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ON THE ODD–PRIMARY STABLE J–HOMOMORPHISM

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Dedicated to the memory of Professor José Adem

0. Introduction

Whitehead conjectured that the stable J-homomorphism

$$\pi_*^s(SO) \longrightarrow \pi_*^s(SG) \cong \pi_*^s(Q_0S^0) \longrightarrow \pi_*^s(S^0),$$

where the last map is induced by the infinite loop space structure of Q_0S^0, is onto for $* \geq 1$. As is well-known, Kahn-Priddy [KP] proved that this is the case for the 2-primary part but Knapp [K] showed that this is not the case for the odd-primary part.

Having Kahn and Priddy's 2-primary positive solution in mind, we proved the following result in our previous paper [M2] (Here "geometrically flasque" is a geometric condition to be recalled in §2, and "double transfer"is also recalled in §2.):

THEOREM (1) of [M2]. *Consider the surjective composite*

$$\pi_*^s(SO_+) \to \pi_*^s(SO) \to \pi_*^s(S^0)_{(2)},$$

where the first map is induced by sending the disjoint basepoint to a basepoint in SO and the second map is induced by the Whitehead J-map $J : SO \to SG = Q_1S^0 \simeq Q_0S^0$. Let $\alpha \in \pi_*^s(S^0)_{(2)}$ be neither Hopf invariant one, nor (possibly) the generator of the image J in $\pi_{15}^s(S^0)$. Then, if $\alpha \in \pi_*^s(S^0)_{(2)}$ has a geometrically flasque lift $\tilde{\alpha} \in \pi_n^s(SO_+)$, it factors through the double transfer.

Now, there are three objects to this paper. The first is to modify this conjecture of Whitehead for the odd-primary situation and prove it (Theorem 1 in §1). The second is to generalize and modify previously quoted Theorem 1 of [M2] to the odd-primary case in the context of the here established modified odd-primary Whitehead conjecture (Theorem 2 in §2). In both of these parts, we used the affirmative solution of the Adams conjecture [Q] [S] [F], which was avoided in [M2]. The third is to prove that ζ_k and $\zeta_{k,i}$ of [R] factor through the double transfer. More precisely, we will actually prove that they factor through $P \wedge \Sigma\mathbb{C}P_+^\infty \xrightarrow{\lambda\wedge t} \Sigma^\infty S^0$ (Theorem 3 and Corollary 4 in §3; see §3 for the notations).

Throughout this paper all the spaces are localized at a prime p which is always odd, except in Lemma 1 where p could be an any prime number. For any space (or spectrum) X, $H_*(X)$ is its mod-p homology and X^n is its n-skeleton.

The author would like to express his gratitude to the referee for pointing out several mistakes of English usage in the first draft of this paper.

1. Stable J-homomorphism

We fix an odd prime p. Then at p, the image of the stable J-homomorphism $\pi_*^s(SO) \to \pi_*^s(S^0)$ is the same as that of the complex stable J-homomorphism $\pi_*^s(U) \to \pi_*^s(S^0)$, induced by the complex J-map $U \to SO \to SG$. Actually, Knapp [K] employed the stable complex J-homomorphism, in order to show that the odd-primary Whitehead conjecture is wrong.

To modify the Whitehead conjecture for the odd-primary case, we note that the complex J-map factors as

$$U \to J \to SG.$$

Here $J \equiv BGL(\mathbb{F}_q)^+$, where q is a prime which generates $(\mathbb{Z}/p^2\mathbb{Z})^*$; the first map $U \to J$ shows up as a part in the Quillen fiber sequence [Q2]

$$U \xrightarrow{\psi^q - 1} U \longrightarrow J \longrightarrow BU \xrightarrow{\psi^q - 1} BU;$$

and the second map $J \to SG$ is given by the affirmative solution of the Adams conjecture [Q] [S] [F]. Such a map $J \longrightarrow SG$ is known to be a homotopy section, i.e. the composite

$$J \longrightarrow SG \simeq Q_0S^0 \xrightarrow{r} J,$$

where $r : Q_0S^0 \equiv (B\Sigma_\infty)^+ \longrightarrow (BGL(\mathbb{F}_q, \infty))^+ \equiv J$ is induced by the permutation representation, is a homotopy equivalence. Among such maps, there is a canonical categorical map $J : J \longrightarrow SG$ [To] [Ma] [tD]. Actually, our modification of the odd-primary Whitehead conjecture is based upon this map J, and it is stated as the following Theorem 1:

