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EHP SPECTRAL SEQUENCE IN THE LAMBDA ALGEBRA

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1. Introduction

The lambda algebra Λ of Kan et al. ([2]) is a bigraded differential algebra over $\mathbb{Z}_2$ (for each prime p there is a similar algebra, but here we will only consider $p = 2$) generated by $\lambda_i \in \Lambda^{1,i}$ for $i \geq 0$ subject to the relations

$$(a) \qquad \lambda_i\lambda_{2i+1+m} = \sum_{\nu\geq 0} \binom{m-\nu-1}{\nu} \lambda_{m+i-\nu}\lambda_{2i+1+\nu} \text{ for } m \geq 0.$$

The differential δ is given by

$$(b) \qquad \delta(\lambda_k) = \sum_{\nu\geq 0} \binom{k-\nu-1}{\nu+1} \lambda_{k-\nu-1}\lambda_\nu.$$

The relations in (a) can be viewed as the "dual" of the Adem relations

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-j-i}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j, \quad a < 2b$$

in the mod 2 Steenrod algebra via the correspondence $\mathrm{Sq}^i\mathrm{Sq}^j \leftrightarrow \lambda_{j-1}\lambda_{i-1}$ (see [12]).

From (a) one sees that $\{\lambda_I = \lambda_{i_1} \ldots \lambda_{i_s} | 2i_j \geq i_{j+1}\}$ ($\lambda_I = 1$ if $I = [\,]$) is a $\mathbb{Z}_2$-base for Λ. Such monomials are called admissible monomials. For $n \geq 1$ let $\Lambda(n)$ be the $\mathbb{Z}_2$-submodule of Λ having the set of admissible monomials $\lambda_{i_1} \ldots \lambda_{i_s}$ with $i_1 \leq n-1$ as a $\mathbb{Z}_2$-base. It is easy to show, from (b) and (a), that each $\Lambda(n)$ is a subcomplex of Λ. The homology $H^{*,*}(\Lambda(n))$ is the E_2-term of an unstable Adams spectral sequence for computing the 2-adic homotopy of the sphere S^n, and the homology $H^{*,*}(\Lambda)$ is the E_2-term of the classical Adams spectral sequence for the 2-adic stable homotopy of spheres [2], [3], [10]. Since Λ is filtered by the subcomplexes $\Lambda(n)$, the spectral sequence $\{E_r^{n,s,t}\}$ obtained by applying $H^{*,*}(\,)$ to $\cdots \Lambda(n) \subset \Lambda(n+1) \subset \cdots$ with

$$(c) \qquad E_1^{n,s,t} \cong H^{s,t}(\Lambda(n)/\Lambda(n-1))$$

converges to $H^{*,*}(\Lambda)$. For each $r \geq 1$ the differential d_r of the spectral sequence goes from $E_r^{n,s,t}$ to $E_r^{n-r,s+1,t-1}$. This spectral sequence is called "the algebraic" EHP spectral sequence for Λ. It corresponds, in a suitable sense,

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to the geometric EHP spectral sequence obtained by applying the homotopy functor to the sequence of iterated loop spaces

$$\cdots \subset \Omega^n S^n \subset \Omega^{n+1} S^{n+1} \subset \cdots \subset \Omega^\infty S^\infty$$

where at each stage we have the James fibration $\Omega^n S^n \to \Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}$ ([9], [13]).

Our purpose here is not to make calculations with this spectral sequence. We refer to [4], [13] for some of these calculations. Rather, we propose another viewpoint toward this spectral sequence. In [8], we constructed, for each $t \geq 1$, a finite dimensional subcomplex $\Lambda[t]$ of Λ such that $\Lambda[t] \subset \Lambda[t+1]$ and $\bigcup_t \Lambda[t] = \Lambda$. These subcomplexes will be described later. We want to show here that the spectral sequence obtained by applying $H^{*,*}(\,)$ to the fibration $\cdots \subset \Lambda[t] \subset \Lambda[t+1] \subset \cdots$ is isomorphic to the EHP spectral sequence. The subcomplexes $\Lambda[t]$ arise in a very natural way, and so do the subcomplexes $\Lambda(n)$, in the homological algebra about the unstable modules over the Steenrod algebra A. We proceed to describe this homological algebra.

Recall that an unstable module M over A is a graded A-module with $M_n = 0$ for $n < 0$ and $\mathrm{Sq}^i x = 0$ for $i > |x|$. Let $\mathcal{U}$ be the category of unstable left A-modules with degree zero A-maps as morphisms. This category is an abelian category and has enough projectives and injectives ([6], [10], [11], [14]). For $N \in \mathcal{U}$, its k^{th} suspension $\Sigma^k N$, defined by $(\Sigma^k N)_n = N_{n-k}$, also lies in $\mathcal{U}$ if $k \geq 0$. The s^{th} left derived functor of $\mathrm{Hom}_{\mathcal{U}}(-, \Sigma^t N)$ is denoted by $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, N)$ for $s \geq 0$, $t \geq 0$ and $M \in \mathcal{U}$. In particular we have the groups $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, \mathbb{Z}_2)$.

Usually, to compute $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, \mathbb{Z}_2)$, take any projective resolution

$$C : 0 \longleftarrow M \overset{\epsilon}{\longleftarrow} C_0 \overset{d_1}{\longleftarrow} C_1 \overset{d_1}{\longleftarrow} C_2 \longleftarrow \cdots$$

of M in $\mathcal{U}$ from which we get, for each $t \geq 1$, a complex

$$C(M)^{*,t} : \mathrm{Hom}_{\mathcal{U}}(C_0, \Sigma^t \mathbb{Z}_2) \overset{d_1^*}{\longrightarrow} \mathrm{Hom}_{\mathcal{U}}(C_1, \Sigma^t \mathbb{Z}_2) \overset{d_2^*}{\longrightarrow} \cdots$$

where $C(M)^{s,t} = \mathrm{Hom}_{\mathcal{U}}(C_s, \Sigma^t \mathbb{Z}_2)$. Then $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, \mathbb{Z}_2) \cong \ker d_{s+1}^* / \ker d_s^*$. For $M = \Sigma^n \mathbb{Z}_2$, one can construct, as in [3], a particular resolution $\overline{C}$ such that $\overline{C}(\Sigma^n \mathbb{Z}_2)^{*,*}$ is identified with $\Lambda(n)$; more precisely, $\overline{C}(\Sigma^n \mathbb{Z}_2)^{s,t} = \Lambda(n)^{s,t-s-n}$. So $H^{*,*}(\Lambda(n))$ is just $\mathrm{Ext}_{\mathcal{U}}^{*,*}(\Sigma^n \mathbb{Z}_2, \mathbb{Z}_2)$. Furthermore, this resolution is geometrically realizable so that $H^{*,*}(\Lambda(n))$ can be identified as the E_2-term of an unstable Adams spectral sequence for $2^\pi * (S^n)$ ([3], [9], [10]).

