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PHANTOM MAPS AND RATIONAL EQUIVALENCES, II

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Let X be a *finite type domain*; that is, a connected CW-complex with finitely generated homology groups in each degree. A *phantom* map from X to another space Y is a pointed map whose restriction to the n-skeleton of X is null homotopic for each integer n. Denote by $\mathrm{Ph}(X,Y)$ the set of pointed homotopy classes of phantom maps from X to Y. Call Y a *finite type target* (or a *countable type target*) if $\pi_n Y$ is finitely generated (or countable) for each $n \geq 2$.[1] In this paper, the source of a phantom map will always be a finite type domain but its range will sometimes be a finite type target, sometimes a countable type target and sometimes an arbitrary target.

1. The influence of ΣX on Ph(X, Y)

A map $g\colon X \to X'$ obviously induces a function from $\mathrm{Ph}(X',Y)$ to $\mathrm{Ph}(X,Y)$, given by precomposition with g. However, it is not obvious that a map from ΣX to $\Sigma X'$ should induce a function from $\mathrm{Ph}(X',Y)$ to $\mathrm{Ph}(X,Y)$, or that it should imply any sort of relationship between $\mathrm{Ph}(X,Y)$ and $\mathrm{Ph}(X',Y)$. Nevertheless, the following theorem shows that this indeed happens in many cases.

THEOREM (1). *Let X and X' be finite type domains and let Y be a finite type target. Assume that there exists a map*

$$\Sigma X \xrightarrow{\ f\ } \Sigma X'$$

that induces a monomorphism in rational homology. Then

$$\mathrm{Ph}(X',Y) = 0. \quad \Longrightarrow \quad \mathrm{Ph}(X,Y) = 0.$$

Moreover, f induces a surjection from $\mathrm{Ph}(X',Y)$ to $\mathrm{Ph}(X,Y)$ *provided its rationalization induces a homomorphism*

$$[\Sigma X', Y^{(n)}]_o \xrightarrow{\ f_o^*\ } [\Sigma X, Y^{(n)}]_o,$$

for each integer n. This happens, for example, if f becomes a co-H-map when rationalized or if the universal cover of Y is an H_0-space. ∎

A few more definitions and remarks are in order; proofs will be given later. We write $\mathrm{Ph}(X,Y) = 0$ when this set of homotopy classes has just one element. In Theorem 1 and elsewhere $Y^{(n)}$ denotes the Postnikov approximation of Y up through dimension n. Notice that since ΣX is 1-connected, $Y^{(n)}$ can be replaced by its universal cover in the group of pointed homotopy

[1] It is unnecessary to place any restriction on $\pi_1 Y$.

classes $[\Sigma X, Y^{(n)}]$. The rationalization of this nilpotent group can be thus identified with $[\Sigma X, Z]$ where Z denotes the rationalization of the universal cover of $Y^{(n)}$. Finally, by an H_0- space we mean a space which has the rational homotopy type of an H-space.

COROLLARY (2). *Let X and X' be finite type domains and let Y be a finite type target. Assume that there exist maps between ΣX and $\Sigma X'$, in both directions, that induce isomorphisms in rational homology. Then the sets* $\mathrm{Ph}(X, Y)$ *and* $\mathrm{Ph}(X', Y)$ *have the same cardinality; namely either 1 or* $2^{\aleph_0}$. ∎

Some special cases of these two results were discovered in [9]. In particular it was shown there that Theorem 1 was true when the map f is a suspension. In that paper we also characterized those finite type domains X which have no essential phantom maps into finite type targets. The result is the following:

THEOREM (3). *If X is a finite type domain, then the following statements are equivalent*:

(i) $\mathrm{Ph}(X, Y) = 0$ *for every finite type target* Y.

(ii) $\mathrm{Ph}(X, S^n) = 0$ *for every* n.

(iii) *There exists a map from ΣX to a bouquet of spheres $\vee S^{n_\alpha}$ that induces an isomorphism in rational homology.* ∎

Notice that one can always construct a rational equivalence from a bouquet of spheres *into* a suspension. [2] Thus the implication $(iii) \Rightarrow (i)$ can be regarded as a special case of Corollary 2. Indeed, it was this result that suggested Corollary 2. The conclusions of Theorem 1 and Corollary 2 fail to hold if the finite type hypothesis on the target Y is relaxed. Here is a relevant example:

EXAMPLE (A). *Let $X = \mathbb{R}P^\infty$, $X' =$ a point, and $Y = \bigvee_{n \geq 1} \Sigma \mathbb{R}P^n$. There is a rational equivalence from X to X' and $Ph(X', Y) = 0$. However, $Ph(X, Y)$ is uncountably large.*

Although the target Y in this example seems huge, it is nonetheless a countable type target. The set $\mathrm{Ph}(X, Y)$ contains the universal phantom map out of X, which is essential, by [5], and so the cardinality of this set is uncountable by [8], Theorem 2.