(We warn the reader that we are going to use the notation J in four different ways: first, as the fiber of $\psi^q - 1$; second as the map $J : J \longrightarrow SG$; third as the map $J : R(\mathbb{Z}/p, \mathbb{F}_q) \longrightarrow A(\mathbb{Z}/p)[\frac{1}{q}]$ which will be introduced soon; fourth as the generalized homology theory given by the spectrum whose associated infinite loop space is the space J. This abuse of the notation should not cause any confusion.)

THEOREM (1). *The composite*

$$\pi_*^s(J) \xrightarrow{J} \pi_*^s(SG) \cong \pi_*^s(Q_0S^0) \to \pi_*^s(S^0)_{(p)},$$

whose last map is induced by the infinite loop space structure of Q_0S^0, is onto for $* \geq 1$.

Proof. We are going to show that the Kahn-Priddy map $B\mathbb{Z}/p \longrightarrow Q_0S^0$ factors as a composite

$$B\mathbb{Z}/p \xrightarrow{\text{Kahn-Priddy}} Q_0S^0 \xrightarrow{r} J \xrightarrow{J} SG \equiv Q_0S^0,$$

up to multiplication by a unit of $\mathbb{Z}_{(p)}$. This would prove the claim, as an immediate consequence of the Kahn-Priddy theorem.

To analyze the situation, we prefer to work at the categorical level; Let $A(\mathbb{Z}/p)$ be the Burnside ring of $\mathbb{Z}/p$, let $R(\mathbb{Z}/p, \mathbb{F}_q)$ be the modular representation ring of $\mathbb{Z}/p$ over $\mathbb{F}_q$, and let $I(\mathbb{Z}/p)$ and $I(\mathbb{Z}/p, \mathbb{F}_q)$ be their augmentation ideals. We are interested in two maps

$$r : A(\mathbb{Z}/p) \longrightarrow R(\mathbb{Z}/p, \mathbb{F}_q), \quad r : I(\mathbb{Z}/p) \longrightarrow I(\mathbb{Z}/p, \mathbb{F}_q)$$

and

$$J : R(\mathbb{Z}/p, \mathbb{F}_q) \longrightarrow A(\mathbb{Z}/p)\left[\frac{1}{q}\right], \quad J : I(\mathbb{Z}/p, \mathbb{F}_q) \longrightarrow 1 + I(\mathbb{Z}/p)\left[\frac{1}{q}\right],$$

where r is induced by the permutation representation and J is induced by regarding a finite dimensional $\mathbb{F}_q$-vector space with $\mathbb{Z}/p$-action as a finite $\mathbb{Z}/p$-set (by forgetting its $\mathbb{F}_q$-vector space structure) [tD]. Then we have a commutative diagram

$$\begin{tikzcd}
I(\mathbb{Z}/p) \arrow[r, "r"] \arrow[d, "\alpha"] & I(\mathbb{Z}/p, \mathbb{F}_q) \arrow[r, "J"] \arrow[d, "\alpha"] & 1 + I(\mathbb{Z}/p))\left[\frac{1}{q}\right] \arrow[r, "-1"] \arrow[d, "\alpha"] & I(\mathbb{Z}/p)\left[\frac{1}{q}\right] \arrow[d, "\alpha"] \\
\left[B\mathbb{Z}/p, Q_0S^0\right] \arrow[r, "r"] & [B\mathbb{Z}/p, J] \arrow[r, "J"] & [B\mathbb{Z}/p, SG] \arrow[r, "s"] & \left[B\mathbb{Z}/p, Q_0S^0\right],
\end{tikzcd}$$

