Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

THE HOMOTOPY TYPE OF CERTAIN CONFIGURATION SPACES

BY W. S. MASSEY

Dedicated to the memory of my friend, José Adem

We determine the homotopy type of the configuration space $F_3(R^n)$, which is the set of all ordered triples (p_1,p_2,p_3) of distinct points of R^n. For $n = 1, 2, 4$, or 8, $F_3(R^n)$ is homeomorphic to the product of $F_2(R^n)$ and R^n minus two points. For other values of n, it is not even of the same homotopy type as this product.

1. Introduction

Let M be a connected manifold; the *configuration space*, $F_k(M)$, is the space of all ordered k-tuples $(x_1, x_2, \ldots, x_k)$ of distinct points of M; it is topologized as a subspace of the product space $M \times M \times \ldots \times M$ (k factors). Apparently the term "configuration space" originated in classical mechanics. Configuration spaces are used in the theory of braids (see the opening chapters of Joan Birman's book [2]), the homotopy classification of higher dimensional links, (see Massey [5] or Koschorke [4]) and in the description of iterated loop spaces (see G. Segal [6], J.P. May [8], and F.R. Cohen [9]). The most important case in all these applications of configuration spaces is the case where $M = R^n$.

It is easily verified that $F_2(R^n)$ has the homotopy type of an $(n-1)$-dimensional sphere. This paper is concerned with the next problem, to determine the homotopy type of $F_3(R^n)$.

2. Some known results

Let $p : F_m(M) \to F_n(M)$ be defined for $m > n$ by $p(x_1, x_2, \ldots, x_m) \to (x_1, x_2, \ldots, x_n)$. Fadell and Neuwirth [3] proved that this map defines $F_m(M)$ as a locally trivial fibre space over $F_n(M)$. They left open the question as to whether or not $F_m(M)$ is fibre bundle over $F_n(M)$ (in the sense of Steenrod, [7]).

In this paper we will be concerned with this fibration in case $m = 3$, $n = 2$, and $M = R^n$:

$$p : F_3(R^n) \to F_2(R^n)$$

It is readily seen that the fibre is R^n with two points removed, and that the fibration admits a cross section $s : F_2(R^n) \to F_3(R^n)$, e.g., define $s(x_1, x_2) = (x_1, x_2, \frac{1}{2}(x_1 + x_2))$. Since $F_2(R^n)$ has the homotopy type of S^{n-1}, and R^n with two points removed has the homotopy type of $S^{n-1} \vee S^{n-1}$, one can use this information to determine the homotopy groups and homology groups of $F_3(R^n)$. At this stage, the following question then arises: For what values of n, if any, is the fibration $p : F_3(R^n) \to F_2(R^n)$ globally trivial, i.e., is $F_3(R^n)$ homeomorphic to the product of $F_2(R^n)$ and the fibre? If the answer is negative, we might still hope that $F_3(R^n)$ is of the same homotopy type as the product space.

It turns out that we can give rather neat answers to these questions. We will also exhibit a rather easily described compact space which has the same homotopy type as $F_3(R^n)$, and give an explicit description of a CW-complex with a minimum number of cells having this homotopy type.

3. Statement of results

Let e be a unit vector in R^n, e.g., we could take $e = (1, 0, \ldots, 0)$.

THEOREM (I). *The fibre space $p : F_3(R^n) \to F_2(R^n)$ is a fibre bundle (in the sense of Steenrod [7]) with fibre $R^n - \{e, -e\}$ and structure group the subgroup of $GL_n^+(R)$ which leaves the vector e fixed.*

By $GL_n^+(R)$ we mean the subgroup of $GL_n(R)$ consisting of matrices having positive determinant. By a well known theorem, the group of this bundle can be reduced to the maximal compact subgroup of the structural group; in this case, the maximal compact subgroup is the rotation group $SO(n-1)$ acting in the subspace orthogonal to the vector e.