In the next result, the cardinality hypothesis on the target Y is relaxed at the cost of insisting that the map $f \colon \Sigma X \to \Sigma X'$ be a homotopy equivalence. Our goal is to answer the following:

[2] Of course, it is not always possible to get a rational equivalence from a suspension ΣX to a bouquet of spheres. Perhaps the simplest example of this occurs when $X = \mathbb{C}P^\infty$.

QUESTION (4). *Suppose that X and X' are two finite type domains which become homotopy equivalent after one suspension. Does it follow that the sets* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{Ph}(X',Y)$ *are isomorphic for* __*all*__ *targets Y ?*

We have been unable to answer this question in general. However, there are some special cases of it that we understand.

PROPOSITION (5). *Let X and X' be finite type domains which become homotopy equivalent after one suspension. Then*

(i) $\mathrm{Ph}(X,Y) = 0$ *for all targets Y if and only if* $\mathrm{Ph}(X',Y) = 0$ *for all targets Y.*

(ii) *For all countable type targets Y, the sets* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{Ph}(X',Y)$ *have the same cardinality; namely either 1 or* $2^{\aleph_0}$.

(iii) *If the universal cover of Y is an H-space, then the abelian groups* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{Ph}(X',Y)$ *are naturally isomorphic.*

(iv) *If the universal cover of Y is a finite type H_0-space, then the abelian groups* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{Ph}(X',Y)$ *are naturally isomorphic.*

(v) *If there exists a co-H-equivalence $f : \Sigma X \to \Sigma X'$, then it induces a bijection between* $\mathrm{Ph}(X',Y)$ *and* $\mathrm{Ph}(X,Y)$ *for all targets Y.*

(vi) *If there exists an equivalence $f : \Sigma X \to \Sigma X'$ which becomes a co-H-map when rationalized, then it induces a bijection between* $\mathrm{Ph}(X',Y)$ *and* $\mathrm{Ph}(X,Y)$ *for all finite type targets Y.* ▮

As illustrations of statements (ii) and (v), here are two examples:

EXAMPLE (B). *Let W be a finite type domain and let $X = \Omega\Sigma W$ and $X' = \bigvee_{j\geq 1}(\bigwedge^j W)$, where $\bigwedge^j W$ denotes the j-fold smash product. Then by Proposition 5(ii)*

$$\mathrm{Ph}(X,Y) \approx \mathrm{Ph}(X',Y) = \prod_{j\geq 1} \mathrm{Ph}(\textstyle\bigwedge^j W, Y)$$

for all countable type targets.

In this example there is a well known homotopy equivalence $\Sigma X \to \Sigma X'$ (which does not, however, desuspend unless W is contractible). Thus $\mathrm{Ph}(\Omega\Sigma W, Y) = 0$ precisely when each $\mathrm{Ph}(\bigwedge^j W, Y) = 0$; in particular, the latter happens when W is a finite CW-complex. It may be of interest to note that for any $N \geq 1$, there exist a finite type domain W and a finite type target Y such that $\mathrm{Ph}(\bigwedge^j W, Y) = 0$ for $j < N$ while $\mathrm{Ph}(\bigwedge^N W, Y) \neq \mathbf{0}$. For example, let $W = K(\mathbb{Z}, 3)$ and let $Y = \Omega S^{3N+2}$.

EXAMPLE (C). *Let A be an acyclic space and let $X' = X \vee A$. The canonical inclusion $g\colon X \to X'$ then suspends to a co-H-equivalence. By Proposition*

5(v), the induced map $\Sigma g^*\colon Ph(X',Y) \longrightarrow Ph(X,Y)$ *is a bijection for finite type domains X and all targets Y.*

For explicit examples of acyclic $K(\pi,1)$'s , see [1]. Some of these $K(\pi,1)$'s are finite CW-complexes while others have π not finitely generated. It is worth pointing out that while the latter are finite type domains in our sense, they are not finite type domains in the sense used in [5], where it was required that the domains X have finitely many cells in each dimension.

2. The kernel of g^*: $\mathrm{Ph}(X',Y) \to \mathrm{Ph}(X,Y)$

Let $g\colon X \to X'$ be a map of finite type domains that induces an isomorphism in rational homology. Theorem 1 says that $g^*\colon \mathrm{Ph}(X',Y) \to \mathrm{Ph}(X,Y)$ is surjective if Y is a finite type target but it tells us nothing about the kernel of g^*. Here is one extreme case:

EXAMPLE (D). *Let* $X = S^3$, $X' = K(\mathbb{Z},3)$, *and* $Y = S^4$. *There is a rational equivalence* $g : X \to X'$ *with degree 1 on the bottom cell. However,* $Ph(X,Y) = 0$ *while* $Ph(X',Y)$, *and hence the kernel of* g^*, *is uncountably large.*

The key point in this example is that while the rationalized spaces $S^3_{(0)}$ and $K(\mathbb{Z},3)_{(0)}$ are homotopy equivalent, there is a rational equivalence between the original spaces in only one direction. [3]

The next theorem shows that by imposing certain restrictions on the spaces X, X' and Y, one may infer that $g^*\colon \mathrm{Ph}(X',Y) \to \mathrm{Ph}(X,Y)$ is a bijection.