where vertical maps α are essentially induced from the group completions $(B\Sigma_\infty)^+ \equiv Q_0S^0$ and $BGL(\mathbb{F}_q, \infty)^+ \equiv J$, and s is induced by shifting components $SG = Q_1S^0 \simeq Q_0S^0$. (For more technical details, see [Ma] [tD] [To]. The relationship between the Burnside rings, infinite loop spaces, and the homology operations was studied in [M1].) Now, as the Kahn-Priddy map is $\alpha([\mathbb{Z}/p] - p) \in [B\mathbb{Z}/p, Q_0S^0]$, the composite map in question is

$$sJr\alpha([\mathbb{Z}/p] - p) = \alpha(Jr([\mathbb{Z}/p] - p) - 1).$$

But, the purely categorical datum

$$Jr([\mathbb{Z}/p] - p) = \frac{Jr([\mathbb{Z}/p])}{Jr(p)}$$

can be calculated quite easily from the fixed point sets data of $Jr([\mathbb{Z}/p]) = \mathbb{F}_q[\mathbb{Z}/p]$:

$$\begin{aligned} |\mathbb{F}_q[\mathbb{Z}/p]^{\mathbb{Z}/p}| &= q \\ |\mathbb{F}_q[\mathbb{Z}/p]| &= q^p \end{aligned}$$

From this, we easily see

$$Jr([\mathbb{Z}/p]-p) = \frac{q^p + \frac{q^p-q}{p}([\mathbb{Z}/p]-p)}{q^p} = 1 + \frac{q^p-q}{q^p p}([\mathbb{Z}/p]-p).$$

Therefore,

$$\alpha(Jr([\mathbb{Z}/p]-p)-1) = \alpha(\frac{q^p-q}{q^p p}([\mathbb{Z}/p]-p)) = \frac{q^p-q}{q^p p}\alpha([\mathbb{Z}/p]-p)).$$

Now the point is that $\frac{q^p-q}{q^p p}$ is p-locally a unit. (Recall that q is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$.) From this, we immediately see that the Kahn-Priddy map $\alpha([\mathbb{Z}/p]-p)$ p-locally factorizes the questioned composite $\frac{q^p-q}{q^p p}\alpha([\mathbb{Z}/p]-p)$. □

Remark. Of course, the above proof implies Q_0S^0 is a direct summand of QJ.

2. The double transfer

Let $\lambda : \Sigma^\infty B\Sigma_{p+} \to \Sigma^\infty S^0$ be the stable adjoint of the composite

$$B\Sigma_p \to Q_pS^0 \to QS^0,$$

where the first map is the Dyer-Lashof map, and the second map is the obvious inclusion in $QS^0 = \coprod_{i\in\mathbb{Z}} Q_iS^0$. If we use $\alpha : A(\Sigma_p) \longrightarrow [B\Sigma_p,\ Q_0S^0]$ as in §1, then $\lambda = \alpha([\Sigma_p/\Sigma_{p-1}])$. It is easy to see that the Kahn-Priddy map factors through λ. Therefore λ also induces a surjection in the p-primary part of the positive dimension homotopy groups.

In this section we call

$$\lambda\wedge\lambda : \Sigma^\infty B\Sigma_{p+}\wedge B\Sigma_{p+} \longrightarrow \Sigma^\infty S^0$$

the double transfer and we hope to show many elements of $\pi_*^s(S^0)_{(p)}$ factors through the double transfer (for the motivation and the 2-primary case, refer [M2]). To simplify the notation we put $P \equiv \Sigma^\infty B\Sigma_{p+}$.

Now the fundamental concept of [M2] was the following:

Definition. Suppose X is a space. Then $\alpha \in \pi_n^s(X_+)$ is called G. F. (=Geometrically Flasque) if α has a framed bordism representative $f : M^n \to X$ such that

$$\Sigma M^n \simeq \Sigma N \vee S^{n+1},$$

where N is the $n-1$ skeleton of M^n.