In preparation for the statement of the next theorem, define

$$\mathcal{S}_{n-1} = \{(x_1, x_2) \in F_2(R^n) \mid |x_1| = 1 \ \text{ and } \ x_2 = -x_1\}.$$

Then $\mathcal{S}_{n-1}$ is an $(n-1)$-sphere which is a deformation retract of $F_2(R^n)$. A retraction $r : F_2(R^n) \to \mathcal{S}_{n-1}$ is defined by

$$r(x_1, x_2) = (\frac{x_1 - x_2}{|x_1 - x_2|}, \frac{x_2 - x_1}{|x_1 - x_2|}).$$

Let

$$E_n = p^{-1}(\mathcal{S}_{n-1}) \subset F_3(R^n).$$

Then $p \mid E_n : E_n \to \mathcal{S}_{n-1}$ is a fibre bundle with the same fibre and structure group as $p : F_3(R^n) \to F_2(R^n)$, and the latter bundle is induced from the former by the retraction $r : F_2(R^n) \to \mathcal{S}_{n-1}$. Also, the spaces $F_3(R^n)$ and E_n have the same homotopy type.

THEOREM (II). *The bundle $E_n \to \mathcal{S}_{n-1}$ is associated to the tangent bundle of the sphere $\mathcal{S}_{n-1}$ for $n > 1$.*

COROLLARY *For $n = 1, 2, 4$, and 8, the bundle $p : F_3(R^n) \to F_2(R^n)$ is a product bundle, and $F_3(R^n)$ is homeomorphic to $F_2(R^n) \times [R^n - \{e, -e\}]$.*

Proof of Corollary. The tangent bundle to an $(n-1)$-sphere is a product bundle for $n = 2, 4$, or 8. The case $n = 1$ has to be treated separately, but it is entirely trivial. (This corollary also follows easily from some results of F.R. Cohen; see Propositions 6.4 and 6.5 on p. 257 of [9])

It remains to discuss the homotopy type of $F_3(R^n)$ in case $n \neq 1, 2, 4$, or 8.

THEOREM (III). *$F_3(R^n)$ has the same homotopy type as the following space: Take two copies of $S^{n-1} \times S^{n-1}$ and identify them along their diagonals.*

By the diagonal of $S^{n-1} \times S^{n-1}$ we mean $\{(x,x) \mid x \in S^{n-1}\}$, as usual.

THEOREM (IV). *If $n \neq 1, 2, 4$, or 8, then $F_3(R^n)$ does not have the same homotopy type as the product space $F_2(R^n) \times [R^n - \{e, -e\}]$.*

In the course of proving theorem (IV), we will explicitly construct a CW-complex which is of the same homotopy type as the space described in theorem (III) and having a minimum number of cells.

4. Proof of Theorem I

In this section we will concern ourselves with the fibration $p : F_3(R^n) \to F_2(R^n)$ for $n > 1$; the case $n = 1$ is rather trivial. Let A_n denote the group of all orientation preserving affine transformations of Euclidean n-space. A_n is a connected, non-compact Lie group, and its operation on R^n is "two point transitive", in the sense that given two ordered pairs (x_1, x_2) and (y_1, y_2) of *distinct* points of R^n, there exists an element $g \in A_n$ such that $g \cdot x_i = y_i$ for $i = 1, 2$. The group A_n also operates on $F_2(R^n)$ in a obvious way, and in view of the preceding statement, the operation is transitive.

LEMMA (1). *Let $x_0 \in F_2(R^n)$. Then there exists an open neighborhood U of x_0 in $F_2(R^n)$ and a differentiable function $s : U \to A_n$ such that $s(x_0) = 1$ and for any $x \in U$, $(sx) \cdot x_0 = x$.*

Proof. Define $q : A_n \to F_2(R^n)$ by $q(g) = g \cdot x_0$ for any $g \in A_n$. Then q is a continuous map of A_n onto $F_2(R^n)$. Let G denote the isotropy subgroup of the point x_0. Then G is a closed subgroup of A_n, and it is easily proved that q induces a homeomorphism of the coset space A_n/G onto $F_2(R^n)$. Also, $q : A_n \to F_2(R^n)$ is a principal G-bundle (see Steenrod [7], §7). Choose a neighborhood U of $x_0 \in F_2(R^n)$ such that there exists a differentiable cross-section $s : U \to A_n$ of the map q. Since $q(1) = x_0$, it is clear that we may choose the cross-section s so that $s(x_0) = 1$.