THEOREM (6). *Assume that X, X', and Y satisfy the following requirements:*

(i) $X = \Sigma^k P$, $k \geq 0$, *where P is a nilpotent finite type domain with* $\pi_1 P$ *finite and with* $\pi_n P = 0$ *for* $n \gg 0$.

(ii) X' *is a nilpotent finite type domain with a finite fundamental group.*

(iii) $Y = \Omega^\ell Z$, $\ell \geq -1$, *where Z is a finite CW-complex with a finite fundamental group.*

Then the induced map $g^* : \mathrm{Ph}(X',Y) \longrightarrow \mathrm{Ph}(X,Y)$ *is a bijection for any rational homotopy equivalence* $g : X \to X'$. ∎

The proof of this result uses a theorem of Zabrodsky, [12], which asserts that $\mathrm{Ph}(X,Y) \approx [X_{(0)}, Y]$ in certain cases.[4] Although the restrictions on the spaces in our theorem seem severe, there are some examples which show

[3] The spaces $X = S^3 \vee K(\mathbb{Z},5)$ and $X' = K(\mathbb{Z},3) \vee S^5$ provide an example where $X_{(0)} \simeq X'_{(0)}$ but there are no rational equivalences between X and X', in *either* direction.

[4] Zabrodsky claims in Theorem D, ibid, that this holds when $X = P$ and Y is merely finite dimensional. However, when one takes $P = K(\mathbb{Z},3)$ and $Y = K(\mathbb{Q},3)$, his claim is seen to be false. If one further requires Y to be a finite complex, as we do in Theorem 6, then his restricted result is valid.

the result is actually quite sharp. Example (D) showed what happens when condition (i) on X is removed. The next example shows what happens when condition (iii) on Y is ignored:

EXAMPLE (E). *Given a finite abelian group G, there exists a space X which satisfies 6(i), a self map $g : X \to X$ which is rational equivalence, and a finite type target Y such that G is isomorphic to the kernel of $g^* : Ph(X,Y) \to Ph(X,Y)$.*

In this example the domain X is a certain $K(\pi,3)$. However, the target Y, which is rationally a product of $K(\mathbb{Q},4)$'s, does not meet the requirements of Theorem 6(iii) for some nonempty set of primes.

As an application of Theorem 6, consider the following:

EXAMPLE (F). *Let $X = \Sigma^2 K(\mathbb{Z},2n-1)$, $X' = K(\mathbb{Z},2n+1)$, where $n \geq 2$, and let $g : X \to X'$ be the double adjoint of the equivalence $K(\mathbb{Z},2n-1) \to \Omega^2 K(\mathbb{Z},2n+1)$. Then g is a rational equivalence and by Theorem 6 it induces a bijection*

$$\mathrm{Ph}(K(\mathbb{Z},2n+1),S^{2n+2}) \xrightarrow{\approx} \mathrm{Ph}(\Sigma^2 K(\mathbb{Z},2n-1),S^{2n+2})\,.$$

Let us pursue this example a bit further. Since

$$\mathrm{Ph}(\Sigma^2 K(\mathbb{Z},2n-1),S^{2n+2}) \approx \mathrm{Ph}(K(\mathbb{Z},2n-1),\Omega^2 S^{2n+2}) \approx \mathbb{R}\,,$$

as rational vector spaces ([12], [10], [11]), it follows that the functor $\Omega^2(\ )$ induces a bijection

$$\mathrm{Ph}(K(\mathbb{Z},2n+1),S^{2n+2}) \xrightarrow{\approx} \mathrm{Ph}(K(\mathbb{Z},2n-1),\Omega^2 S^{2n+2})$$

of nontrivial pointed sets. Iterating this process one sees that $\Omega^{2n-2}(\ )$ induces a bijection

$$\mathrm{Ph}(K(\mathbb{Z},2n+1),S^{2n+2}) \xrightarrow{\approx} \mathrm{Ph}(K(\mathbb{Z},3),\Omega^{2n-2} S^{2n+2})$$

of nontrivial pointed sets.[5]

EXAMPLE (G). *Given a proper set of primes S, there exists a finite type domain X and a map $g : X \to \mathbb{C}P^\infty$ which is a p-equivalence for every prime p in S, such that the kernel of the induced epimorphism $g^* : Ph(\mathbb{C}P^\infty,S^3) \to Ph(X,S^3)$ is isomorphic to the product $\prod_{p\notin S}\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the p-adic integers.*