Remark. Of course, if α is in the image of

$$\pi_n(\Omega\Sigma(X_+)) \to \pi_n(Q(X_+)) \simeq \pi_n^s(X_+),$$

then it is $G.F$. But, usually the set of $G.F$. elements is much larger than this image. For instance, when X is a point (iethe case of the framed bordism groups $\pi_n^s(S^0)$) any element $\alpha \in \pi_n^s(S^0)$ is $G.F.$, since Kervaire-Milnor [KM] showed that a framed bordism representative of α can be taken to be either by a homotopy sphere or the Kervaire manifold.

Now we are going to prove

THEOREM (2). *Consider the surjective composite*

$$\pi_*^s(J_+) \to \pi_*^s(J) \to \pi_*^s(S^0)_{(p)},$$

where the first map is induced by sending the disjoint basepoint to a basepoint in J and the second map in induced by the canonical map $J : J \to SG = Q_1S^0 \simeq Q_0S^0$. Let $\alpha \in \pi_*^s(S^0)_{(2)}$ be neither Hopf invariant one, nor (possibly) the generator of the image J in $\pi_{11}^s(S^0)_{(3)}$. Then, if $\alpha \in \pi_*^s(S^0)_{(2)}$ has a geometrically flasque lift $\tilde{\alpha} \in \pi_n^s(SO_+)$, it factors through the double transfer.

Proof. By Lemma 2 of [M2], we only have to prove the claim for those elements which are in the image of $J : \pi_*(J) \longrightarrow \pi_*^s(S^0)$, the image J-elements (recall that p is odd). On the other hand, several authors calculated the J-homology of the lens spaces and (essentially its stable summand) P [H] [D] [T]. In particular, their calculation indicated an isomorphism

$$J_{qi-1}(P^{q(\nu(i)+1)}) \xrightarrow{\simeq} J_{qi-1}(P),$$

where $q = 2(p-1)$. Then existence of the double transfer lifts of the image J-elements immediately follow from the following Lemma 1. $\square$

In the following lemma, we put $l = |\frac{n-1}{2}|$ to simplify the notation.

LEMMA (1). *If $\alpha \in \pi_n(J) = J_n(S^0)$ comes from $J_n(P^l)$, then $J(\alpha) \in \pi_n^s(S^0)$ factors through the double transfer. Here p could be any prime, including 2.*

Proof. Note that the stable n-dual of P^l is a suspension spectrum of a space, and the n-dual of the (restriction of) Kahn-Priddy map $\lambda : \Sigma^\infty P^l \longrightarrow \Sigma^\infty S^0$ can be taken by a space level map:

$$D\lambda : S^n \longrightarrow \Sigma^n DP^l.$$

Now we have the following commutative diagram:

$$\begin{tikzcd}
J_n(P^l) \arrow[r, "\simeq"] \arrow[d, "\lambda_*"] & {[\Sigma^n DP^l, J]^*} \arrow[r, "J"] \arrow[d, "(D\lambda)^*"] & {[\Sigma^n DP^l, Q_0S^0]^*} \arrow[r, "\simeq"] \arrow[d, "(D\lambda)^*"] & \pi_n^s(P^l) \arrow[d, "\lambda_*"] \\
J_n(S^0) \arrow[r, "\simeq"] & {[S^n, J]^*} \arrow[r, "J"] & {[S^n, Q_0S^0]^*} \arrow[r, "\simeq"] & \pi_n^s(S^0)
\end{tikzcd}$$

To explain the situation we recall a couple of definitions. An image J element $\alpha \in J_n(S^0) \subset \pi_n^s(S^0)$, where the inclusion is induced by the J map, is said to have the stable J-sphere of origin $\leq t$, if it is in the image of $\lambda_* : J_n(P^{t-1}) \to J_n(S^0)$. An element $\beta \in \pi_n^s(S^0)$ is said to have the stable sphere of origin $\leq t$, if it is in the image of $\lambda_* : \pi_n^s(P^{t-1}) \to \pi_n^s(S^0)$. Then this commutative diagram says if the stable J-sphere of origin of an Image J element $\alpha \in \pi_n^s(S^0)$ is lower than $l+2$, then it is actually equal to the stable sphere of origin. But, if the stable sphere of origin is this low, then it factors through the double transfer, by the Kahn-Priddy theorem and an argument using S-duality (cf. Proposition 1 of [M2]). □