Using this neighborhood U of the point $x_0 = (x_{01}, x_{02}) \in F_2(R^n)$, we will now define a diffeomorphism

$$f : U \times [R^n - \{x_{01}, x_{02}] \to p^{-1}(U)$$

by the following formula:

$$f[(x_1, x_2), x_3] = (x_1, x_2, s(x_1, x_2) \cdot x_3). \tag{1}$$

Here $(x_1, x_2) \in U, x_3 \in [R^n - \{x_{10}, x_{20}]$, and s is the function of lemma (1). Recall that this function satisfies the condition

$$s(x_1, x_2) \cdot (x_{10}, x_{20}) = (x_1, x_2)$$

for any $x = (x_1, x_2) \in U$. This is equivalent to the following two equations:

$$\begin{aligned} s(x_1,x_2) \cdot x_{10} &= x_1, \\ s(x_1,x_2) \cdot x_{20} &= x_2. \end{aligned}$$

Hence, if $x_3 \neq x_{10}$ and $x_3 \neq x_{20}$, it follows that $s(x_1,x_2) \cdot x_3 \neq x_1$ and $s(x_1,x_2) \cdot x_3 \neq x_2$. Therefore formula (1) does indeed define a differentiable mapping of $U \times [R^n - \{x_{01}, x_{02}\}]$ into $p^{-1}(U) \subset F_3(R^n)$. To see that f is a diffeomorphism, observe that it has an inverse defined by

$$f^{-1}(y_1, y_2, y_3) = (y_1, y_2, [s(y_1, y_2)]^{-1} \cdot y_3)$$

where $(y_1, y_2, y_3) \in p^{-1}(U)$. Finally, notice that

$$p\,f[(x_1,x_2),x_3] = (x_1,x_2).$$

Using these formulas, we can now prove theorem (I). We will use the terminology, etc. of §2 of Steenrod, [7].

Given any point $x_0 = (x_{01}, x_{02}) \in F_2(R^n)$, choose a neighborhood U of x_0 and a function $s : U \to A_n$ as in lemma (1). Choose an element $t \in A_n$ such that

$$t(-e) = x_{01},\, t(e) = x_{02},$$

where $e \in R^n$ is a unit vector. Define a coordinate function

$$\varphi_U : U \times [R^n - \{e, -e\}] \to p^{-1}(U)$$

by the formula

$$\varphi_U[(x_1,x_2),x_3] = (x_1, x_2, s(x_1,x_2) \cdot t \cdot x_3).$$

The coordinate functions thus defined satisfy all the conditions needed to define a coordinate bundle; see Steenrod, loc.cit. The group of the bundle is the set of all $g \in A_n$ such that $g \cdot (\pm e) = \pm e$. This implies that g leaves the origin fixed, and hence belongs to the subgroup $GL_n^+(R)$ of A_n.

Remark. This proof depends essentially on the fact that the Lie group A_n is 2-point transitive on R^n. The author's colleagues G. Margulis and G.D. Mostow have orally described proofs that no connected Lie group can operate on R^n in a manner which is 3-point transitive. Thus one can not hope to generalize this proof to $F_k(R^n)$ for $k > 3$.

