

## ON THE UNSTABLE ADAMS SPECTRAL SEQUENCE FOR $SO$ and $U$ , AND SPLITTINGS OF UNSTABLE EXT GROUPS\*

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*This paper is dedicated to the memory of José Adem*

### 1. Introduction

Let  $\mathcal{U}$  denote the category of unstable modules over the mod 2 Steenrod algebra  $A$ , and let  $Ext_U^{s,t}(-) = Ext_U^s(-, \Sigma^t Z_2)$ . Let  $M_\infty = \tilde{H}^*(\Sigma CP_+^\infty)$  denote the unstable  $A$ -module with nonzero classes  $x_i$  such that  $i$  is odd and positive, and

$$Sq^{2j} x_{2k+1} = \binom{k}{j} x_{2(j+k)+1}.$$

Then  $H^*(U) \approx U(M_\infty)$ , where the left side is the mod 2 cohomology of the infinite unitary group, and the right side the free unstable  $A$ -algebra generated by  $M_\infty$ . Thus there is an unstable Adams spectral sequence (UASS), defined as in [6], converging to  $\pi_*(U)$  with  $E_2^{s,t} \approx Ext_U^{s,t}(M_\infty)$ . We shall construct an algebraic spectral sequence (SS) which we conjecture agrees with this UASS. In Section 3, we perform the minor modifications required to yield the analogous results for the infinite special orthogonal group  $SO$ .

Part of our conjecture is a splitting result for  $Ext(M_\infty)$ . Let  $M_n$  denote the subspace of  $M_\infty$  spanned by those  $x_i$  such that  $\alpha(i) \leq n$ , where  $\alpha(i)$  denotes the number of 1's in the binary expansion of  $i$ . This provides a filtration of  $M_\infty$  by  $A$ -submodules, associated to which is a trigraded SS with

$$E_1^{n,s,t} = Ext^{s,t}(M_n/M_{n-1}),$$

$d_r : E_r^{n,s,t} \rightarrow E_r^{n+r,s+1,t}$ , and  $Ext^{s,t}(M_\infty)$  filtered with  $n$ th subquotient  $E_\infty^{n,s,t}$ .

CONJECTURE (1.1). *This SS collapses to an isomorphism*

$$Ext^{s,t}(M_\infty) \approx \bigoplus_{n \geq 1} Ext^{s,t}(M_n/M_{n-1}).$$

Conjecture (1.1), which should be of much interest in its own right, is implied by our main conjecture, because the algebraic SS which we construct has  $\bigoplus Ext(M_n/M_{n-1})$  as its  $E_2$ -term. The following analogue for  $SO$  may be of even more interest.

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$$\begin{array}{ccccccc}
 \xrightarrow{j} & Ext^{s-1,t}(M_1) & \xrightarrow{\delta} & 0 & & & \\
 & & & \downarrow i & & & \\
 \xrightarrow{j} & Ext^{s-2,t-1}(M_2/M_1) & \xrightarrow{\delta} & D_0^{s,t} & \xrightarrow{j} & Ext^{s,t}(M_1) & \xrightarrow{\delta} & 0 \\
 & & & \downarrow i & & & & \downarrow i \\
 \xrightarrow{j} & Ext^{s-3,t-2}(M_3/M_2) & \xrightarrow{\delta} & D_1^{s,t} & \xrightarrow{j} & Ext^{s-1,t-1}(M_2/M_1) & \xrightarrow{\delta} & D_0^{s+1,t} & \xrightarrow{j} \\
 & & & \downarrow i & & & & \downarrow i & \\
 \xrightarrow{j} & Ext^{s-4,t-3}(M_4/M_3) & \xrightarrow{\delta} & D_2^{s,t} & \xrightarrow{j} & Ext^{s-2,t-2}(M_3/M_2) & \xrightarrow{\delta} & D_1^{s+1,t} & \xrightarrow{j} \\
 & & & \downarrow i & & & & \downarrow i & 
 \end{array}$$

If  $t - s < 2^{n+1} - 2$ , then  $i_n^{s,t}$  is bijective, since both groups surrounding it are 0. Thus limits are attained in this SS, and so the following corollary is immediate.