The s^{th} left derived functor of $\mathrm{Hom}_{\mathcal{U}}(-, \Sigma^t \mathbb{Z}_2)$, when applied to M, is canonically isomorphic to the s^{th} right derived functor of $\mathrm{Hom}_{\mathcal{U}}(M, -)$ when applied to $\Sigma^t \mathbb{Z}_2$. Thus $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, \mathbb{Z}_2)$ can also be computed as follows. Take any injective resolution

$$I : 0 \longrightarrow \Sigma^t\mathbb{Z}_2 \xrightarrow{\epsilon} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \longrightarrow \cdots$$

of $\Sigma^t\mathbb{Z}_2$ in $\mathcal{U}$ from which we get a complex

$$I(M)^{*,t} : \mathrm{Hom}_{\mathcal{U}}(M, I_0) \xrightarrow{d_{1*}} \mathrm{Hom}_{\mathcal{U}}(M, I_1) \xrightarrow{d_{2*}} \ldots$$

where $I(M)^{s,t} = \mathrm{Hom}_{\mathcal{U}}(M, I_S)$. Then $\mathrm{Ext}^{s,t}_{\mathcal{U}}(M, \mathbb{Z}_2) \cong \ker d_{s+1*} \ker d_{s*}$.

For an integer $k \geq 1$, let $\overline{k} = [k/2]$, the greatest integer less than or equal to $k/2$. For a sequence of integers $J = (j_1, \ldots, j_s)$ let $|J| = j_1 + \ldots + j_s$. For $t \geq 1$, let $\Lambda[t]$ be the $\mathbb{Z}_2$-submodule of Λ generated by monomials $\lambda_I = \lambda_{i_1} \ldots \lambda_{i_S}$ (not necessarily admissible) such that

(d) either (i) $\lambda_I = 1$

or (ii) $i_S + 1 \leq \overline{t}$ and for $1 \leq j < s$

$i_j + 1 \leq \overline{t - |I_j| - (s-j)}$ where $I_j = (i_{j+1}, \ldots, i_S)$.

Clearly $\Lambda[t]$ is finite dimensional over $\mathbb{Z}_2$. The following are prove in [8].

(i) Each $\Lambda[t]$ is a subcomples of Λ.

(ii) The set of admissible monomials λ_I that satisfy (d) is a $\mathbb{Z}_2$-base for $\Lambda[t]$.

(iii) There is a particular injective resolution $\overline{I}$ of $\Sigma^t\mathbb{Z}_2$ in $\mathcal{U}$ such that $\overline{I}$ $(\Sigma^n\mathbb{Z}_2)^{s,t} \cong \Lambda[t]^{s,t-n-s}$.

So $\mathrm{Ext}^{*,*}_{\mathcal{U}}(\Sigma^n\mathbb{Z}_2, \mathbb{Z}_2)$ can also be computed from these $\Lambda[t]$. To give some ideas about these subcomplexes we tabulate below $\Lambda[t]$ for $1 \leq t \leq 6$ with their admissible basis elements

$$\begin{array}{lllllll}
\Lambda[1]: & 1 & & & & & \\
\Lambda[2]: & 1 & \lambda_0 & & & & \\
\Lambda[3]: & 1 & \lambda_0 & \lambda_0\lambda_0 & & & \\
\Lambda[4]: & 1 & \lambda_0 & \lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0 & & \\
 & & \lambda_1 & & & & \\
\Lambda[5]: & 1 & \lambda_0 & \lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0\lambda_0 & \\
 & & \lambda_1 & \lambda_1\lambda_0 & & & \\
\Lambda[6]: & 1 & \lambda_0 & \lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0\lambda_0 & \lambda_0\lambda_0\lambda_0\lambda_0\lambda_0 \\
 & & \lambda_1 & \lambda_1\lambda_0 & \lambda_1\lambda_0\lambda_0 & & \\
 & & \lambda_2 & \lambda_1\lambda_1. & & &
\end{array}$$

Note that $\Lambda[t] \subset \Lambda[t+1]$ and $\xrightarrow[t]{\lim} \Lambda[t] = \Lambda$. Thus Λ is filtered by the subcomplexes $\Lambda[t]$. This filtration gives rise to a spectral sequence $\{\overline{E}^{*,*,*}_r\}$ with

$$\overline{E}^{t,s,\tau}_1 = H^{s,\tau}(\Lambda[t]/\Lambda[t-1]) \qquad (e)$$

and it converges to $H^{*,*}(\Lambda)$ too. For each $r \geq 1$, the differential d_r goes from $\overline{E}_r^{t,s,\tau}$ to $\overline{E}_r^{t-r,s+1,\tau-1}$. In this note we prove the following result.

THEOREM (1.1). *The spectral sequence $\{\overline{E}_r^{*,*,*}\}$ defined by (e) is isomorphic to the EHP spectral sequence $\{E_r^{*,*,*}\}$ defined in (c). More precisely, $\overline{E}_r^{t,s,\tau} \cong E_r^{t-\tau-s,s,\tau}$.*

In other words, the filtration $\cdots \subset \Lambda(n) \subset \Lambda(n+1) \subset \cdots$ of Λ, each $\Lambda(n)$ being obtained by a "projective resolution", and the filtration $\cdots \subset \Lambda[t] \subset \Lambda[t+1] \subset \cdots$ of Λ, each $\Lambda[t]$ being obtained by an "injective resolution", give rise to the same spectral sequence for computing $H^{*,*}(\Lambda)$.

(1.1) is proved in §2. In §3 we also consider an "injective" type EHP spectral sequence for $\mathbb{R}P^\infty$ and discuss its relations with the two spectral sequences in Theorem (1.1). The additional writing of §3 to the original version of this note is owing to referee's suggestion to say more about the $\Lambda[t]$'s.

2. Proof

Before proving (1.1) we first make a few simple calculations, only to illustrate the comparison between these spectral sequences, particulary in the filtration degree changes of the differentials. The "projective" EHP spectral sequence and the "injective" EHP spectral sequence will be abreviated as EHP S.S and $\overline{\text{EPH}}$ S.S respectively.

The basis for $\Lambda^{1,*}$ is $\{\lambda_k | k \geq 0\}$. Recall that in Λ we have

$$(b) \qquad \delta(\lambda_k) = \sum_{\nu \geq 0} \binom{k-\nu-1}{\nu+1} \lambda_{k-\nu-1}\lambda_\nu .$$

It is easy to see that $\delta(\lambda_{2^i-1}) = 0$ for all $i \geq 0$. Thus for each $i \geq 0$, $h_i = \{\lambda_{2^i-1}\}$ persists to E_∞ in both the EHP S.S. and the $\overline{\text{EPH}}$ S.S h_i has filtration 2^i in the EHP S.S and has filtration 2^{i+1} in the $\overline{\text{EPH}}$ S.S. More precisely, $h_i \in E_\infty^{2^i,1,2^i-1}$ (from (c)) and $h_i \in \overline{E}_\infty^{2^{i+1},1,2^i-1}$ (from (e)).