5. Proof of Theorem II

As usual let S^{n-1} denote the unit sphere in R^n, and let

$$E_n' = \{(x,y) \in S^{n-1} \times R^n \mid y \neq \pm x\}.$$

Define $p' : E'_n \to S^{n-1}$ by $p'(x,y) = x$. Then we have a commutative diagram, as follows,

$$\begin{array}{ccc} E_n & \xrightarrow{p|E_n} & S_{n-1} \\ \downarrow & & \downarrow \\ E'_n & \xrightarrow{p'} & S^{n-1} \end{array}$$

where the vertical arrows are homeomorphisms. To verify this, note that

$$E_n = \{(x_1,x_2,x_3) \in F_3(R^n) \mid |x_1| = 1 \text{ and } x_2 = -x_1\}.$$

Thus to prove theorem (II), it suffices to prove that $p' : E'_n \to S^{n-1}$ is a fibre bundle associated to the tangent bundle of S^{n-1}. We will leave the details of the proof to the reader, and will only offer the following suggestion: Consider the following somewhat similar problem. Let $T_n = \{(x,y) \in S^{n-1} \times R^n \mid y \cdot x = 0\}$. Define $q : T_n \to S^{n-1}$ by $q(x,y) = x$. Prove that $q : T_n \to S^{n-1}$ is the tangent bundle of S^{n-1}.

6. Proof of Theorem III

In this section, we will assume $n > 1$. The discussion of the case $n = 1$ is rather trivial. Since the space E'_n has the same homotopy type as $F_3(R^n)$, it suffices to prove that E'_n has the homotopy type of the space described in the statement of the theorem. Now E'_n is a fibre bundle with fibre $R^n - \{e, -e\}$ and structural group $SO(n-1)$; and $SO(n-1)$ acts on the fibre by rotations in the subspace of R^n perpendicular to the unit vector e. Let

$$\begin{aligned} S^{n-1} \vee S^{n-1} &= \{x \in R^n \mid |x - e| = 1\} \\ &\cup \{x \in R^n \mid |x + e| = 1\} \end{aligned}$$

Then $S^{n-1} \vee S^{n-1}$ is the union of two $(n-1)$-spheres of radius 1 whose only common point is the origin. It is clear that $S^{n-1} \vee S^{n-1}$ is a deformation retract of $R^n - \{e, -e\}$, and that the action of $SO(n-1)$ on the fibre $R^n - \{e, -e\}$ carries $S^{n-1} \vee S^{n-1}$ into itself. Hence there is a sub-bundle $E''_n \subset E'_n$ with fibre $S^{n-1} \vee S^{n-1}$ which is a deformation retract of E'_n. In the notation used in the proof of theorem (II),

$$\begin{aligned} E''_n &= \{(x,y) \in S^{n-1} \times R^n \mid |y - x| = 1\} \\ &\cup \{(x,y) \in S^{n-1} \times R^n \mid |y + x| = 1\}. \end{aligned}$$

If we let

$$E_n^+ = \{(x,y) \in S^{n-1} \times R^n \mid |y - x| = 1\},$$

and

$$E_n^- = \{(x,y) \in S^{n-1} \times R^n \mid |y + x| = 1\},$$

then $E''_n = E^+_n \cup E^-_n, E^+_n \cap E^-_n$ is an $(n-1)$-sphere which is a cross-section of the bundle, and both E^+_n and E^-_n are $(n-1)$-sphere bundles over S^{n-1} which are associated to the tangent bundle of S^{n-1}. We assert that *both E^+_n and E^-_n are product bundles, and hence are homeomorphic to $S^{n-1} \times S^{n-1}$*. The reason for this is the well-known fact that the tangent bundle to S^{n-1} plus a trivial line bundle is a product bundle. The bundles E^+_n and E^-_n can both be regarded as the unit sphere bundle of the tangent bundle plus a trivial line bundle to S^{n-1}.