In this example, each prime at which the domain X *fails* to meet condition 6(i) contributes a copy of the p-adic integers to the kernel of g^*. These

[5] Letting $X = \bigvee_{n\geq 1} K(\mathbb{Z},2n+1)$ it follows that there exists a phantom map f out of X with the property that $\Omega^n f$ is essential for every n.

examples suggest that if one wishes to relax the conditions on the spaces in Theorem 6 and still obtain the conclusion that g^* is a bijection, then one must place more restrictions on the map g. To this end, we say that a map $g\colon X \to X'$ is *almost a homology equivalence* if it induces an isomorphism in rational homology (in all degrees) and an isomorphism in integral homology in all but finitely many degrees; equivalently, if the integral homology groups of C_g, the mapping cone of g, are torsion in all degrees and almost always 0. The following strikes us as plausible:

CONJECTURE (7). *Let X and X' be finite type domains and let Y be a finite type target. If $g\colon X \to X'$ is almost a homology equivalence, then $g^*\colon \mathrm{Ph}(X', Y) \to \mathrm{Ph}(X, Y)$ is a bijection.*

There is a small shred of evidence in favor of a positive solution to the conjecture, which we now present. Let

$$[C_g, Y] \xrightarrow{h^*} [X', Y] \xrightarrow{g^*} [X, Y]$$

be the usual exact sequence of pointed sets associated with the cofiber sequence

$$X \xrightarrow{g} X' \xrightarrow{h} C_g .$$

Since g is almost a homology equivalence, it follows from obstruction theory that $[C_g, Y]$ is a finite set. If we were in a situation where g^* and h^* were homomorphisms and $[X', Y]$ were a torsion-free group, we would then be able to conclude that h^* has trivial image, hence that $\ker g^* = 0$. The following result describes two such situations:

PROPOSITION (8). *Let X, X', and Y satisfy the following requirements:*

(i) *X is a nilpotent finite type domain with a finite fundamental group.*

(ii) *$X' = \Sigma^k P'$, $k \geq 0$, where P' is a nilpotent finite type domain with $\pi_1 P'$ finite and with $\pi_n P' = 0$ for $n \gg 0$.*

(iii) *$Y = \Omega^\ell Z$, $\ell \geq 0$, where Z is a finite CW-complex with a finite fundamental group.*

If $g : X \to X'$ is almost a homology equivalence and either (a) Y is an H-space (e.g. $\ell \geq 1$) or (b) $k \geq 1$ and g is a co-H-map of co-H-spaces, then $g^*\colon \mathrm{Ph}(X', Y) \longrightarrow \mathrm{Ph}(X, Y)$ is a bijection.

This ends the discussion of the results in this paper. In the next section we prove the results in the order they were presented. Proofs are also given for Examples (E) and (G). The first named author thanks his colleague Bob Bruner for many helpful comments on this paper.

3. Proofs

Proof of Theorem (1). Recall that the set of pointed homotopy classes of phantoms maps from one pointed space, X, to another, Y, can be identified with the $\varprojlim^1$ term of a certain sequence of nilpotent groups, namely

$$\mathrm{Ph}(X,Y) \quad \approx \quad \varprojlim{}^1[\Sigma X, Y^{(n)}].$$

The basic reference here is [3], Chapter IX.

Now assume that $f : \Sigma X \longrightarrow \Sigma X'$ is a map that induces a monomorphism in rational homology. The proof of the first part of Theorem 1 deals with the following commutative diagram:

$$\begin{array}{ccc} [\Sigma X, Y] & \xleftarrow{\ f^*\ } & [\Sigma X', Y] \\ \Big\downarrow & & \Big\downarrow \\ [\Sigma X, Y^{(n)}] & \xleftarrow{\ f^*\ } & [\Sigma X', Y^{(n)}]\,. \end{array}$$

The vertical maps in this diagram are homomorphisms induced by the inclusion $Y \to Y^{(n)}$. The horizontal maps are not necessarily homomorphisms since the map f is not necessarily a co-H map and Y need not be an H-space.

Let G'_n denote $[\Sigma X', Y^{(n)}]$. Since each G'_n is a countable group, the hypothesis $\mathrm{Ph}(X', Y) = 0$ implies that the tower $\{G'_n\}$ is Mittag-Leffler, by Theorem 2 of [8]. Recall that for an inverse tower of groups $\{H_k\}$, the Mittag-Leffler property ensures that for each n, the images in H_n of the terms farther out in the sequence do not become smaller and smaller without end; instead they stabilize at some point. That is, for some N sufficiently large,

$$\mathrm{image}\{H_n \leftarrow H_N\} \quad = \quad \mathrm{image}\{H_n \leftarrow H_{N+k}\},$$

for all $k \geq 0$. We need the following lemma from [8].