For $k \neq 2^i - 1$ let $k = 2^j(2l+1) - 1$; so $l \geq 1, j \geq 0$. Note that λ_k is a basis element in $\Lambda(k+1)/\Lambda(k)$. We observe that the smallest ν such that $\binom{k-\nu+1}{\nu+1} \equiv 1 \pmod 2$ is $\nu = 2^j - 1$. So, by (b)

$$\delta(\lambda_k) \equiv \lambda_{2^{j+1}l-1}\lambda_{2^j-1} \mod \Lambda(2^{j+1}l-1).$$

This means that, in the EHP S.S, $d_r(\lambda_k) = 0$ for $r < 2^j$ and

$$\lambda_k \in E_{2^j}^{k+1,1,k} \xrightarrow{d_{2^j}} \lambda_{2^{j+1}l-1}h_j \in E_{2^j}^{k+1-2^j,2,k-1}.$$

We also note, from (d), that λ_k is a basis element in $\Lambda[2k+2]/\Lambda[2k+1]$ and that $\lambda_{k-\nu-1}\lambda_\nu$ is a basis element in $\Lambda[2k-\nu+1]/\Lambda[2k-\nu]$. So

$$\delta(\lambda_k) \equiv \lambda_{2^{j+1}l-1}\lambda_{2^j-1} \mod \Lambda[2k-2^j+2].$$

This means that, in the $\overline{\mathrm{EPH}}$ S.S, $d_r(\lambda_k) = 0$ for $r < 2^j$ and

$$\lambda_k \in \overline{E}_{2^j}^{2k+2,1,k} \xrightarrow{d_{2^j}} \lambda_{2^{j+1}l-1}h_j \in \overline{E}_{2^j}^{2k-2^j+2,2,k-1}.$$

Thus $\delta(\lambda_k)$ decreases filtration degree by 2^j and projects to $\lambda_{2^{j+1}l-1}h_j$ in both spectral sequences.

The result obtained above is that $\{h_i | i \geq 0\}$ is a $\mathbb{Z}_2$-base for $H^{1,*}(\Lambda)$ which is well known.

We proceed to the proof of (1.1).

It follows from the definition of $\Lambda(n)$ that we have the following.

(2.1) $\{\lambda_{n-1}\lambda_{i_1}\dots\lambda_{i_s} | (n-1, i_1, \dots, i_s) \text{ admissible}\}$ is a $\mathbb{Z}_2$-base for $\Lambda(n)/\Lambda(n-1)$.

To determine basis elements in $\Lambda[t]/\Lambda[t-1]$ we first note that condition (d) (ii) in §1 for monomials $\lambda_{i_1}\dots\lambda_{i_s} \in \Lambda[t]$ (we stress that $\lambda_{i_1}\dots\lambda_{i_s}$ is not necessarily admissible) can be simply stated as

(d) $i_k + 1 \leq \overline{t - i_{k+1} - \dots - i_s - (s-k)}$ for $1 \leq k \leq s$ where, for $k = s$, $i_{k+1} + \dots + i_s + (s-k) = 0$.

This condition for $k = 1$ is $i_1 + 1 \leq \overline{t - i_2 - \dots - i_s - (s-1)} \leq \frac{t - i_2 - \dots - i_s - (s-1)}{2}$ which is equivalent to $2i_1 + i_2 + \dots + i_s + s + 1 \leq t$.

LEMMA (2.2) *An admissible monomial* $\lambda_{j_1}\dots\lambda_{j_s}$ *lies in* $\Lambda[t]$ *if and only if* $2j_1 + j_2 + \dots + j_2 + s + 1 \leq t$.

In other words, to see whether an admissible monomial $\lambda_{j_1}\dots\lambda_{j_s}$ lies in $\Lambda[t]$, it suffices to show if it satisfies condition (d) for $k = 1$. This is not true for inadmissible monomials. For example, this is not true for $\lambda_{2^i-1}\lambda_{2^{i+2}-1}$.

COROLLARY (2.3). *The set of admissible monomials* $\lambda_{j_1}\dots\lambda_{j_2}$ *with* $2j_1 + j_2 + \dots + j_s + s + 1 = t$ *is a* $\mathbb{Z}_2$*-base for* $\Lambda[t]/\Lambda[t-1]$.

This is straightforward from (2.2).

Proof of Lemma (2.2). We need only show that if $2j_1 + j_2 + \dots + j_s + s + 1 \leq t$ then $\lambda_{j_1}\dots\lambda_{j_s} \in \Lambda[t]$. As already noted above, $2j_1 + j_2 + \dots + j_s + s + 1 \leq t$ implies condition (d) for $k = 1$. Since $\lambda_{j_1}\dots\lambda_{j_s}$ is admissible, $2j_l \geq j_{l+1}$. Then, for $2 \leq k \leq s$,

$$\begin{aligned} &2(j_k + 1) + j_{k+1} + \dots + j_s + s - k \\ &= 2j_k + j_{k+1} + \dots + j_s + s + 2 - k \\ &\leq 2j_{k-1} + j_k + j_{k+1} + \dots + j_s + s + 2 - k \\ &\leq 2j_{k-2} + j_{k-1} + j_k + j_{k+1} + \dots + j_s + s + 2 - k \end{aligned}$$

$$\begin{aligned}&\vdots\\ &\leq 2j_1 + j_2 + \cdots + j_s + s + 2 - k\\ &\leq 2j_1 + j_2 + \cdots + j_s + s \qquad (k \geq 2)\\ &\leq t - 1 < t \qquad \text{(by assumption)}\end{aligned}$$

that is,

$$\begin{aligned} j_k + 1 \quad &< \quad \frac{t - j_{k+1} - \cdots - j_s - (s-k)}{2} \quad \text{which implies}\\ j_k + 1 \quad &\leq \quad t - j_{k+1} - \cdots - j_s - (s-k).\end{aligned}$$

So $\lambda_{j_1} \ldots \lambda_{j_s}$ also satisfies condition (d) for $2 \leq k \leq s$. Thus $\lambda_{j_1} \ldots \lambda_{j_s} \in \Lambda[t]$. Q.E.D.

Given an admissible monomial $\lambda_J = \lambda_{j_s}$, we have

$$\delta(\lambda_{j_1} \ldots \lambda_{j_s}) = \lambda_{j_1}\delta(\lambda_{j_2} \ldots \lambda_{j_s}) + \delta(\lambda_{j_1})\lambda_{j_2} \ldots \lambda_{j_s}$$

in Λ. From (a), (b) in §1 and, by induction on s, it is easy to verify the following.