Thus E''_n is a space obtained by taking two copies $S^{n-1} \times S^{n-1}$ and identifying along an $(n-1)$-sphere. The problem is to precisely describe how the identification is to be made. This requires that we describe precisely the trivialization of the bundles E^+_n and E^-_n. For this purpose, we define homeomorphisms

$$\varphi, \psi : S^{n-1} \times R^n \to S^{n-1} \times R^n$$

by the following simple formulas:

$$\begin{aligned} \varphi(x,y) &= (x, y-x), \\ \psi(x,y) &= (x, y+x). \end{aligned}$$

Then φ and ψ are inverses of each other, and they leave the first coordinate unchanged. In this notation,

$$S^{n-1} \times S^{n-1} = \{(x,y) \in S^{n-1} \times R^n \mid |y| = 1\}$$

is the product bundle over S^{n-1}. It can be quickly verified that φ maps E^+_n onto $S^{n-1} \times S^{n-1}$, and ψ maps E^-_n onto $S^{n-1} \times S^{n-1}$. Thus φ and ψ provide the needed trivializations of the bundles E^n_+ and E^n_-. Also, φ maps $E^n_+ \cap E^n_-$ onto the "anti-diagonal", $\{(x, -x) \mid x \in S^{n-1}\}$, while ψ maps $E^n_+ \cap E^n_-$ onto the diagonal $\{(x,x) \mid x \in S^{n-1}\}$ of $S^{n-1} \times S^{n-1}$. It follows that E''_n is homeomorphic to the space obtained by taking two copies of $S^{n-1} \times S^{n-1}$, and identifying the diagonal of one copy with the anti-diagonal of the other. But there are obvious self-homeomorphisms of $S^{n-1} \times S^{n-1}$ which interchange the diagonal and the anti-diagonal: apply the antipodal map on one of the factors. This leads immediately to the statement of theorem (III).

7. Proof of Theorem IV

We will describe an explicit construction of a CW-complex having the homotopy type of the space E''_n. The $(n-1)$-skeleton will be $S^{n-1} \vee S^{n-1} \vee S^{n-1}$, a wedge of three spheres, and there will be two cells of dimension $2n-2$. The homotopy classes of the attaching maps of these two cells will be described explicitly.

As a preliminary step in this construction, we will describe a CW-complex having the same homotopy type as $S^{n-1} \times S^{n-1}$ but which is different from the usual description of $S^{n-1} \times S^{n-1}$ as a CW-complex.

First, let L denote the usual CW-complex on $S^{n-1} \times S^{n-1}$: the $(n-1)$-skeleton is

$$L^{n-1} = S_1^{n-1} \vee S_2^{n-1},$$

the wedge of two $(n-1)$-spheres, and there is a single $(2n-2)$-cell. The homotopy class of the attaching map is the Whitehead product $\beta = [\iota_1, \iota_2]$, where $\iota_k : S_k^{n-1} \to S_1^{n-1} \vee S_2^{n-1}$ is the (homotopy class of) the inclusion map for $k = 1, 2$. We may as well assume that the attaching map is chosen in its homotopy class so that L is actually homeomorphic to $S_1^{n-1} \times S_2^{n-1}$. Similarly, let K denote a CW-complex which has the same $(n-1)$-skeleton,

$$K^{n-1} = S_1^{n-1} \vee S_2^{n-1},$$

but now the attaching map for the single $(2n-2)$-cell is

$$\alpha = [\iota_1, \iota_2] - [\iota_1, \iota_1] = [\iota_1, \iota_2 - \iota_1].$$

(It is assumed that all spheres are appropriately oriented). Next, define a map

$$f : K^{n-1} \to L = S_1^{n-1} \times S_2^{n-1}$$

such that the sphere S_1^{n-1} is mapped onto the sphere S_1^{n-1} with degree $+1$, and the sphere S_2^{n-1} is mapped onto the diagonal. In terms of the induced homomorphism

$$f_* : \pi_{n-1}(K^{n-1}) \to \pi_{n-1}(L)$$

we are requiring that

$$\begin{aligned} f_*(\iota_1) &= \iota_1, \\ f_*(\iota_2) &= \iota_1 + \iota_2. \end{aligned}$$

We now homotopically deform the map f so that K^{n-1} is mapped into L^{n-1}; we will denote the deformed map by the same symbol f. Consider the induced homomorphism

$$f_* : \pi_{2n-3}(K^{n-1}) \to \pi_{2n-3}(L^{n-1}).$$

Then

$$\begin{aligned} f_*(\alpha) &= f_*[\iota_1, \iota_2 - \iota_1] = [f_*\iota_1, (f_*\iota_2) - (f_*\iota_1)] \\ &= [\iota_1, (\iota_1 + \iota_2) - \iota_1] = [\iota_1, \iota_2] = \beta. \end{aligned}$$