COROLLARY (1.5). *There is a SS with*

$$\mathcal{E}_2^{n,s,t} = Ext^{s-n,t-n}(M_{n+1}/M_n),$$

with  $d_r^{n,s,t} : \mathcal{E}_r^{n,s,t} \rightarrow \mathcal{E}_r^{n-r+1,s+1,t}$  and  $\mathcal{E}_\infty^{n,s,t}$  the  $n$ th subquotient of a filtration of  $D^{s,t}$ .

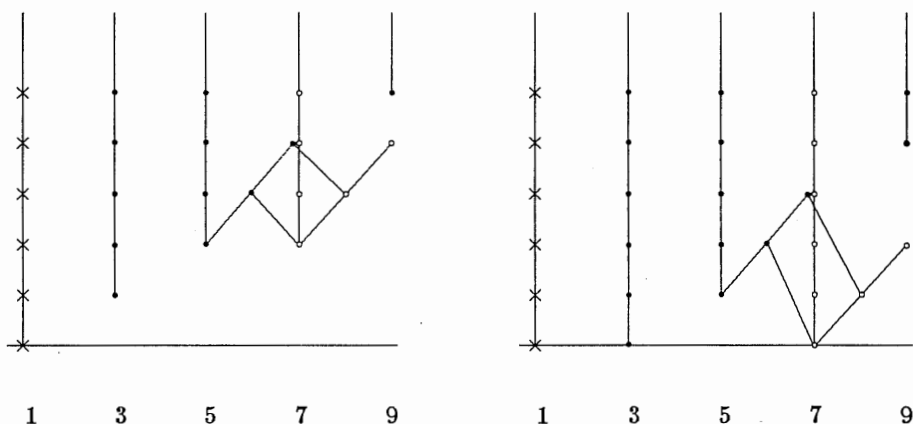
Thus this algebraic SS, which has a very complicated  $\mathcal{E}_2$ -term, converges to the nice ASS for  $u$ . But its  $\mathcal{E}_2$ -term is not the  $E_2$ -term of the UASS converging to  $\pi_*(U)$ , for it effectively pushes the chart of  $Ext^{s,t}(M_{n+1}/M_n)$  up by  $n$  units, by letting  $Ext^{s-n,t-n}(M_{n+1}/M_n)$  contribute to the limit  $(s, t)$ -group. In order to have a chance of obtaining the correct  $E_2$ -term, we regrade.

Define a new SS with

$$E_2^{s,t} = \bigoplus_n \mathcal{E}_2^{n,s+n,t+n} = \bigoplus_n Ext^{s,t}(M_{n+1}/M_n), \tag{1.6}$$

with  $d_r$  on the  $n$ th summand of  $E_r^{s,t}$  equal to  $d_r^{n,s+n,t+n}$  of the SS of (1.5). The  $E_\infty$ -term of this SS is a regraded version of  $\mathcal{E}_\infty$  of the SS of (1.5). When an element of  $D^{s,t}$  is pulled back to  $D_n^{s,t}$  for smallest possible  $n$ , it will now be seen in filtration  $s - n$  rather than  $s$ , as it was in  $\mathcal{E}$ . These  $n$ 's are a nonincreasing function of  $s$  as we move up a tower (fixed  $t - s$ ) of  $D^{s,t}$ , and will eventually stabilize. But changes in this  $n$  cause what look like filtration jumps in the SS of (1.6).

We illustrate with the situation through  $t - s = 9$ , where this jump first happens. The left SS is that of (1.5), and the right that of (1.6). Classes with  $n = 0$  are indicated by  $\times$ , with  $n=1$  by  $\bullet$ , and with  $n = 2$  by  $\circ$ . As usual, coordinates are  $(t - s, s)$ .