(2.4) Suppose $\delta(\lambda_J) \neq 0$ in Λ, and let
$$\delta(\lambda_J) = \delta(\lambda_{j_1} \ldots \lambda_{j_s}) = \sum_\nu \lambda_{I_\nu} + \sum_\mu \lambda_{J_\mu}$$
be its admissible expansion where the first entry of I_ν is i_1 for all ν and the first entry J_μ is strictly less than i_1 for all μ. Then $i_1 \leq j_1$.

To prove (1.1) we first note that we may start the two spectral sequences with E_0 and $\overline{E}_0$ given respectively by

$$E_0 = \bigoplus_{n \geq 1} \Lambda(n)/\Lambda(n-1) \ \text{ and } \ \overline{E}_0 = \bigoplus_{t \geq 1} \Lambda[t]/\Lambda[t-1].$$

Define a $\mathbb{Z}_2$-map $f_0 : E_0 \to \overline{E}_0$ by $f_0(\lambda_{j_1} \ldots \lambda_{j_s}) = \lambda_{j_1} \ldots \lambda_{j_s}$ for any admissible monomial $\lambda_{j_1} \ldots \lambda_{j_s}$. By (2.1) the domain $\lambda_{j_1} \ldots \lambda_{j_s}$ is a basis element in $\Lambda(j_1 + 1)/\Lambda(j_1)$, and by (2.3), the range $\lambda_{j_1} \ldots \lambda_{j_s}$ is a basis element in $\Lambda[t]/\Lambda[t-1]$ where $t = 2j_1 + j_2 + \cdots + j_s + s + 1$. It is clear that $E_0 \cong \overline{E}_0$.

For a basis element $\lambda_J = \lambda_{j_1} \ldots \lambda_{j_s}$ in $\Lambda(j_1 + 1)/\Lambda(j_1)$, let $\delta(\lambda_J) = \sum_\nu \lambda_{I_\nu} + \sum_\mu \lambda_{I_\mu}$ and i_1 be as in (2.4). So $\delta(\lambda_J) \equiv \sum_\nu \lambda_{I_\nu}$ is a basis element in $\Lambda(i_1 + 1)/\Lambda(i_1)$. Thus $\delta(\lambda_J)$ decreases filtration degree by $j_1 - i_1$ in the EHP spectral sequence, and it projects to $\sum \nu\lambda_{I_\nu}$ in $\Lambda(i_1 + 1)/\Lambda(i_1)$. By (2.3), $\lambda_J =$

$\lambda_{j_1} \cdots \lambda_{j_s}$ is a basis element in $\Lambda[t]/\Lambda[t-1]$ where $t = 2j_1 + j_2 + \cdots + j_s + s + 1$. It is not difficult to see that to prove f_0 induces isomorphisms $E_r \cong \overline{E}_r$ it suffices to show the following.

(2.5) Each λ_{I_ν} is a basis element in $\Lambda[t']/\Lambda[t'-1]$ and $\sum_\mu \lambda_{j_\mu} \in \Lambda[t'-1]$ where $t' = t - (j_1 - i_1)$; so $\delta(\lambda_J) \equiv \sum_\nu \lambda_{I_\nu}$ in $\Lambda[t']/\Lambda[t'-1]$. Thus $\delta(\lambda_J)$ also decreases filtration degree by $j_1 - i_1$ in the $\overline{\text{EHP}}$ spectral sequence, and it projects to $\sum_\nu \lambda_{I_\nu}$ in $\Lambda[t']/\Lambda[t'-1]$ too.

Let $I_\nu = (i_1, i_2(\nu), \ldots, i_{s+1}(\nu))$. Then $i_1 + i_2(\nu) + \cdots + i_{s+1}(\nu) = j_1 + \cdots + j_s - 1$. We have

$$\begin{aligned} &2i_1 + i_2(\nu) + \cdots + i_{s+1}(\nu) + s + 2 \\ &= i_1 + i_1 + i_2(\nu) + \cdots + i_{s+1}(\nu) + s + 2 \\ &= i_1 + j_i + j_2 + \cdots + j_s - 1 + s + 2 \\ &= (i_1 - j_1) + 2j_1 + j_2 + \cdots + j_s + s + 1 \\ &= (i_1 - j_1) + t \\ &= t - (j_1 - i_1) \\ &= t'. \end{aligned}$$

Since I_ν is admissible, it follows that λ_{I_ν} is a basis element in $\Lambda[t']/\Lambda[t'-1]$ by (2.3). Let $J_\mu = (j_1(\mu), j_2(\mu), \ldots, j_{s+1}(\mu))$. By making a similar calculation we see $2j_1(\mu) + j_2(\mu) + \cdots + j_{s+1}(\mu) + s + 2 = t - (j_1 - j_1(\mu))$ as $j_1(\mu) + j_2(\mu) + \cdots + j_{s+1}(\mu)$ is also equal to $j_1 + \cdots + j_s - 1$. Since $j_1(\mu) < i_1$ (by (2.4)), $t - (j_1 - j_1(\mu)) < t - (j_1 - i_1) = t'$. Again, since J_μ is admissible, $\lambda_{J_\mu} \in \Lambda[t'-1]$ by (2.2). This proves (2.5) and therefore Theorem (1.1).

3. An injective type spectral sequence for the infinite real projective space $\mathbb{R}P^\infty$

Corresponding to the "projective" EHP spectral sequence for $H^{*,*}(\Lambda)$ is a similar spectral sequence for $\mathrm{Ext}_A^{*,*}(\mathbb{R}P^\infty) = \mathrm{Ext}_A^{*,*}(\tilde{H}^*(\mathbb{R}P^\infty), \mathbb{Z}_2)$, the E_2-term of the mod 2 Adams spectral sequence for computing $2^{\pi_*^S}$. This spectral sequence is considered in [4], and will be recalled in a moment. This spectral sequence may be referred to as the "projective" EHP spectral sequence for $\mathbb{R}P^\infty$. Corresponding to the "injective" EHP spectral sequence for $H^{*,*}(\Lambda)$ is also an "injective" type EHP spectral sequence for $\mathrm{Ext}_A^{*,*}(\mathbb{R}P^\infty)$. While there are no natural maps from the projective EHP spectral sequence for $\mathbb{R}P^\infty$ to the projective EHP spectral sequence for Λ (as will be seen), there is a natural map from the injective EHP spectral sequence for $\mathbb{R}P^\infty$ to the injective EHP spectral squence for Λ, and so via Theorem (1.1), to the usual EHP spectral sequence for Λ. Because of this it is perhaps worthwhile to record here this new spectral sequence for $\mathbb{R}P^\infty$ also. We will propose a problem concerning this spectral sequence.

