleftarrow$
$4C[20] \longleftarrow$	$\nu A[14] Q_0$		$\nu C[20] \longleftarrow$	$C[20] Q_0$	
	$\nu C[18] \longleftarrow$	$C[18] Q_0$	$2\nu C[20] \longleftarrow$	$2C[20] Q_0$	
	$\eta C[20] \longleftarrow$	$A[14] V_{0,1}$	$4\nu C[20] \longleftarrow$	$4C[20] Q_0$	

26	27	28	29	30
$\longleftarrow \sigma^2 Q_0^3$		$A[8] C[20] \longleftarrow$	$\eta C[20] V_{0,1}$	
$\nu^2 C[20] \longleftarrow$	$\nu C[20] Q_0$			
$\longleftarrow \eta^2 Q_0^2 V_{0,1}^2$				
$\nu A[19] Q_0 \longleftarrow$	$\sigma Q_0 V_{0,1}^2$			$A[30] \longleftarrow$
				$\eta^2 C[20] Q_0^2 \longleftarrow$
				$\nu^2 C[20] Q_0 \longleftarrow$

31	32	33	34
$4\nu Q_0 V_{0,1}^3 = \Phi_1^2 \Phi_3$	$A[32,1] \longleftarrow$	ηQ_0^8	$\nu A[31] \longleftarrow$
$A[31] \longleftarrow$	$C[20] \overline{Q}_1$	$\eta A[32,1] \longleftarrow$	$\eta^2 Q_0^8$
$\eta A[30] \longleftarrow$	$A[16] Q_0 Q_1$	$\eta A[32,2] \longleftarrow$	$\eta^2 C[20] Q_0^3$
$\longleftarrow 2\sigma \left(Q_0^6 + 2Q_0 Q_1 V_{0,1}\right)$	$A[32,2] \longleftarrow$	$\alpha_1 Q_0^6$	$B[34] \longleftarrow$
$\longleftarrow \sigma V_{0,1}^3$	$A[32,3] \longleftarrow$	$\nu^3 Q_0 Q_1 V_{0,1}$	$2B[34] \longleftarrow$
$\longleftarrow \beta_1 Q_0^5$		$\nu A[30] \longleftarrow$	$A[30] Q_0$
			$A[14] C[20] \longleftarrow$

<u>35</u>	<u>36</u>	<u>37</u>	<u>38</u>
$\longleftarrow A[31]Q_0$	$A[36] \longleftarrow$	$\eta C[20]Q_0\overline{Q}_1$	$\nu A[31]Q_0 \longleftarrow$
$\nu A[32,3] \longleftarrow$	$A[32,3]Q_0$	$A[37] \longleftarrow$	$A[30]V_{0,1}$
		$\sigma A[30] \longleftarrow$	$A[30]\langle Q_0^2\rangle$
$\longleftarrow \beta_1 Q_0^3 Q_1$	$A[8]C[20]Q_0^2 \longleftarrow$	$\alpha_1 Q_0^7$	$A[14]C[20]Q_0 \longleftarrow$
$\longleftarrow 2\beta_1 Q_0^3 Q_1$		$\eta A[32,2]Q_0 \longleftarrow$	$\eta\alpha_1 Q_0^7$
$\eta A[14]C[20] \longleftarrow$	$A[32,2]Q_0$	$\nu A[30]Q_0 \longleftarrow$	$C[18]Q_0^2\overline{Q}_1$
$\longleftarrow 2\nu C[20]Q_0V_{0,1}$			$B[38] \longleftarrow$
			$2B[38] \longleftarrow$
			$\eta\sigma A[30] \longleftarrow$