Note that when the SS of (1.5) is pictured as above, all differentials look like  $d_1$ 's, for they all go from a group contributing to  $D^{s,t}$  to a group contributing to  $D^{s+1,t}$ . But the subscripts of differentials in the SS of (1.5) are related to changes of summand.

Our second main conjecture for  $U$  is as follows.

CONJECTURE (1.7). *The SS of (1.6) agrees with the UASS of  $U$ .*

As evidence, we observe that if Conjecture (1.1) is true, then the SS of (1.6) has the correct  $E_2$ -term, and its  $E_\infty$  could be correct, for it has 0 in even stems and  $Z_2$ 's of strictly increasing filtrations in odd stems. Further evidence is given by the fact that our proof of Proposition (1.4) will involve the cohomology of the spaces in the Postnikov system for  $U$ .

As pointed out by Mark Mahowald, this algebraic SS for  $\pi_*(U)$  can be demystified somewhat by thinking of it as arising from the destabilization of an Adams resolution of  $u$ . This is discussed in Section 4, where we also explain how the unstable  $A$ -modules  $M_{n+1}/M_n$  can be obtained as derived functors of the destabilization functor applied to a shifted version of  $H^*u$ . In Section 5, we present a generalization of our results and conjectures. It is not clear whether the situation for  $SO$  and  $U$  contains essential ingredients not present in the much more general context of Section 5.

The reader attempting to prove Conjectures (1.1) and (1.2) should keep in mind that they are not true if an arbitrary module is allowed in the second variable. For example,  $Ext^{0,0}(\tilde{H}^*RP^\infty, Q_1) = 0$ , while  $Ext^{0,0}(Q_1, Q_1) \neq 0$ . Also, the conjectures are not true for stable Ext.

We would like to thank Haynes Miller for pointing out a mistake in an early draft, and Bill Massey for allowing us to use his unpublished work [5].

**2. Construction of the spectral sequence**

In this section, we prove Proposition (1.4), which we have already seen implies the existence of the SS's of (1.5) and (1.6).

Define  $F_n = F'(2n)$ , and define  $K_n$  so that there is a short exact sequence (SES) in  $\mathcal{U}$

$$0 \rightarrow K_n \rightarrow F_n \xrightarrow{\cdot Sq^3} J_{n-1} \rightarrow 0. \tag{2.1}$$

Since  $J_m = 0$  for  $m < 2$ , and

$$Ext^{s,t}(F_n) = \begin{cases} Z_2 & t - s = 2n, s \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

the Ext sequence of (2.1) yields

PROPOSITION (2.2).

i) if  $n = 1$  or  $2$ , then  $Ext^{s,t}(K_n) = \begin{cases} Z_2 & t - s = 2n, s \geq 0 \\ 0 & \text{otherwise;} \end{cases}$

ii) if  $n \geq 3$ , then  $Ext^{s,t}(J_{n-1}) \approx Ext^{s-1,t}(K_n)$  unless  $t - s = 2n$  and  $s \geq 0$ , in which case  $Ext^{s,t}(J_{n-1}) = Z_2$  and  $Ext^{s-1,t}(K_n) = 0$ .

We will need the following elementary result, where  $\Omega$  is the loop functor in  $\mathcal{U}$  as defined in [6].

PROPOSITION (2.3).  $Ext^{s,t}(\Omega K_n) \approx Ext^{s,t+1}(K_n)$ .