We simply write P for $\mathbb{R}P^\infty$. Consider the complex $\widetilde{H}_*(P) \otimes \Lambda$ which is bigraded by $(\widetilde{H}_*(P) \otimes \Lambda)^{s,t} = \sum_{k \geq 1} \widetilde{H}_k(P) \otimes \Lambda^{s,t-k}$. $\widetilde{H}_*(P) \otimes \Lambda$ is a differential right Λ-module with differential δ given by

$$\text{(f)} \qquad \delta(e_k) = \sum_{\nu \geq 0} \binom{k-\nu-1}{\nu+1} e_{k-\nu-1} \otimes \lambda_\nu$$

where e_k is the generator of $\widetilde{H}_k(P) = \mathbb{Z}_2 (k \geq 1)$. Then

$$H^{s,t-s}(\widetilde{H}_*(P) \otimes \Lambda, \delta) \cong \operatorname{Ext}_A^{s,t}(P) \quad ([3]).$$

Define a map

$$\varphi : \widetilde{H}_*(P) \otimes \Lambda \longrightarrow \Lambda \text{ by } \varphi(e_k \otimes \lambda_I) = \lambda_k \lambda_I$$

for any monomial $\lambda_I = \lambda_{i_1} \ldots \lambda_{i_s} \in \Lambda$. It is easy to see from (b) (in §1) and (f) that φ is a chain map. The induced map in Ext groups

$$\varphi_* : H^{s,t-s}(\widetilde{H}_*(P) \otimes \Lambda, \delta) = \operatorname{Ext}_A^{s,t}(P) \longrightarrow \operatorname{Ext}_A^{s+1,t+1}(\widetilde{H}(S^0)) = H^{s+1,t-s}(\Lambda)$$

is the induced map of the transfer map $P \xrightarrow{t} S^0$ ([5]) in E_2 terms of the Adams spectral sequences. The algebraic Kahn-Priddy Theorem ([7]) asserts that φ_* is onto which corresponds to the topological result—Kahn-Priddy Theorem ([5]) asserting $t_* :_2 \pi_*^s(P) \to_2 \pi_*^s(S^0)$ is onto in positive stems.

Define a filtration $F(1) = 0 \subset F(2) \subset \ldots \subset F(i) \subset F(i+1) \subset \ldots$ of the complex $\widetilde{H}_*(P) \otimes \Lambda$ by $F(i) = \sum_{k \leq i-1} e_k \otimes \Lambda$. Clearly $F(i)/F(i-1) \cong \Sigma^{i-1}\Lambda$. This filtration gives rise to a spectral sequence $\{E_r^{i,s,t}(P)\}_{r \geq 1}$ converging to $\operatorname{Ext}_A^{*,*}(P)$ with

$$E_1^{i,s,t}(P) = H^{s,t}(F(i)/F(i-1)) \cong H^{s,t-i+1}(\Lambda).$$

For each $r \geq 1$ the differential d_r goes from $E_r^{i,s,t}(P)$ to $E_r^{i-r,s+1,t-1}(P)$. This spectral sequence is considered and used in [4] to compute $\operatorname{Ext}_A^{s,*}(P)$ for small s. We should remark that $F(i)$ in [4] is $F(i+1)$ here, and that $\Lambda(i)$ there is $\Lambda(i+1)$ here. Note that the filtration $\{F(i)\}$ of $\widetilde{H}_*(P) \otimes \Lambda$ above is defined in a fashion similar to the filtration $\ldots \subset \Lambda(i) \subset \Lambda(i+1) \subset \ldots$ of Λ in §1. We may thus refer to $\{E_r^{i,s,t}(P)\}_{r\geq 1}$ as the "projective" EHP spectral sequence for P. The chain map $\varphi : \widetilde{H}_*(P) \otimes \Lambda \to \Lambda$ in (g), however, is not a filtration-preserving map, that is, $\varphi(F(i)) \not\subset \Lambda(i)$. For example, $e_1 \otimes \lambda_4 \in F(2)$, but $\varphi(e_1 \otimes \lambda_4) = \lambda_1\lambda_4 = \lambda_2\lambda_3$ is a basis element in $\Lambda(3)/\Lambda(2)$. Thus φ does not induce a map between the projective EHP spectral sequences for P and Λ.

Corresponding to the filtration $\cdots \subset \Lambda[t] \subset \Lambda[t+1] \subset \cdots$ of Λ is a filtration $\cdots \subset [t] \subset F[t+1] \subset \cdots$ of $\widetilde{H}_*(P) \otimes \Lambda$ defined, similar to (d), as follows. $F[t] = 0$ for $t \leq 3$. For $t \geq 4$, $F[t]$ is the $\mathbb{Z}_2$-submodule of $\widetilde{H}_*(P) \otimes \Lambda$ generated by elements of the form $e_k \otimes \lambda_{i_1} \ldots \lambda_{i_s} (\lambda_{i_1} \ldots \lambda_{i_s}$ is no necessarily admissible) such that the corresponding monomial $\lambda_k\lambda_{i_1} \ldots \lambda_{i_s}$ in Λ satisfies the condition (d), that is,

$$\begin{aligned} i_s + 1 &\leq \overline{t} \\ i_j + 1 &\leq \overline{t - i_{j+1} - \cdots - i_s - (s-j)} \end{aligned}$$

and

$$k + 1 \leq \overline{t - i_1 - i_2 - \cdots - i_s - s}.$$

The proof of [8] showing that $\Lambda[t]$ is a subcomplex of Λ also shows that each $F[t]$ is a subcomplex of $\widetilde{H}_*(P) \otimes \Lambda$, owing to the fact that the differential formulae (b) and (f) are similar and the fact that neither $\lambda_k\lambda_{i_1} \ldots \lambda_{i_s}$ nor $\lambda_{i_1} \ldots \lambda_{i_s}$ is required to be admissible. So we have a spectral sequence $\{\overline{E}_r^{t,s,\tau}(P)\}_{r\geq 1}$ for $\mathrm{Ext}_A^{*,*}(P)$ with

$$\overline{E}_1^{t,s,\tau}(P) = H^{s,\tau}(F[t]/F[t-1]).$$

For each $r \geq 1$, the differential d_r goes from $\overline{E}_r^{t,s,\tau}(P)$ to $\overline{E}_t^{t-r,s+1,\tau-1}$. We will call this spectral sequence the "injective" EHP spectral sequence for P.