39	40	41	42
$\leftarrow A[19]Q_0^2\overline{Q}_1$	$A[40,1] \longleftarrow$	$\eta A[16]Q_0V_{1,2}$	$\eta^2C[20]Q_0^2\overline{Q}_1 \longleftarrow$
$A[39,1] \longleftarrow$	$A[32,3]Q_0^2$	$\eta C[20]^2 \longleftarrow$	$A[14]C[20]V_{0,1}$
$A[39,2] \longleftarrow$	$A[32,2]V_{0,1}$	$\eta A[40,1] \longleftarrow$	$B[34]Q_0^2$
$\leftarrow 2\beta_1Q_0^5V_{0,1}$	$A[40,2] \longleftarrow$	$\alpha_1Q_0^8$	$\eta^2C[20]^2 \longleftarrow$
$A[39,3] \longleftarrow$	$A[32,1]V_{0,1}$	$\eta A[40,2] \longleftarrow$	$\eta\alpha_1Q_0^8$
$\eta A[14]C[20]Q_0 \longleftarrow$	$A[8]C[20]Q_0^3$		$\eta A[37]Q_0 \longleftarrow$
$\leftarrow \gamma_1Q_0^6$	$C[20]^2 \longleftarrow$	$\eta C[20]Q_0V_{0,1}^2$	
$\leftarrow 2\gamma_1Q_0^6$	$2C[20]^2 \longleftarrow$	$\eta A[32,2]V_{0,1}$	$C[42] \longleftarrow$
$\begin{bmatrix} A[39,3] \\ +\sigma A[32,1] \end{bmatrix} \longleftarrow$	$A[32,1]Q_0^2$		$2C[42] \longleftarrow$ $4C[42] \longleftarrow$
$\eta B[38] \longleftarrow$	$A[36]Q_0$		
$\leftarrow \nu A[32,3]Q_0$	$\eta A[39,3] \longleftarrow$	$\eta A[32,1]V_{0,1}$	
	$\begin{bmatrix} \eta A[39,3] \\ +\eta\sigma A[32,1] \end{bmatrix} \longleftarrow$	$A[37]Q_0$	

<u>43</u>	<u>44</u>	<u>45</u>	<u>46</u>
$\longleftarrow 4\nu Q_0\overline{Q}_1V_{0,1}^3$	$A[36]Q_0^2 \longleftarrow$	$\eta A[8]Q_0Q_1\langle V_{1,2}\rangle$	$\eta^2C[44] \longleftarrow$
$A[31]\overline{Q}_1 \longleftarrow$	$A[8]V_{0,2}V_{1,2}$	$\eta C[44] \longleftarrow$	$\nu A[31]Q_0^3$
	$C[20]^2Q_0 \longleftarrow$	$\eta A[16]Q_0^4Q_1$	$\eta A[45,1] \longleftarrow$
$\longleftarrow A[39,2]Q_0$	$C[44] \longleftarrow$	$\nu C[18]Q_0^3Q_1$	$\eta A[45,2] \longleftarrow$
	$2C[44] \longleftarrow$	$\nu A[30]Q_0^3$	$\eta D[45] \longleftarrow$
$\longleftarrow \eta A[30]Q_0V_{0,1}$	$4C[44] \longleftarrow$	$\sigma A[30]Q_0^2$	$\eta^2C[20]^2Q_0 \longleftarrow$
$\eta B[38]Q_0 \longleftarrow$	$A[8]C[20]Q_0^2V_{0,1}$	$A[45,1] \longleftarrow$	$B[34]Q_0V_{0,1}$
$\longleftarrow \gamma_1Q_0^5V_{0,1}$		$D[45] \longleftarrow$	$2C[18]\left(Q_0^3V_{0,2}+Q_0^5V_{0,1}\right)$
$\longleftarrow 2\gamma_1Q_0^5V_{0,1}$		$2D[45] \longleftarrow$	$4C[18]\left(Q_0^3V_{0,2}+Q_0^5V_{0,1}\right)$
$\longleftarrow 4\gamma_1Q_0^5V_{0,1}$		$4D[45] \longleftarrow$	$B[38]Q_0^2$
		$8D[45] \longleftarrow$	$C[42]Q_0$
	$\left[\begin{array}{l}\eta\sigma A[32,1]Q_0\\ +\eta A[39,3]Q_0\end{array}\right] \longleftarrow$	$\eta A[32,1]Q_0^3$	
		$A[45,2] \longleftarrow$	$\eta^2\left(Q_0Q_1\langle Q_2\rangle+Q_0^7\langle Q_0^4\rangle\right)$