Since $f_*(\alpha) = \beta$, the map f can be extended to a map $F : K \to L$ such that the $(2n-2)$-cell of K is mapped onto the $(2n-2)$-cell of L with degree $+1$. Obviously, F must be a homotopy equivalence. Also, the map F can be deformed homotopically so that the sphere $S_2^{n-1} \subset K$ is mapped onto the sphere $S_1^{n-1} \subset L$ with degree $+1$, and S_2^{n-1} is mapped onto the diagonal of

$L = S^{n-1} \times S^{n-1}$ with degree $+1$. Thus we have achieved our goal: K is of the same homotopy type as $S^{n-1} \times S^{n-1}$, and the "diagonal" is part of the $(n-1)$-skeleton of K, at least up to homotopy.

Remark. It may be possible to construct K so that it is *homeomorphic* to $S^{n-1} \times S^{n-1}$; however, we have no need for this stronger condition.

It is now clear how to construct a CW-complex of the same homotopy type as E_n'': take two copies of K, and identify them along the sphere S_2^{n-1}.

Changing notation, we have proved that E_n'' is of the same homotopy type as a CW-complex K_n defined as follows:

$$K_n = (S_1^{n-1} \vee S_2^{n-1} \vee S_3^{n-1}) \cup e_1^{2n-2} \cup e_2^{2n-2}$$

where the top dimensional cells are adjoined by maps representing

$$\begin{aligned} \alpha_1 &= [\iota_1, \iota_2] - [\iota_1, \iota_1] = [\iota_1, \iota_2 - \iota_1], \\ \alpha_2 &= [\iota_3, \iota_2] - [\iota_3, \iota_3] = [\iota_3, \iota_2 - \iota_3]. \end{aligned}$$

For sake of comparison, $(S^{n-1} \vee S^{n-1}) \times S^{n-1}$ is a CW-complex

$$L_n = (S_1^{n-1} \vee S_2^{n-1} \vee S_3^{n-1}) \cup e_1^{2n-2} \cup e_2^{2n-2}$$

where the attaching maps are

$$\begin{aligned} \beta_1 &= [\iota_1, \iota_2] \\ \beta_2 &= [\iota_3, \iota_2] \end{aligned}$$

Note that K_n and L_n have the same $(n-1)$-skeletons, $K_n^{n-1} = L_n^{n-1} = S_1^{n-1} \vee S_2^{n-1} \vee S_3^{n-1}$.

We will now prove that K_n and L_n do not have the same homotopy type, provided $n \neq 1, 2, 4$, or 8 (in particular, $n \geq 3$). The proof is by contradiction; assume there exists a homotopy equivalence $f : K_n \to L_n$ with homotopy inverse $g : L_n \to K_n$. We may assume that f and g are cellular maps; it follows that they define a homotopy equivalence between the pairs (K_n, K_n^{n-1}) and (L_n, L_n^{n-1}). These maps induce isomorphisms of the corresponding long exact sequences of homotopy groups. We are particularly interested in the following part of these long exact sequences:

$$\begin{array}{ccc} \pi_{2n-2}(K_n, K_n^{n-1}) & \xrightarrow{\partial_1} & \pi_{2n-3}(K_n^{n-1}) \\ \big\downarrow f_* & & \big\downarrow f_* \\ \pi_{2n-2}(L_n, L_n^{n-1}) & \xrightarrow{\partial_2} & \pi_{2n-3}(L_n^{n-1}) \end{array}$$