3. Double transfer lifts of ζ_k.

Let $\lambda_k = b_k$, the p-fold Massey product $\langle h_k, \cdots, h_k\rangle$. Then, R. Cohen [C] proved that, for any odd prime p and $k \geq 1$, $h_0\lambda_k$ is represented by a permanent cycle ζ_k of order p.

Now the purpose of this section is to prove that ζ_k factors through, not only the double transfer, but even the composite

$$P \wedge \Sigma\mathbb{C}P_+^\infty \xrightarrow{\lambda\wedge t} \Sigma^\infty S^\infty,$$

where $t : \Sigma\mathbb{C}P_+^\infty \longrightarrow \Sigma^\infty S^\infty$ is the S^1-transfer (Note that $\lambda \wedge t$ factorizes the double transfer as t factorizes λ.):

THEOREM (3). *For any odd prime ζ_k can be lifted to $P \wedge \Sigma\mathbb{C}P_+^\infty$ so that the lift too has order p.*

COROLLARY (4). *$\zeta_{k,i}$ of [C] also factors through $\lambda \wedge t$.*

Proof of Corollary 4, assuming Theorem 3. This is trivial from the definition: $\zeta_{k,i} = \langle \zeta_k, p, \alpha_i\rangle$. □

Proof of Theorem 3. (This is a straightforward generalization of the original papers [C] [Mh] [CJM] for the context of the double transfer as was done in [M2]) We begin by recalling the Snaith decompositions [Sn] as presented by F.Cohen-Mahowald-Milgram [CMM]:

$$\Omega^2\Sigma^2 S^{2m+1} \simeq \bigvee_{k\geq 1} \Sigma^{2mk} t(V_k),$$

where $t(V_k) = C_{2,k} \ltimes_{\Sigma_k} S^{1(k)}$ (for the notation refer [C]). We also recall a theorem of R.Cohen [C] which claims that for any $m \geq 0$ and r with $1 \leq r \leq p$,

$$t(V_{p(mp+r)}) \simeq \Sigma^{2(mp+r)(p-1)} B(m),$$

where $B(m)$ is the m-th mod-p Brown-Gitler spectrum [BG] constructed by R.Cohen [C].

We are now going to construct three maps:

$$\begin{aligned}
\tilde{\xi}_k &: \Sigma^{2(p^{k+1}+1)(p-1)-3}M(p) \longrightarrow P \wedge \Sigma^{-2p^{k+1}-2p+1}t(V_{p^{k+2}+p}) \\
\tilde{h} &: P \wedge \Sigma^{-2p^{k+1}-2p+1}t(V_{p^{k+2}+p}) \longrightarrow P \wedge \Sigma^{(2p-4)p^{k+1}}t(V_{p^{k+1}}) \\
\tilde{g}_{k+1} &: P \wedge \Sigma^{(2p-4)p^{k+1}}t(V_{p^{k+1}}) \longrightarrow P \wedge \Sigma\mathbb{C}P^{\infty}_{+}.
\end{aligned}$$

Then our lift of ζ_k will be given, up to multiplication by a non-zero element in $\mathbb{Z}/p$, by the composite

$$\tilde{g}_{k+1} \circ \tilde{h} \circ \tilde{\xi}_k \circ i,$$

where i is the inclusion of the bottom cell of the mod p Moore space.