$$
\begin{array}{lclclclcl}
\underline{47} & & \underline{48} & & \underline{49} & & \underline{50} & & \underline{51} \\[4pt]
\longleftarrow A[39,1]Q_0^2 & & \eta A[47] & \longleftarrow & \eta A[40,2]V_{0,1} & & 2C[18]Q_0^3Q_1V_{0,1} & \longleftarrow & 4\nu Q_0^3\overline{Q}_1V_{0,1}^3 \\
\nu C[44] & \longleftarrow & C[44]Q_0 & & 8D[45]Q_0 & \longleftarrow & 2B[38]Q_0^3 & & \\
\longleftarrow A[39,3]V_{0,1} & & \eta B[47] & \longleftarrow & A[45,2]Q_0 & & & & \\
\longleftarrow A[39,2]V_{0,1} & & & & \eta C[20]^2Q_0^2 & \longleftarrow & \eta^2C[20]Q_0^2\overline{Q}_1\langle V_{0,1}\rangle & & \\
\longleftarrow A[31]Q_0^4 & & \nu D[45] & \longleftarrow & D[45]Q_0 & & & & \\
\longleftarrow \eta A[14]C[20]Q_0^3 & & 2\nu D[45] & \longleftarrow & 2D[45]Q_0 & & & & \\
\eta^2D[45] & \longleftarrow & A[36]\overline{Q}_1 & & & & & & \\
A[47] & \longleftarrow & A[40,2]V_{0,1} & & & & & & \\
B[47] & \longleftarrow & C[20]^2V_{0,1} & & & & & & \\
2B[47] & \longleftarrow & 2C[20]^2V_{0,1} & & & & & & \\
R(0,1;0,2) & & \nu A[45,1] & \longleftarrow & A[45,1]Q_0 & & & & \\
\quad = \eta A[30]Q_0\overline{Q}_1 & & & & & & & & \\[8pt]
\Phi_1\Phi_2[\Phi_1V_6+\Phi_2V_{0,1}^2] & & & & & & & & \\
\quad = \beta_1Q_0Q_1V_{1,2} & & & & & & & &
\end{array}
$$

5. Relations and Toda brackets

In our spectral sequence all d^4-differentials are given by multiplication by ν while multiplication by σ determines certain d^8-differentials. Thus, the analysis of our spectral sequence determines the relations in π_*^S given by multiplication by ν and σ. We give two new relations in Theorem (5.1) which are not readily discernible from the *BP* spectral sequence or the classical Adams spectral sequence. We then generalize two theorems from [6, Chapter 2, Section 4] which show how differentials in an Atiyah-Hirzebruch spectral sequence determine Toda brackets in π_*^S. We apply these theorems in Theorem (5.6) to give Toda bracket decompositions of elements of π_*^S which are defined as images of d^8, d^{12} and d^{16}-differentials.

THEOREM (5.1). (a) *$\sigma\alpha_n = \eta\gamma_n$ for $n \geq 0$.*
(b) *$\sigma A[39,2] = 0$.*

Proof. (a) This relation is a consequence of Theorem (3.5)(b). It can also be inferred from the Adams–Novikov spectral sequence.

(b) Since $d^8\left(A[32,2]Q_0^2\right) = d^8\left(A[32,2]V_{0,1}\right) = A[39,2]$, it follows that $A[39,2]Q_0^2 \sim A[39,2]V_{0,1}$ in $E^8_{8,39}$. Thus, $\sigma A[39,2] = d^8\left(A[39,2]\left(Q_0^2 + V_{0,1}\right)\right) = 0$ and $\sigma A[39,2] \in \nu \cdot \pi_{43}^S = 0$. □

The following theorem is a generalization of [6, Theorem (2.4.4)(b),(c)]. Let (B,ω) be a ring spectrum. Let $\phi : S^m \to B$ and $\psi : S^n \to B$ be two maps whose product $\omega \circ (\phi \wedge \psi)$ is null–homotopic. We use the notation $B_{\phi\psi}$ to denote a map $B_{\phi\psi} : D^{m+n+1} \to B$ such that

$$\partial B_{\phi\psi} = B_{\phi\psi} \mid S^{m+n} = \omega \circ (\phi \wedge \psi).$$

THEOREM (5.2). *Let B be a ring spectrum with torsion free homology. Consider the Atiyah-Hirzebruch spectral sequence:*

$$E^r_{n,t} = H_n B \otimes \pi_t^S \Longrightarrow \pi_{n+t} B. \tag{5}$$

Assume that α, β, $\xi \in \pi_*^S$ and X, $Y \in H_*B$ satisfy the following conditions:

(i) $\alpha \cdot \beta = 0$;

(ii) $\alpha \cdot \xi = 0$;

(iii) $d^s(Y) = \beta$;

(iv) $d^r(X) = \alpha Y$.

Then $\langle \xi, \alpha, \beta \rangle$ is defined, ξX survives to E^{r+s} and $\langle \xi, \alpha, \beta \rangle$ contains an element which projects to $d^{r+s}(\xi X)$.