LEMMA (1.1). *Let $H_1 \leftarrow H_2 \leftarrow H_3 \leftarrow \cdots$ be a tower of countable groups with the property that for each n, the image of H_{n+1} has finite index in H_n. Then the tower $\{H_k\}$ is Mittag-Leffler if and only if the canonical map $\varprojlim H_k \to H_n$ has finite index for each n.* ∎

When we say a homomorphism $h : A \to B$ has finite index we mean simply that the image $h(A)$ has finite index in B. The tower $\{G'_n\}$ has the finite index property mentioned in Lemma 1.1 since each of the k-invariants of ΩY has finite order. Since $[\Sigma X', Y]$ maps onto $\varprojlim G'_n$, it follows that the image of $[\Sigma X', Y]$ has finite index in each G'_n.

Let G_n denote $[\Sigma X, Y^{(n)}]$. Our goal is to show that the tower $\{G_n\}$ must also be Mittag-Leffler. To this end it would suffice to show that the left vertical, $[\Sigma X, Y] \to G_n$, has finite index, by Lemma 1.1. To establish this, we will examine the bottom map in more detail.

LEMMA (1.2). *Assume that $f : \Sigma X \longrightarrow \Sigma X'$ induces a monomorphism in rational homology where X and X' are finite type domains. Then for any space Y and natural number n, the induced map,*

$$[\Sigma X, Y^{(n)}] \xleftarrow{\;f^*\;} [\Sigma X', Y^{(n)}]$$

becomes surjective when rationalized. ∎

It is well known that in a finitely generated nilpotent group G, a subgroup H has finite index if and only if $H_o = G_o$. Therefore when rationalized, the right side of the square becomes an epimorphism. Assuming the lemma is true, the left side must likewise rationalize to an epimorphism. The finiteness of the index of $[\Sigma X, Y] \to G_n$ then follows.

Proof of Lemma (1.2). First recall that G_n is naturally isomorphic to $[\Sigma X, U]$ where U denotes the universal cover of $Y^{(n)}$. Hence the rationalization of G_n can be identified with $[\Sigma X, U_{(0)}]$. Similar remarks apply to G'_n. It is then clear that the map f induces a function between the rationalizations of the groups G_n and G'_n. Of course, this last remark would be obvious if Y were nilpotent or if $f^* : G_n \to G'_n$ were a homomorphism, but we have not assumed either hypothesis.

We would like to replace the map f by some finite skeletal approximation of it. However, since such an approximation will not, in general, also induce a monomorphism in rational homology we have to work a little harder. Let K be a complex of dimension $n+1$ with the following properties:

i) $K_n = X_n$.

ii) $H_n(K;\mathbb{Z}) \approx H_n(X;\mathbb{Z})$.

iii) $H_{n+1}(K;\mathbb{Z}) \approx H_{n+1}(X;\mathbb{Z})$/torsion.

Both homology isomorphisms are to be induced by a map of K into X_{n+1}. Such a K, as well as the map, exist by Theorem 2.1 of [2]. Let K' denote the $n+1$ - skeleton of X'. It follows that there is a map, say f_K, from ΣK to $\Sigma K'$ that fits into a commutative diagram,

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\;f\;} & \Sigma X' \\ \uparrow & & \uparrow \\ \Sigma K & \xrightarrow{\;f_K\;} & \Sigma K'\,. \end{array}$$

The vertical maps here are inclusions. Notice that they are $n+1$ -connected. Hence when the functor $[\ \ , Y^{(n)}]$ is applied to this square, the verticals become group isomorphisms. Notice also that the map f_K induces a monomorphism in rational homology. Since ΣK and $\Sigma K'$ are finite suspensions, they each have the rational homotopy type of a bouquet of spheres. Hence the map f_K when rationalized, has a left inverse. Consequently, f_K induces a surjection of sets between the rationalized groups $[\Sigma K', Y^{(n)}]_o$ and $[\Sigma K, Y^{(n)}]_o$. The lemma follows, as does the first part of Theorem 1.

Consider now the second conclusion of Theorem 1; that is assume that the map f induces a homomorphism between the rationalized groups,

$$[\Sigma X', Y^{(n)}]_o \xrightarrow{f_o^*} [\Sigma X, Y^{(n)}]_o.$$

Note that this homomorphism is surjective by Lemma 1.2.