The remarkable thing is that the filtration $\{F[t]\}$ of $\overline{H}_*(P) \otimes \Lambda$ and the filtration $\{\Lambda[t]\}$ of Λ are preserved by the chain map $\varphi : \overline{H}_*(P) \otimes \Lambda \to \Lambda$ in (g), that is, $\varphi(F[t]) \subset \Lambda[t]$ for all t, and this is straightforward from the definitions. So φ induces a map

$$\varphi_* : \overline{E}_r^{t,s,\tau}(P) \longrightarrow \overline{E}_r^{t,s+1,\tau}$$

of the injective EHP spectral sequences. Now Theorem (1.1) says that one can identify the injective EHP S.S with the projective EHP S.S for $H^{*,*}(\Lambda)$ via $\overline{E}_r^{t,s+1,\tau} \cong E_r^{t-\tau-s-1,s+1,\tau}$. Thus φ induces a map

$$\varphi_* : \overline{E}_r^{t,s,\tau}(P) \longrightarrow E_r^{t-\tau-s-1,s+1,\tau}$$

from the injective EHP S.S for $\mathrm{Ext}_A^{*,*}(P)$ to the usual EHP S.S for $H^{*,*}(\Lambda)$. In particualr, we have a map

$$\widetilde{\varphi}_* : \overline{E}_\infty^{t,s,\tau}(P) \longrightarrow \overline{E}_\infty^{t,s+1,\tau} \cong E_\infty^{t-\tau-s-1,s+1,\tau}.$$

Because of the algebraic Kahn-Priddy Theorem, it is tempting to make the following conjecture.

CONJECTURE. *$\widetilde{\varphi}_*$ is onto all $t \geq 4$, $s \geq 0$ and $\tau \geq 1$.*

PROPOSITION (3.1). *The conjecture is true for $0 \leq s \leq 2$.*

Before proving (3.1) we note the following lemma which follows from (2.3) and the definitions of $\Lambda[t]$ and $F[t]$.

LEMMA (3.2). (1) *If $\lambda_I = \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_s}$ $(s \geq 2)$ is an inadmissible monomial in Λ such that $(i_2, \ldots, i_s)$ is admissible then $\lambda_I \neq 0$ in $\Lambda[t]\Lambda[t-1]$ where $t = 2i_2 + i_3 + \cdots + i_s + s$.*

(2) *Let $e_k \otimes \lambda_{i_1}\ldots\lambda_{i_s}$ $(s \geq 0)$ be and element in $\widetilde{H}_*(P) \otimes \Lambda$ such that $(i_1, \ldots, i_s)$ is admissible.*

(i) If $(k, i_1, \ldots, i_s)$ is admissible then $e_k \otimes \lambda_{i_1}\ldots\lambda_{i_s}$ is a basis element in $F[t]/F[t-1]$ where $t = 2k + i_1 + \cdots + i_s + s + 2$.

(ii) If $(k, i_1, \ldots, i_s)$ is inadmissible then $e_k \otimes \lambda_{i_1}\ldots\lambda_{i_s} \neq 0$ in $F[t']/F[t'-1]$ where $t' = 2i_1 + i_2 + \cdots + i_s + s + 1$.

Now we prove (3.1). Recall ([1], [15]) that $H^{1,*}(\Lambda)$, $H^{2,*}(\Lambda)$ and $H^{3,*}(\Lambda)$ for $* > 0$ are generated respectively by the following sets of classes:

$$(3.1) \qquad \{h_i \mid i \geq 1\}, \qquad \{h_j h_i \mid 0 \leq j < i-1 \text{ or } 0 < j = 1\},$$
$$\{h_k h_j h_i \mid 0 \leq k < j-1 < i-3 \text{ or } 0 \leq k = j \leq i-2 \text{ or } 3 \leq k+3 \leq j = 1\}$$
$$\cup \{c_i \mid i \geq 0\}.$$

These classes are represented respectively by the following cycles in Λ where in the first set of variables i, j, k are positive except for $\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}$ and the restrictions on these numbers are as in (3.3):

(3.4)

$$(1) \qquad \lambda_{2^i-1} \in \Lambda^{1,2^i-1}, \qquad \lambda_{2^j-1}\lambda_{2^i-1} \in \Lambda^{2,2^i+2^j-2},$$
$$\lambda_{2^k-1}\lambda_{2^j-1}\lambda_{2^i-1} \in \Lambda^{3,2^i+2^j+2^k-3},$$
$$\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1} \in \Lambda^{3,2^{i+3}+2^{i+1}+2^i-3},$$
$$(2) \qquad \lambda_0\lambda_{2^i-1} \in \Lambda^{2,2^i-1} \ (i \geq 2), \quad \lambda_0^2\lambda_{2^i-1} \in \Lambda^{3,2^i-1} \ (i \geq 2),$$
$$\lambda_0\lambda^2_{2^i-1} \in \Lambda^{3,2^{i+1}-2} \ (i \geq 3),$$
$$\lambda_0\lambda_{2^j-1}\lambda_{2^i-1} \in \Lambda^{3,2^i+2^j-2} \ (2 \leq j < i-1).$$

To prove (3.1) we will first prove it for the classes in (3.3) represented by the cycles in (3.4)(1).

Consider the following elements in $\widetilde{H}_*(P) \otimes \Lambda$ corresponding to the elements in (3.4)(1):

(3.5)
$$\begin{aligned}
&e_{2^i-1} \in (\widetilde{H}_*(P) \otimes \Lambda)^{0,2^i-1},\\
&e_{2^j-1} \otimes \lambda_{2^i_1} \in (\widetilde{H}_*(P) \otimes \Lambda)^{1,2^i+2^j-2},\\
&e_{2^k-1} \otimes \lambda_{2^j-1}\lambda_{2^i-1} \in (\widetilde{H}_*(P) \otimes \Lambda)^{2,2^i+2^j+2^k-3},\\
&e_{2^{i+1}+2^i-1} \otimes \lambda^2_{2^{i+2}-1} \in (\widetilde{H}_*(P) \otimes \Lambda)^{2,2^{i+3}+2^{i+1}+2^i-3}.
\end{aligned}$$

It is easy to see that these are also cycles in $\widetilde{H}_*(P) \otimes \Lambda$ representing classes in $\mathrm{Ext}_A^{*,*}(P)$ which we will denote by:

(3.6)
$$\begin{aligned}
&\widehat{h}_i \in \mathrm{Ext}_A^{0,2^i-1}(P),\\
&\widehat{h_j h_i} \in \mathrm{Ext}_A^{1,2^i+2^j-1}(P),\\
&\widehat{h_k h_j h_i} \in \mathrm{Ext}_A^{2,2^i+2^j+2^k-1}(P),\\
&\widehat{c}_i \in \mathrm{Ext}_A^{2,2^{i+3}+2^{i+1}+2^i-1}(P).
\end{aligned}$$

Applying (2.3) and (3.2) to the elements in (3.4)(1) and (3.5) we have the following