Proof. By (i) and (ii), $\langle \xi, \alpha, \beta \rangle$ is defined. We can represent X, Y by $\underline{X}$, $\underline{Y}$, respectively, such that $\partial \underline{Y} = \beta$ and $\partial \underline{X} = (\alpha \wedge \underline{Y}) \cup B_{\alpha\beta}$. Then we can represent ξX by $R = (\xi \wedge \underline{X}) \cup \left(B_{\xi\alpha} \wedge \underline{Y}\right)$ with $\partial R = \left(\xi \wedge B_{\alpha\beta}\right) \cup \left(B_{\xi\alpha} \wedge \beta\right)$. Thus ξX survives to E^{r+s} and ∂R projects to $d^{r+s}(\xi X)$. Moreover, $\partial R \in \langle \xi, \alpha, \beta \rangle$. □

Theorem (5.2) gives Toda bracket decompositions of all ~~elements~~ $\zeta \in \pi_*^S$ which are defined as $\zeta = d^8\left(\xi V_{0,1}\right) = d^8\left(\xi Q_0^2\right)$ or $\zeta = d^{12}\left(\xi Q_1\right)$ or $\zeta = d^{12}\left(\xi \overline{Q}_1\right)$.

COROLLARY (5.3). (a) *Assume that ξQ_0^2 survives to E^8 and $d^8\left(\xi Q_0^2\right) = d^8\left(\xi V_{0,1}\right) = \zeta$. Then $2\nu\xi = 0$ and $\langle \nu, 2\nu, \xi \rangle$ contains an element which projects to ζ.*

(b) *Assume that $\nu\xi = 0$. Then ξQ_1 survives to E^{12} and $\langle \sigma, \nu, \xi \rangle$ contains an element which projects to $d^{12}\left(\xi Q_1\right)$.*

(c) *Assume that $\sigma\xi = 0$. Then $\xi \overline{Q}_1$ survives to E^{12} and $\langle \nu, \sigma, \xi \rangle$ contains an element which survives to $d^{12}\left(\xi \overline{Q}_1\right)$.*

The following theorem is a generalization of [6, Theorem (2.4.5)].

THEOREM (5.4). *Consider the Atiyah–Hirzebruch spectral sequence (5) of a spectrum B with torsion free homology. Assume that α, β, γ, $\xi \in \pi_*^S$ and X, Y, $Z \in H_*B$ satisfy the following conditions:*

(i) *$\langle\gamma,\xi,\alpha,\beta\rangle$ is defined;*
(ii) *$d^s(Y)=\beta$;*
(iii) *$d^r(X)=\alpha Y$;*
(iv) *$d^t(Z)=\gamma$.*

Then ξXZ survives to E^{r+s+t} and $\langle\gamma,\xi,\alpha,\beta\rangle$ contains an element which projects to $d^{r+s+t}(\xi XZ)$.

Proof. If $\langle\phi,\psi,\lambda\rangle$ contains 0, let $B_{\langle\phi,\psi,\lambda\rangle}$ denotes a map with domain a disc and $\partial B_{\langle\phi,\psi,\lambda\rangle}\in\langle\phi,\psi,\lambda\rangle$. Represent X, Y, Z by $\underline{X}$, $\underline{Y}$, $\underline{Z}$, respectively, such that $\partial\underline{Y}=\beta$, $\partial\underline{Z}=\gamma$ and $\partial\underline{X}=(\alpha\wedge\underline{Y})\cup B_{\alpha\beta}$. Then we can represent ξXZ by

$$R=(\underline{Z}\wedge\xi\wedge\underline{X})\cup(B_{\gamma\xi}\wedge\underline{X})\cup(\underline{Z}\wedge B_{\xi\alpha}\wedge\underline{Y})\cup(B_{\langle\gamma,\xi,\alpha\rangle}\wedge\underline{Y})\cup(\underline{Z}\wedge B_{\langle\xi,\alpha,\beta\rangle}).$$

Since $\partial R=(\gamma\wedge B_{\langle\xi,\alpha,\beta\rangle})\cup(B_{\gamma\xi}\wedge B_{\alpha\beta})\cup(B_{\langle\gamma,\xi,\alpha\rangle}\wedge\beta)\in\langle\gamma,\xi,\alpha,\beta\rangle$, it follows that ξXZ survives to E^{r+s+t} and $\langle\gamma,\xi,\alpha,\beta\rangle$ contains an element which projects to $d^{r+s+t}(\xi XZ)$. □

In our spectral sequence, Theorem (5.4) gives Toda bracket decompositions of all elements $\zeta\in\pi_*^S$ which are defined as $\zeta=d^{12}(\xi Q_0V_{0,1})$ or $\zeta=d^{16}\left(\xi\langle Q_0^2\rangle V_{0,1}\right)$ or $\zeta=d^{16}(\xi Q_0Q_1)$ or $\zeta=d^{20}\left(\xi\langle Q_0^2\rangle Q_1\right)$ or $\zeta=d^{16}\left(\xi Q_0\underline{Q}_1\right)$ or $\zeta=d^{20}\left(\xi\langle Q_0^2\rangle\underline{Q}_1\right)$.