*Proof.* Letting  $\Omega_1$  denote the first derived functor of  $\Omega$ , [6, pp. 43-44] and (2.1) imply that  $\Omega_1 K_n = 0$ , since it injects to  $\Omega_1 F_n = 0$ . Then the exact sequence

$$\rightarrow Ext^{s,t}(\Omega K_n) \rightarrow Ext^{s,t+1}(K_n) \rightarrow Ext^{s-1,t}(\Omega_1 K_n) \rightarrow Ext^{s+1,t}(\Omega K_n) \rightarrow$$

(see, e.g., [2, 3.7]) implies the desired result.  $\square$

The Postnikov tower for  $U$  is a tower

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ K(Z, 5) & \xrightarrow{i} & U[1, 5] & \xrightarrow{k} & K(Z, 8) \\ & & \downarrow p & & \\ K(Z, 3) & \xrightarrow{i} & U[1, 3] & \xrightarrow{k} & K(Z, 6) \\ & & \downarrow p & & \\ & & K(Z, 1) & \xrightarrow{k} & K(Z, 4), \end{array} \tag{2.4}$$

in which each  $\xrightarrow{i} \xrightarrow{p} \xrightarrow{k}$  is a fiber sequence, and the map  $U \rightarrow U[1, 2n + 1]$  induces an isomorphism in  $\pi(-)$  for  $i \leq 2n + 1$ , while  $\pi_i(U[1, 2n + 1]) = 0$  for  $i > 2n + 1$ .

There are isomorphisms of  $A$ -algebras  $H^*(K(Z, 2n - 1)) \approx U(\Omega F_n)$  and, by [7] or [3],  $H^*(U[1, 2n - 1]) \approx U(X_n)$  for a certain unstable  $A$ -module  $X_n$ , which can be thought of as  $\Sigma^{-1}QH^*(BU[2, 2n])$ , where  $Q$  denotes the indecomposable quotient. The morphisms of  $H^*(-)$  of (2.4) are induced by morphisms of these unstable  $A$ -modules as below.

$$\begin{array}{ccccccc}
 \Omega F_{n+1} & \xleftarrow{i_{n+1}} & X_{n+1} & \xleftarrow{f_{n+1}} & F_{n+2} & & \\
 & & \uparrow & & & & \\
 \Omega F_n & \xleftarrow{i_n} & X_n & \xleftarrow{f_n} & F_{n+1} & & (2.5) \\
 & & \uparrow & & & & \\
 & & X_{n-1} & \xleftarrow{f_{n-1}} & F_n & & 
 \end{array}$$

The following result is culled from [7] (see esp. [7, 8.6] for (2.8) and [7, 7.1] for (2.9)), with  $K_n, J_n$ , and  $M_n$  as above.

**THEOREM (2.6)** *In the above diagram,  $i_n f_n = \cdot Sq^3$ ,*

$$ker(f_n) = K_{n+1} \quad \text{and} \quad im(f_{n+1}) \approx J_{n+1}, \tag{2.7}$$

and there are SES's in  $\mathcal{U}$

$$0 \rightarrow M_n \rightarrow X_{n+1} \rightarrow \Omega ker(f_n) \rightarrow 0 \tag{2.8}$$

and

$$0 \rightarrow im(f_{n+1}) \rightarrow X_{n+1} \rightarrow M_{n+1} \rightarrow 0. \tag{2.9}$$

Note that (2.8) and (2.9) imply that

$$M_n \approx im(X_n \rightarrow X_{n+1}). \tag{2.10}$$

Taking quotients by  $M_n$  of the latter modules in (2.9), then using (2.8) to rewrite the middle group, and finally using (2.7) yields a SES

$$0 \rightarrow J_{n+1} \rightarrow \Omega K_{n+1} \rightarrow M_{n+1}/M_n \rightarrow 0. \tag{2.11}$$

From (2.11) and (2.3), we obtain a LES

$$\begin{array}{ccccccc}
 \rightarrow Ext^{s-n-1, t-n+1}(K_{n+1}) & \rightarrow & Ext^{s-n-1, t-n}(J_{n+1}) & \rightarrow & Ext^{s-n, t-n}(M_{n+1}/M_n) & & \\
 & & \rightarrow & & Ext^{s-n, t-n+1}(K_{n+1}) & \rightarrow & .
 \end{array} \tag{2.12}$$

We will deduce the exact sequence of Proposition (1.4) from this.