$e_{2^i-1} \neq 0$ in $F[t]/F[t-1]$ and $\lambda_{2^i-1} \neq 0$ in $\Lambda[t]/\Lambda[t-1]$ where $t = 2^{i+1}$,
$e_{2^j-1} \otimes \lambda_{2^i-1} \neq 0$ in $F[t]/F[t-1]$ and $\lambda_{2^j-1}\lambda_{2^i-1} \neq 0$ in $\Lambda[t]/\Lambda[t-1]$
where $t = \begin{cases} 2^{i+1} & \text{if } j < i-1 \\ 2^{i+1}+2^i & \text{if } j = i, \end{cases}$
$e_{2^k-1} \otimes \lambda_{2^j-1}\lambda_{2^i-1} \neq 0$ in $F[t]/F[t-1]$
and $\lambda_{2^k-1}\lambda_{2^j-1}\lambda_{2^i-1} \neq 0$ in $\Lambda[t]/\Lambda[t-1]$
where $t = \begin{cases} 2^{i+1}+2^i & \text{if } k+3 \leq j = 1 \\ 2^{i+1} & \text{otherwise,} \end{cases}$
$e_{2^{i+1}+2^i-1} \otimes \lambda^2_{2^{i+2}-1} \neq 0$ in $F[t]/F[t-1]$ and $\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1} \neq 0$ in $\Lambda[t]/\Lambda[t-1]$, where $t = 2^{i+3}+2^{i+2}+2^{i+1}$.

This implies that in the injective EHP spectral sequences for $\mathrm{Ext}_A^{*,*}(P)$ and $H^{*,*}(\Lambda)$ the classes in (3.6) and the classes in (3.3) represented by cycles in

(3.4)(1) project to the following classes in $\overline{E}_\infty(P)$ and $\overline{E}_\infty$ respectively:

$$e_{2^i-1} \in \overline{E}_\infty^{2^{i+1},0,2^i-1}(P), \qquad \lambda_{2^i-1} \in \overline{E}_\infty^{2^{i+1},1,2^i-1},$$

$$e_{2^j-1} \otimes \lambda_{2^i-1} \in \begin{cases} \overline{E}_\infty^{2^{i+1},1,2^i+2^j-2}(P) & \text{for } j < i-1 \\ \overline{E}_\infty^{2^{i+1}+2^i,1,2^{i+1}-2}(P) & \text{for } j = 1, \end{cases}$$

$$\lambda_{2^j-1}\lambda_{2^i-1} \in \begin{cases} \overline{E}_\infty^{2^{i+1}}, 2, 2^i+2^j-2 & \text{for } j < i-1 \\ \overline{E}_\infty^{2^{i+1}} + 2^i, 2, 2^{i+1}-2 & \text{for } j = 1, \end{cases}$$

$$e_{2^k-1} \otimes \lambda_{2^j-1}\lambda_{2^i-1} \in \begin{cases} \overline{E}_\infty^{2^{i+1}+2^i,2,2^i+2^j+2^k-3}(P) & \text{for } k+3 \leq j = 1 \\ \overline{E}_\infty^{2^{i+1},2,2^i+2^j+2^k-3}(P) & \text{otherwise,} \end{cases}$$

$$\lambda_{2^k-1} \otimes \lambda_{2^j-1}\lambda_{2^i-1} \in \begin{cases} \overline{E}_\infty^{2^{i+1}+2^i,3,2^i+2^j+2^k-3} & \text{for } k+3 \leq j = 1 \\ \overline{E}_\infty^{2^{i+1},3,2^i+2^j+2^k-3} & \text{otherwise,} \end{cases}$$

$$e_{2^{i+1}+2^i-1} \otimes \lambda^2_{2^{i+2}-1} \in \overline{E}_\infty^{2^{i+3}+2^{i+2}+2^{i+1}+2^{i+3}+2^{i+1}+2^i-3}(P),$$

$$\lambda_{2^{i+1}+2^i-1} \otimes \lambda^2_{2^{i+2}-1} \in \overline{E}_\infty^{2^{i+3}+2^{i+2}+2^{i+1},3,2^{i+3}+2^{i+1}+2^i-3}.$$

This proves (3.1) for the classes in (3.3) corresponding to the elements in (3.4)(1).

To prove (3.1) for the remaining classes represented by the cycles in (3.4)(2), we first note that these classes are also represented respectively by the following cycles:

$$\lambda_{2^i-1}\lambda_0, \quad \lambda_{2^i-1}\lambda_0^2, \quad \lambda^2_{2^i-1}\lambda_0 \quad \text{and} \quad \lambda_{2^i-1}\lambda_{2^j-1}\lambda_0.$$

The following lemma is not difficult to verify from (a), (b) in §1, and from (2.3) and (3.2)(1).

LEMMA (3.7).

(1) $\delta(\lambda_{2^i})+\lambda_{2^i-1}\lambda_0 = \lambda_0\lambda_{2^i-1} = \lambda_{2^i-2}\lambda_1 + \sum_{l=2}^{i-1} \lambda_{2^i-2^l}\lambda_{2^l-1}$ *where* $\lambda_{2^i-2}\lambda \neq 0$ *in* $\Lambda[2^{i+1}]/\Lambda[2^{i+1}-1]$ *and* $\sum_{l=2}^{i-1} \lambda_{2^i-2^l}\lambda_{2^l-1} \in \Lambda[2^{i+1}-1]$.

(2) $\delta(\lambda_{2^i}\lambda_0 + \lambda_0\lambda_{2^i}) + \lambda_{2^i-1}\lambda_0^2 = \lambda_0^2\lambda_{2^i-1} = \lambda_{2^i-3}\lambda_1^2 + \sum_\nu \lambda_{I_\nu}$ *where each* $I_\nu = (i_1(\nu), i_2(\nu), i_3(\nu))$ *is admissible with* $i_1(\nu) \geq 1$, $\lambda_{2^i-3}\lambda_1^2 \neq 0$ *in* $\Lambda[2^{i+1}]/\Lambda[2^{i+1}-1]$ *and* $\sum_\nu \lambda_{I_\nu} \in \Lambda[2^{i+1}-1]$.

(3) $\delta(\lambda_{2^i}\lambda_{2^i-1}+\lambda_{2^i-1}\lambda_{2^i})+\lambda_{2^i-1}^2\lambda_0=\lambda_0\lambda_{2^i-1}^2=\lambda_{2^i-2}\lambda_1\lambda_{2^i-1}+\sum_\mu \lambda_{J_\mu}$ *where each* $J_\mu=(j_1(\mu),\,j_2(\mu),\,j_3(\mu))$ *is admissible with* $j_1(\mu)\geq 1$, $\lambda_{2^i-2}\lambda_1\lambda_{2^i-1}\neq 0$ *in* $\Lambda[2^{i+1}+2^i]/\Lambda[2^{i+1}+2^i-1]$ *and* $\sum_\mu \lambda_{J_\mu}\in\Lambda[2^{i+1}+2^i-1]$.