COROLLARY (5.5). (a) *Assume that $\langle\nu,2\nu,\xi,\nu\rangle$ is defined. Then $\xi Q_0V_{0,1}$ survives to E^{12} and $\langle\nu,2\nu,\xi,\nu\rangle$ contains an element which projects to $d^{12}(\xi Q_0V_{0,1})$.*
(b) *Assume that $\langle\nu,2\nu,\xi,\sigma\rangle$ is defined. Then $\xi\langle Q_0^2\rangle V_{0,1}$ survives to E^{16} and $\langle\nu,2\nu,\xi,\sigma\rangle$ contains an element which projects to $d^{16}\left(\xi\langle Q_0^2\rangle V_{0,1}\right)$.*
(c) *Assume that $\langle\sigma,\nu,\xi,\nu\rangle$ is defined. Then ξQ_0Q_1 survives to E^{16} and $\langle\sigma,\nu,\xi,\nu\rangle$ contains an element which projects to $d^{16}(\xi Q_0Q_1)$.*
(d) *Assume that $\langle\sigma,\nu,\xi,\sigma\rangle$ is defined. Then $\xi\langle Q_0^2\rangle Q_1$ survives to E^{20} and $\langle\sigma,\nu,\xi,\sigma\rangle$ contains an element which projects to $d^{20}\left(\xi\langle Q_0^2\rangle Q_1\right)$.*
(e) *Assume that $\langle\nu,\sigma,\xi,\nu\rangle$ is defined. Then $\xi Q_0\overline{Q}_1$ survives to E^{16} and $\langle\nu,\sigma,\xi,\nu\rangle$ contains an element which projects to $d^{16}\left(\xi Q_0\underline{Q}_1\right)$.*
(f) *Assume that $\langle\nu,\sigma,\xi,\sigma\rangle$ is defined. Then $\xi\langle Q_0^2\rangle\overline{Q}_1$ survives to E^{20} and $\langle\nu,\sigma,\xi,\sigma\rangle$ contains an element which projects to $d^{20}\left(\xi\langle Q_0^2\rangle\overline{Q}_1\right)$.*

The following Toda brackets follow from Corollaries (5.3), (5.5) and [6, Theorem (2.4.2)] which we denote as (2.4.2) below.

THEOREM (5.6). *The following Toda brackets are defined in π_*^S and contain the indicated elements.*

TODA BRACKET	REF.	TODA BRACKET	REF.
$A[14] \in \langle 2\nu, \nu, 2\nu, \nu\rangle$	$(5.5)(a)$	$A[39,2] \in \langle \nu, 2\nu, A[32,2]\rangle$	$(5.3)(a)$
$\eta A[14] \in \langle \nu, 2\nu, A[8]\rangle$	$(5.3)(a)$	$A[39,3] \in \langle \nu, 2\nu, A[32,1]\rangle$	$(5.3)(a)$
$A[19] \in \langle \sigma, A[8], \nu\rangle$	$(2.4.2)$	$2C[20]^2 \in \langle \nu, 2\nu, \eta A[32,2]\rangle$	$(5.3)(a)$
$2C[20] \in \langle \nu, 2\nu, \nu^3, \nu\rangle$	$(5.5)(a)$	$A[45,1] \in \langle \nu, 2\nu, B[34], \nu\rangle$	$(5.5)(a)$
$\eta C[20] \in \langle \nu, 2\nu, A[14]\rangle$	$(5.3)(a)$	$\eta C[44] \in \langle \sigma, \nu A[31], \nu\rangle$	$(2.4.2)$
$\eta A[30] \in \langle \sigma, \nu, A[16], \nu\rangle$	$(5.5)(c)$	$\eta A[45,1] \in \langle \nu, 2\nu, A[39,3]\rangle$	$(5.3)(a)$
$A[31] \in \langle \nu, \sigma, C[20]\rangle$	$(5.3)(c)$	$\eta A[45,2] \in \langle \nu, 2\nu, A[39,2]\rangle$	$(5.3)(a)$
$\eta A[32,2] \in \langle \nu, 2\nu, \eta^2 C[20], \nu\rangle$	$(5.5)(a)$	$A[47] \in \langle \nu, \sigma, A[36]\rangle$	$(5.3)(c)$
$A[36] \in \langle \nu, \sigma, \eta C[20], \nu\rangle$	$(5.5)(e)$	$B[47] \in \langle \nu, 2\nu, C[20]^2\rangle$	$(5.3)(a)$
$A[37] \in \langle \nu, 2\nu, A[30]\rangle$	$(5.3)(a)$	$\eta^2 D[45] \in \langle \nu, \sigma, A[36]\rangle$	$(5.3)(c)$

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