For a nilpotent group N, the kernel of the rationalization homomorphism, $r : N \to N_o$, is the torsion subgroup, TN. Equivalently, the image of N in N_o is isomorphic to the torsion-free quotient $FN = N/TN$. Hence, from the commutativity of the diagram

$$\begin{array}{ccc} G_n' & \xrightarrow{r'} & (G_n')_o \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f_o^*} \\ G_n & \xrightarrow{r} & (G_n)_o \end{array},$$

in which all arrows except the one on the left side are assumed to be homomorphisms, it follows that f induces a homomorphism of towers, say

$$\bar{f} : \{FG_n'\} \longrightarrow \{FG_n\}.$$

The original towers had the finite index property and so their quotients $\{FG_n'\}$ and $\{FG_n\}$ also have this property. For each n, the image of FG_n' has finite index in FG_n (since $\bar{f}$ rationalizes to an epimorphism) and so by Lemma 2.2 of [9], the induced map

$$\varprojlim{}^1(\bar{f}) : \varprojlim{}^1 FG_n' \longrightarrow \varprojlim{}^1 FG_n$$

is a surjection. Finally note that the quotient map $G_n \to FG_n$ induces a $\varprojlim^1$ isomorphism. To see this, apply the six term $\varprojlim - \varprojlim^1$ sequence to the short exact sequence of towers

$$\{TG_n\} \longrightarrow \{G_n\} \longrightarrow \{FG_n\},$$

and take into account that $\{TG_n\}$ will have a trivial $\varprojlim^1$ term since it is a tower of finite groups. Similar remarks apply to the quotient map $G_n' \to FG_n'$.

Thus we have shown that f induces a surjection from $\mathrm{Ph}(X', Y)$ to $\mathrm{Ph}(X, Y)$ when the rationalization

$$[\Sigma X', Y^{(n)}]_o \xrightarrow{f_o^*} [\Sigma X, Y^{(n)}]_o,$$

is a homomorphism. We trust it is clear to the reader that this happens when f rationalizes to a co-H-map or when the universal cover of Y rationalizes to an H-space.

Proof of Corollary 2. This is an immediate consequence of Theorem 1 and the following:

LEMMA (2.1). *If $\{G_n\}$ is a tower of countable*[6] *groups then $\varprojlim^1 G_n$ has cardinality either 1 or $2^{\aleph_o}$.* ∎

Proof. Assume that $\varprojlim^1 G_n$ is nontrivial. Then by Theorem 2 of [8] it follows that the cardinality of this term is uncountably large. Thus it suffices to show that this cardinality is no larger than $2^{\aleph_o}$. Recall that $\varprojlim^1 G_n$ is a quotient of the direct product ΠG_n and so it has cardinality at most ${\aleph_o}^{\aleph_o}$. Since

$${\aleph_o}^{\aleph_o} \leq (2^{\aleph_o})^{\aleph_o} = 2^{(\aleph_o \times \aleph_o)} = 2^{\aleph_o},$$

the lemma follows.

Proof of Proposition 5.
(i). Assume that $\mathrm{Ph}(X, Y) = 0$ for all targets Y. Then, in particular, the universal phantom map out of X is trivial and so ΣX is a retract of $\bigvee_{n\geq 1} \Sigma X_n$ by Theorem 2 of [5]. Moreover, the proof of that theorem shows that the folding map

$$\nabla\colon \bigvee_{n\geq 1} \Sigma X_n \to \Sigma X ,$$

which for each n restricts to the inclusion $i_n : \Sigma X_n \to \Sigma X$, on the $n-th$ summand, has a right inverse

$$\rho\colon \Sigma X \to \bigvee_{n\geq 1} \Sigma X_n .$$

Given a homotopy equivalence $f\colon \Sigma X \to \Sigma X'$, there is, for each n, an n-equivalence $f_n\colon \Sigma X_n \to \Sigma X'_n$ such that the following diagram commutes up

[6]Countability is necessary here. It will be shown elsewhere that without it any finite abelian group can occur as the $\varprojlim^1$ term of a suitable tower of groups.

to homotopy:

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\ f\ } & \Sigma X' \\ {\scriptstyle i_n}\uparrow & & \uparrow{\scriptstyle i'_n} \\ \Sigma X_n & \xrightarrow{\ f_n\ } & \Sigma X'_n. \end{array}$$

The wedge sum of these f_n's is a map

$$F\colon \bigvee_{n\geq 1} \Sigma X_n \longrightarrow \bigvee_{n\geq 1} \Sigma X'_n$$

which makes the following diagram commute:

$$\begin{array}{ccc} \bigvee_{n\geq 1} \Sigma X_n & \xrightarrow{\ \nabla\ } & \Sigma X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \bigvee_{n\geq 1} \Sigma X'_n & \xrightarrow{\ \nabla'\ } & \Sigma X'. \end{array}$$

(We do not claim that F is a homotopy equivalence). Now define

$$\rho'\colon \Sigma X' \longrightarrow \bigvee_{n\geq 1} \Sigma X'_n$$

by

$$\rho' = F \circ \rho \circ f^{-1},$$

where $f^{-1}\colon \Sigma X' \to \Sigma X$ is a homotopy inverse of f. A simple computation shows that ρ' is a right inverse of ∇' and so, again by Theorem 2 of [5], it follows that $\mathrm{Ph}(X', Y) = 0$ for all targets Y. To complete the proof reverse the roles of X and X' and repeat the argument just given.