This diagram is commutative, and the arrows labelled f_* are isomorphisms. The relative homotopy groups in the left hand column of this diagram are free abelian of rank 2. The image of the homomorphism ∂_1 is the subgroup generated by α_1 and α_2, and the image of ∂_2 is generated by β_1 and β_2. The group $\pi_{n-1}(K_n^{n-1}) = \pi_{n-1}(L_n^{n-1})$ is free abelian of rank 3 with basis $\{\iota_1, \iota_2, \iota_3\}$. The structure of the group $\pi_{2n-3}(K_n^{n-1}) = \pi_{2n-3}(L_n^{n-1})$ is described by a well known theorem of Hilton: it is the direct sum of a free group of rank 3 (with basis the Whitehead products $[\iota_j, \iota_k]$ for $1 \leq j < k \leq 3$) and the subgroups $\pi_{2n-3}(S_k^{n-1})$ for $k = 1, 2, 3$.

Now let π'_{2n-3} denote the subgroup of $\pi_{2n-3}(K_n^{n-1}) = \pi_{2n-3}(L_n^{n-1})$ which is generated by all Whitehead products $[u, v]$ for $u, v \in \pi_{n-1}(K_n^{n-1})$. Note the following facts about this subgroup:

1) π'_{2n-3} is free abelian of rank 6 if n is odd, with basis the Whitehead products $[\iota_j, \iota_k]$ for $1 \leq j \leq k \leq 3$. If n is even, then π'_{2n-3} is the direct sum of a free abelian group of rank 3 and three cyclic groups of order two. In this case, the Whitehead products $[\iota_j, \iota_k]$ still generate π'_{2n-3}, and $[\iota_j, \iota_j]$ is an element of order two. This last assertion depends on the solution of the Hopf invariant one problem by Frank Adams [1].

2) π'_{2n-3} contains the images of the homomorphisms ∂_1 and ∂_2. This is a consequence of the way the attaching maps $\alpha_1, \alpha_2, \beta_1$, and β_2 were chosen.

3) Let $\varphi : K_n^{n-1} \to L_n^{n-1}$ be any continuous map. Then the induced homomorphism $\varphi_* : \pi_{2n-3}(K_n^{n-1}) \to \pi_{2n-3}(L_n^{n-1})$ maps π'_{2n-3} into itself; if φ is a homotopy equivalence, then φ_* induces an automorphism of π'_{2n-3}.

Now consider the following commutative diagram:

$$\begin{array}{ccc} \pi_{n-1}(K_n^{n-1}) \otimes \pi_{n-1}(K_n^{n-1}) & \longrightarrow & \pi'_{2n-3} \\ \Big\downarrow{\scriptstyle f_*\otimes f_*} & & \Big\downarrow{\scriptstyle f_*} \\ \pi_{n-1}(L_n^{n-1}) \otimes \pi_{n-1}(L_n^{n-1}) & \longrightarrow & \pi'_{2n-3}. \end{array}$$

The horizontal arrows denote Whitehead products and the vertical arrows are isomorphisms.

First, we will consider the case where n is odd, $n \geq 3$. Then $n-1$ is even, hence the Whitehead products in the above diagram are commutative i.e. $[u, v] = [v, u]$. We wish to prove that the automorphism $f_* : \pi'_{2n-3} \to \pi'_{2n-3}$ can not map the image of ∂_1 onto the image of ∂_2. In order to better understand this situation, consider the following algebraically isomorphic situation: Let $P_* = \Sigma P_n$ denote the polynomial algebra in three variables, x_1, x_2, and x_3 over the ring of integers; it is to be considered as a graded algebra, with the grading defined by the usual notion of the degree of a homogeneous polynomial. Define an isomorphism $P_1 \approx \pi_{n-1}(K_n^{n-1})$ by letting x_k correspond to ι_k for $k = 1, 2, 3$; also, $P_2 \approx \pi'_{2n-3}$, where the monomial $x_j x_k$ corresponds to the Whitehead product $[\iota_j, \iota_k]$. With this correspondence, the Whitehead product

$$\pi_{n-1}(K_n^{n-1}) \otimes \pi_{n-1}(K_n^{n-1}) \to \pi'_{2n-3}$$

corresponds to the multiplication

$$P_1 \otimes P_1 \to P_2$$

in the polynomial algebra. We will use this correspondence to complete the proof of the theorem. The isomorphism