(4) $\delta(\lambda_{2^j}\lambda_{2^i-1}+\lambda_{2^j-1}\lambda_{2^i}+\lambda_{2^i+2^j-1}\lambda_0)+\lambda_{2^i-1}\lambda_{2^j-1}\lambda_0=\lambda_0\lambda_{2^j-1}\lambda_{2^i-1}=\lambda_{2^i-2^j-2}\lambda_1\lambda_{2^{j+1}-1}+\sum_\mu\lambda_{K_\mu}$ *where each* $K_\mu=(k_1(\mu),\,k_2(\mu),\,k_3(\mu))$ *is admissible with* $k_1(\mu)\geq 1$, $\lambda_{2^i-2^j-2}\lambda_1\lambda_{2^{j+1}-1}\neq 0$ *in* $\Lambda[2^{i+1}]/\Lambda[2^{i+1}-1]$ *and* $\sum_\mu\lambda_{K_\mu}\in\Lambda[2^{i+1}-1]$.

It follows that in the injective EHP S.S for $H^{*,*}(\Lambda)$ we have the following:

(*)
$$h_0h_i \text{ projects to } \lambda_{2^i-2}\lambda_1\in\overline{E}_\infty^{2^{i+1},2,2^i-1},$$
$$h_0^2h_i \text{ projects to } \lambda_{2^i-3}\lambda_1^2\in\overline{E}_\infty^{2^{i+1},3,2^i-1},$$
$$h_0h_i^2 \text{ projects to } \lambda_{2^i-2}\lambda_1\lambda_{2^i-1}\in\overline{E}_\infty^{2^{i+1}+2^i,3,2^{i+1}-2},$$
$$h_0h_jh_i \text{ projects to } \lambda_{2^i-2^j-2}\lambda_1\lambda_{2^{j+1}-1}\in\overline{E}_\infty^{2^{i+1},3,2^i+2^j-2}.$$

Now consider the following corresponding elements in $\overline{H}_*(P)\otimes\Lambda$:

(**)
$$e_{2^i-2}\otimes\lambda_1+\sum_{l=2}^{i-1}e_{2^i-2^l}\otimes\lambda_{2^l-1}\in(\widetilde{H}_*(P)\otimes\Lambda)^{1,2^i-1},$$
$$e_{2^i-3}\otimes\lambda_1^2+\sum e_{i_1(\nu)}\otimes\lambda_{i_2(\nu)}\lambda_{i_3(\nu)}\in(\widetilde{H}_*(P)\otimes\Lambda)^{2,2^i-1},$$
$$e_{2^i-2}\otimes\lambda_1\lambda_{2^i-1}+\sum_\mu e_{j_1}(\mu)\otimes\lambda_{j_2(\mu)}\lambda_{j_3(\mu)}\in(\widetilde{H}_*(P)\otimes\Lambda)^{2,2^{i+1}-2},$$
$$e_{2^i-2^j-2}\otimes\lambda_1\lambda_{2^{j+1}-1}+\sum_\mu e_{k_1(\mu)}\otimes\lambda_{k_1(\mu)}\lambda_{k_3(\mu)}\in(\widetilde{H}_*(P)\otimes\Lambda)^{2,2^i+2^j-2}.$$

LEMMA (3.8). *These are cycles in* $\widetilde{H}_*(P)\otimes\Lambda$.

Proof. Let $\lambda_0\lambda_{2^i}=\sum_w\lambda_{k_1(w)}\lambda_{k_2(w)}$ be the admissible expansion. Clearly $k_1(w)\geq 1$. Then from (a), (b) in §1 and (f), we have the following identities in $\widetilde{H}_*(P)\otimes\Lambda$:

$$\delta(e_2^i)\;=\;e_{2^i-1}\otimes\lambda_0+e_{2^i-2}\otimes\lambda_1+\sum_{l=2}^{i-1}e_{2^i-2^l}\otimes\lambda_{2^l-1},$$

$$
\begin{aligned}
&\delta(e_{2^i}\otimes\lambda_0+\sum_w e_{k_1(w)}\otimes\lambda_{k_2(w)} \\
=\;& e_{2^i-1}\otimes\lambda_0^2+e_{2^i-3}\otimes\lambda_1^2+\sum_\nu e_{i_1(\nu)}\otimes\lambda_{i_2(\nu)}\lambda_{i_3(\nu)}, \\
&\delta(e_{2^i}\otimes\lambda_{2^i-1}+e_{2^i-1}\otimes\lambda_{2^i}) \\
=\;& e_{2^i-1}\otimes\lambda_{2^i-1}\lambda_0+e_{2^i-2}\otimes\lambda_1\lambda_{2^i-1}+\sum_\mu e_{j_1(\mu)}\otimes\lambda_{j_2(\mu)}\lambda_{j_3(\mu)}, \\
&\delta(e_{2^j}\otimes\lambda_{2^i-1}+e_{2^j-1}\otimes\lambda_{2^i}+e_{2^i+2^j-1}\otimes\lambda_0) \\
=\;& e_{2^i-1}\otimes\lambda_{2^j-1}\lambda_0+e_{2^i-2^j-2}\otimes\lambda_1\lambda_{2^{j+1}-1}+\sum_\mu e_{k_1(\mu)}\otimes\lambda_{k_2(\mu)}\lambda_{k_3(\mu)}.
\end{aligned}
$$

The conclusion of the lemma follows since $e_{2^i-1}\otimes\lambda_0$, $e_{2^i-1}\otimes\lambda_0^2$, $e_{2^i-1}\otimes\lambda_{2^i-1}\lambda_0$ and $e_{2^i-1}\otimes\lambda_{2^j-1}\lambda_0$ are cycles in $\widetilde{H}_*(P)\otimes\Lambda$. Q.E.D.

From (3.2)(2) and (3.7) we see the classes in $\mathrm{Ext}_A^{*,*}(P)$ represented by the cycles in (**) project respectively to the following classes in $\overline{E}_\infty(P)$ in the injective EHP S.S for $\mathrm{Ext}_A^{*,*}(P)$:

$$
\begin{aligned}
e_{2^i-2} &\otimes \lambda_1\in\overline{E}_\infty^{2^{i+1},1,2^i-1}(P), \\
e_{2^i-3} &\otimes \lambda_1^2\in\overline{E}_\infty^{2^{i+1},2,2^i-1}(P), \\
e_{2^i-2} &\otimes \lambda_1\lambda_{2^i-1}\in\overline{E}_\infty^{2^{i+1}+2^i,2,2^{i+1}-2}(P), \\
e_{2^i-2^j-2} &\otimes \lambda_1\lambda_{2^{j+1}-1}\in\overline{E}_\infty^{2^{i+1},2,2^i+2^j-2}(P).
\end{aligned}
$$

Comparing these with those classes in (*) we see (3.1) is also true for the classes h_0h_i, $h_0^2h_i$ and $h_0h_i^2$ and $h_0h_jh_i$.

This completes the proof of Proposition (3.1).

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