(ii). Consider the bijection of towers

$$\{[\Sigma X', Y^{(n)}]\} \xrightarrow{\ f^*\ } \{[\Sigma X, Y^{(n)}]\}.$$

induced by a homotopy equivalence $f : \Sigma X \to \Sigma X$. The Mittag-Leffler property is clearly a set theoretic condition and so it follows that one tower has this property if and only if the other does. But these are towers of countable groups and so it follows that one tower has a trivial $\varprojlim^1$ term if and only if the other does. The result then follows using Lemma 2.1.

(iii). Assume that $f : \Sigma X \to \Sigma X'$ is a homotopy equivalence and let U denote the universal cover of Y. If U is an H-space, then so is each Postnikov approximation $U^{(n)}$ and the induced map

$$\{[\Sigma X', U^{(n)}]\} \xrightarrow{\ f^*\ } \{[\Sigma X, U^{(n)}]\}$$

is then an isomorphism between two towers of abelian groups. Clearly it induces an isomorphism between the $\varprojlim^1$ terms.

(iv). If U is an H_0-space, then the induced function, just displayed, becomes an isomorphism of abelian groups when rationalized. The proof is then essentially the one given for the second part of Theorem 1.

The proofs for (v) and (vi) are similar to those given for (iii) and (iv), respectively. ∎

Proof of Theorem 6. If U and V are nilpotent spaces of finite type with finite fundamental groups, we follow [12] in identifying $\mathrm{Ph}(U,V)$ with the image of

$$[U_{(0)},V] \xrightarrow{r^*} [U,V]\,,$$

where $r\colon U \to U_{(0)}$ is a rationalization map. [7] In the commutative diagram

$$\begin{array}{ccc} [X'_{(0)},Y] & \xrightarrow{r'^*} & [X',Y] \\ {\scriptstyle g^*_{(0)}}\downarrow & & \downarrow{\scriptstyle g^*} \\ [X_{(0)},Y] & \xrightarrow{r^*} & [X,Y]\,, \end{array}$$

the map $g^*_{(0)}$ is a bijection since $g_{(0)}\colon X_{(0)} \to X'_{(0)}$ is a homotopy equivalence. It therefore suffices to show that r^* is injective. But according to Theorem D of [12], r^* is injective (and even bijective if $\ell \geq 0$). ∎

Proof of Example (E). We begin with the simplest case where the group G is cyclic of order, say λ. Let Λ denote the set of all primes that divide λ and let Λ' be the set of all primes which do not. Let $X = K(\mathbb{Z},3)$ and let Y be the Zabrodsky mix of $K(\mathbb{Z},4)$ at Λ and ΩS^5 at Λ'. In other words, Y is the homotopy pullback of a diagram

$$\begin{array}{ccc} & & \Omega S^5_{(\Lambda')} \\ & & \downarrow \\ K(\mathbb{Z}_{(\Lambda)},4) & \longrightarrow & K(\mathbb{Q},4). \end{array}$$

Take the arrows in this diagram to be rational equivalences and loop maps. Thus Y is a finite type loop space and $\mathrm{Ph}(X,Y)$ will be isomorphic to the $\varprojlim^1$ term of a tower of finitely generated abelian groups. Now

$$\mathrm{Ph}(X,Y) \approx \varprojlim{}^1 A_n, \quad \text{where } A_n = [K(\mathbb{Z},3),\Omega Y^{(n)}].$$

[7] The finite type hypothesis on V cannot be removed, as shown by the example $U = \mathbb{R}P^\infty$ and $V = \bigvee_{n\geq 1} \Sigma\mathbb{R}P^n$ in Example (A).

By a calculation essentially due to Meier, [7], it follows that

$$\varprojlim{}^1 A_n \approx \mathrm{Ext}(\mathbb{Z}_{(\Lambda)}, \mathbb{Z}) \approx \mathbb{R} + \sum_{p\in\Lambda} \mathbb{Z}_{p^\infty}.$$

Now take $g : X \to X$ to be a map with degree λ on the bottom cell. It is easy to check that g induces multiplication by λ on each $A_n/\text{torsion}$. It follows that g also induces multiplication by λ on the $\varprojlim^1$ term. Clearly $\mathbb{Z}/\lambda$ is the kernel of g^* in this case.