$$f_* : \pi_{n-1}(K_n^{n-1}) \to \pi_{n-1}(L_n^{n-1})$$

has the form

$$f_*(\iota_j) = \sum_{k=1}^{3} n_{jk}\iota_j, \quad j = 1, 2, 3$$

where (n_{jk}) is a 3×3 integer matrix with determinant ± 1. Using this matrix, define an automorphism

$$\varphi : P_1 \to P_1$$

by the formula

$$\varphi(x_j) = \sum_{k=1}^{3} n_{jk}x_k, \quad j = 1, 2, 3.$$

Then φ extends to a degree preserving automorphism of the polynomial algebra P_* in a unique way; we will denote this extended automorphism by the same symbol, φ. Now consider the effect of this extended automorphism in degree 2:

$$\varphi : P_2 \to P_2.$$

Since f_* maps the subgroup generated by α_1 and α_2 onto the subgroup generated by β_1 and β_2, it follows that φ must map the subgroup of P_2 generated by the polynomials $x_1x_2 - x_1^2$ and $x_2x_3 - x_3^2$ onto the subgroup generated by x_1x_2 and x_2x_3. As a consequence of this fact, the algebra isomorphism φ must map the ideal

$$I_1 = (x_1x_2 - x_1^2, x_2x_3 - x_3^2)$$

onto the ideal

$$I_2 = (x_1x_2, x_2x_3).$$

But this is clearly impossible: all polynomials in the ideal I_2 have x_2 as a factor, while the two generators of the ideal I_1 are relatively prime. Clearly these properties are invariant under any automorphism of P_*. This contradiction completes the proof in case n is odd.

It remains to consider the case where n is even, $n \neq 2, 4$, or 8. In this case the Whitehead products involved are anti-commutative rather than commutative: $[u, v] = -[v, u]$, and $[\iota_k, \iota_k]$ is of order two. However, the preceding proof can be adapted to this case by reducing the groups $\pi_{n-1}(K_n^{n-1})$ and π'_{2n-3} modulo two, and letting P_* denote the graded polynomial algebra in the variables x_1, x_2, and x_3 over the ring of integers *modulo two* . With these modifications, the previous proof goes through.

This completes the proof of theorem (IV).

Remark. It can be shown that for n odd, the integral cohomology rings of the spaces $F_3(R^n)$ and $F_2(R^n) \times [R^n - \{e, -e\}]$ are non-isomorphic. This fact also proves theorem (IV) in the case where n is odd. For n even, the cohomology rings of these two spaces are isomorphic.

YALE UNIVERSITY
NEW HAVEN, CT. 06520
U.S.A.

REFERENCES

[1] J.F. ADAMS, *On the non-existence of elements of Hopf invariant one*, Ann. Math. **72** (1960), 20–104.
[2] J. BIRMAN, *Braids, Links, and Mapping Class Groups*, Ann. Math. Studies **82** (1974).
[3] E. FADELL AND L. NEUWIRTH, *Configuration Spaces*, Math. Scand. **10** (1962), 111–118.
[4] U. KOSCHORKE, *Higher order homotopy invariants for higher dimensional link maps*, Springer Lecture Notes in Math. **1172** (1985), 116–129.
[5] W.S. MASSEY, *Homotopy classification of 3-Component links of codimension greater then 2*, Top. and Its Appl. **34** (1990), 169–300.
[6] G. SEGAL, *Configuration spaces and iterated loop spaces*, Inv. Math. **21** (1973), 213–221.
[7] N.E. STEENROD, The Topology of Fibre Bundles, Princeton University Press (1951).
[8] J.P. MAY, *The Geometry of Iterated Loop Spaces*, Springer Lecture Notes in Mathematics **27** (1972).
[9] F.R. Cohen, T.J. LADA, AND J.P. MAY, *The homology of Iterated Loop Spaces*, Springer Lecture Notes in Mathematics **533** (1976).