Now, to construct $\tilde{\xi}_k$, note that

$$\begin{aligned}
\tilde{\xi}_k &\in \{\Sigma^{2(p^{k+1}+1)(p-1)-3}M(p), \Sigma^{-2p^{k+1}-2p+1}t(V_{p^{k+2}+p})\} \\
&\simeq \{\Sigma^{2(p^{k+1}+1)(p-1)-3}M(p), \Sigma^{-2p^{k+1}-2p+1}\Sigma^{2(p^{k+1}+1)(p-1)}B(p^k)\} \\
&\simeq B(p^k)_{2p(p^k+1)-4}(P \wedge \Sigma^{-1}M(p)).
\end{aligned}$$

Consider the commutative diagram:

$$\begin{array}{ccc}
B(p^k)_{2p(p^k+1)-2}(P \wedge \Sigma M(p)) & \longrightarrow & H_{2p(p^k+1)-2}(P \wedge \Sigma M(p)) \\
\Sigma^{-2}\downarrow\cong & & \Sigma^{-2}\downarrow\cong \\
B(p^k)_{2p(p^k+1)-4}(P \wedge \Sigma^{-1}M(p)) & \longrightarrow & H_{2p(p^k+1)-4}(P \wedge \Sigma^{-1}M(p)) \\
\downarrow & & \downarrow\cong \\
B(p^k)_{2p(p^k+1)-4}(P) & \longrightarrow & H_{2p(p^k+1)-4}(P)
\end{array}$$

Here the horizontal maps are induced by the fundamental cohomology class of the mod p Brown Gitler spectrum of [C], and are surjective by the property of the Brown-Gitler spectrum (see Theorem I of [C]); The right side homologies are all isomorphic, via the vertical maps, to $\mathbb{Z}/p$. Now we simply pick $\tilde{\xi}_k \in B(p^k)_{2p(p^k+1)-2}(P \wedge \Sigma M(p))$ as a lift of the generator of $\mathbb{Z}/p \cong H_{2p(p^k+1)-2}(P \wedge \Sigma M(p))$.

To construct $\tilde{h}$, we recall (some suspension-shift of) h in the proof of Theorem 1.1 in p.57 of [C]:

$$\begin{aligned}
& \Sigma^{-2p^{k+1}-2p+1}t(V_{p^{k+2}+p}) \longrightarrow \Sigma^{-2p^{k+1}-2p+1-(2p-4)(p^{k+2}+p)}\Omega^2 S^{2p-1} \\
\longrightarrow\ & \Sigma^{-2p^{k+1}-2p+1-(2p-4)(p^{k+2}+p)}\Omega^2 S^{2p^2-2p+1} \\
\longrightarrow\ & \Sigma^{-2p^{k+1}-2p+1-(2p-4)(p^{k+2}+p)+(2p^2-2p-2)(p^{k+1}+1)+1}t(V_{p^{k+1}}) \\
=\ & \Sigma^{(2p-4)p^{k+1}}t(V_{p^{k+1}}) \longrightarrow \Omega^2 S^{2p-1}.
\end{aligned}$$

Then, $\tilde{h}$ is simply the smash product of h and the identity of P.

Finally, to construct $\tilde{g}_{k+1}$, consider the composite

$$\Sigma^{(2p^2-4)p^{k+1}} t(V_{p^{k+1}}) \longrightarrow \Omega^2 S^{2p-1} \longrightarrow U \longrightarrow \Sigma^\infty \Sigma \mathbb{C}P^\infty_+,$$

where the first map comes from the Snaith splitting as before; the second map is the double loop extension of the generator of $\pi_{2p-3}(U)$; the third map is obtained by regarding a unitary matrix as a S^1-equivariant self map of a sphere, using the Becker-Schultz [BS] identification $G(S^1) \simeq Q(\Sigma\mathbb{C}P^\infty_+)$. The composition of this map and the S^1-transfer $\Sigma\mathbb{C}P^\infty_+ \longrightarrow S^0$ is the stable adjoint of the complex J-homomorphism $U \longrightarrow SG \simeq Q_0 S^0$, as was shown by [K] [MMM].

From the construction, it is easy to see that the composition

$$\tilde{g}_{k+1} \circ \tilde{h} \circ \tilde{\xi}_k \circ i,$$

is a lift of ζ_k in [C], up to multiplication by a non-zero element in $\mathbb{Z}/p$. This completes the proof of Theorem 3. □

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