Consider next the product group $G = (\mathbb{Z}/\lambda)^t$ where $t \geq 1$. Take the new domain and target to be the t-fold product of the previous choices and note that in this special case,

$$\mathrm{Ph}(X^t, Y^t) \approx (\mathrm{Ph}(X, Y))^t.$$

To verify this recall that Y is a loop space and so

$$\mathrm{Ph}(X^t, Y^t) \approx \mathrm{Ph}(\Sigma(X^t), (\Omega^{-1}Y)^t).$$

Now $\Sigma(X^t)$ splits into a wedge of smash products of the form $\Sigma \bigwedge^k X$ while $\Omega^{-1}Y$ has the rational homotopy type of a $K(\mathbb{Q}, 5)$. It follows easily that $\mathrm{Ph}(\Sigma \bigwedge^k X, \Omega^{-1}Y) = 0$ when $k > 1$ and the claim follows. The map g in this case should, of course, be the t-fold external product of the previous map.

The case of an arbitrary finite abelian group G differs from the case just considered only in notational complexity; e.g., λ_1, λ_2, ... etc. We leave these details to the reader.

Proof of Example (G). Let X be the Zabrodsky mix of $\mathbb{C}P^\infty$ at S and ΩS^3 at S'. Take the map $g : X \to \mathbb{C}P^\infty$ to have degree 1 on the bottom cell. We claim there is a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Ph}(\mathbb{C}P^\infty, S^3) & \xrightarrow{g^*} & \mathrm{Ph}(X, S^3) \\
 & & \uparrow\approx & & \uparrow\approx \\
 & & \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) & \longrightarrow & \mathrm{Ext}(\mathbb{Z}_{(S')}, \mathbb{Z}) \\
 & & \uparrow\approx & & \uparrow\approx \\
\prod_{p\in S'} \mathbb{Z}_p & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} + \sum_{p\in S'} \mathbb{Z}_{p^\infty}
\end{array}$$

wherein the middle horizontal is induced by the inclusion $\mathbb{Z}_{(S')} \to \mathbb{Q}$ and the bottom line is a short exact sequence.

To obtain the upper square of this diagram, let

$$A_n = [\Sigma\mathbb{C}P^\infty, (S^3)^{(n)}] \quad \text{and} \quad B_n = [\Sigma X, (S^3)^{(n)}].$$

These are finitely generated abelian groups and so one can use Jensen's formula, ([6], Chapter 1),

$$\varprojlim{}^1 G_n \approx \operatorname{Ext}(\varinjlim \operatorname{Hom}(G_n, \mathbb{Z}), \mathbb{Z}),$$

with $G = A$ or B, to obtain the vertical isomorphisms in the upper square. Notice that $\operatorname{Hom}(A_n, \mathbb{Z}) \approx \operatorname{Hom}(B_n, \mathbb{Z}) \approx \mathbb{Z}$ for $n \geq 3$. In the $\operatorname{Hom}(A_n, \mathbb{Z})$ sequence,

$$\cdots \ \mathbb{Z} \xrightarrow{n_{k-1}} \mathbb{Z} \xrightarrow{n_k} \mathbb{Z} \xrightarrow{n_{k+1}} \mathbb{Z} \ \cdots,$$

each prime must divide infinitely many n_k's because $[\mathbb{C}P^\infty, \Omega S^3_{(p)}] = 0$ for every prime p. Thus the direct limit of the $\operatorname{Hom}(A_n, \mathbb{Z})$ sequence is the rational numbers, $\mathbb{Q}$. In the $\operatorname{Hom}(B_n, \mathbb{Z})$ sequence each prime in S must divide infinitely many n_k's for the same reason ($X \simeq_p \mathbb{C}P^\infty$ at these primes). However, those primes not in S divide none of the n_k's since $B_n \approx_p [\Omega S^3, (\Omega S^3)^{(n)}]$. Thus the limit of the $\operatorname{Hom}(B_n, \mathbb{Z})$ sequence is $\mathbb{Z}_{(S')}$. Notice that the homomorphism from $\operatorname{Hom}(B_n, \mathbb{Z})$ to $\operatorname{Hom}(A_n, \mathbb{Z})$ induced by $g : X \to \mathbb{C}P^\infty$ is an isomorphism when $n = 3$ and is uniquely determined from that point on. The claim about the middle arrow being induced by the inclusion $\mathbb{Z}_{(S')} \to \mathbb{Q}$ follows.

The properties of the lower square and the exactness of the bottom line follow from fairly basic Ext calculations. A good reference for the facts used here is [4], Chapter IX.

Proof of Proposition 8. We refer to the exact sequence of pointed sets

$$[C_g, Y] \xrightarrow{h^*} [X', Y] \xrightarrow{g^*} [X, Y],$$

and the discussion which precedes the statement of Proposition 8. In the case that Y is an H-space, it is clear that g^* and h^* are homomorphisms; and Theorem 2.2 of [10] tells us that $[X', Y]$ is a rational vector space, hence torsion-free. In the case that g is a co-H-map of co-H-spaces, C_g admits a co-H-space structure in such a way that h is a co-H-map. Thus g^* and h^* are again homomorphisms; and this time Theorem 3.1 of [11] tells us that $[X', Y]$ is a rational vector space. ∎

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