When  $n = 0$ , the sequence of (1.4) reduces to the fact that

$$D_0^{s,t} \approx Ext^{s,t}(M_1) \approx \begin{cases} Z_2 & t - s = 1 \text{ and } s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

When  $n = 1$ , both (2.12) and (1.4) reduce to the same isomorphism for  $t - s > 3$ , using (2.2) and the definition of  $D_n^{s,t}$ . The initial towers ( $t - s = 1$ ) in  $D_0^{s,t}$  and  $D_1^{s,t}$  correspond under the morphism of (1.4). The tower in  $D_1^{s,t}$  in  $t - s = 3, s \geq 1$ , maps in (1.4) to a tower in  $Ext(M_2/M_1)$  which in (2.12) maps to the tower in  $Ext(K_2)$ .

If  $n \geq 2$ , exactness of (2.12) is maintained if  $D^{s,t}$  for  $t - s \leq 2n + 1$  is added to both  $Ext^{s-n-1,t-n+1}(K_{n+1})$  and  $Ext^{s-n-1,t-n}(J_{n+1})$ . The latter clearly becomes  $D_n^{s,t}$ , while (2.2) shows that the former becomes  $D_{n-1}^{s,t}$ , as desired.

Three remarks are in order. First, the reader may object that this is not an algebraic SS, as advertised, because it involves the cohomology of the spaces  $U[1, 2n - 1]$ . To this, we counter that the SS comes completely from (2.11), which is just algebraic. Moreover, a completely algebraic derivation of (2.11) can be given, but we felt the one presented here is more likely to lead to a proof of our conjectures.

Second, note that when the exact sequence

$$\Sigma^3 A/ASq^1 \xrightarrow{Sq^3} A/ASq^1 \xrightarrow{Sq^3} \Sigma^{-3} A/ASq^1 \tag{2.13}$$

is destabilized, the resulting sequence fails to be exact. Massey ([5]) computed the homology of the destabilized sequence

$$F'(n + 3) \rightarrow F'(n) \rightarrow F'(n - 3)$$

to be, if  $n = 2m + \epsilon$  with  $\epsilon = 0$  or  $1$ , an  $A$ -module spanned by  $z_{2m+\epsilon}$  and all elements  $Sq^I z_{2m+\epsilon}$ , where  $I = (i_1, \dots, i_r)$  is admissible, and, for all  $j$  satisfying  $1 \leq j \leq r$ , the excess of  $(i_j, \dots, i_r)$  equals  $2^m$  or  $2^m - 2^e$  for some  $e$  satisfying  $1 \leq e < m$ . Our result (2.11) is closely related to this. It says that the sequence

$$F'(2n + 4) \xrightarrow{Sq^3} F'(2n + 1) \xrightarrow{\theta} F'(2n - 1) \tag{2.14}$$

has homology  $M_{n+1}/M_n$ . Here  $\theta$  sends  $Sq^I \iota_{2n+1}$  to  $Sq^I Sq^3 \iota_{2n-1}$ , provided  $\text{excess}(I) \leq 2n + 1$ . Note that  $\theta$  increases degree by 1.

Third, as more evidence for the relationship between our SS and the UASS, we point out that the Postnikov tower of  $U$  resembles an Adams-Postnikov tower, as defined in [6, p. 82], because

$$\ker(H^*(U[1, 2n - 1]) \rightarrow H^*(U)) = \ker(H^*(U[1, 2n - 1]) \rightarrow H^*(U[1, 2n + 1])).$$

### 3. Results for $SO$

The entire argument can be directly adapted to  $SO$ , using results of [8] and [3] instead of [7]. We just list the replacements for the various symbols.

- $M_\infty = \tilde{H}^*(RP^\infty)$ . Thus  $M_\infty$  has classes  $x^i$  for all positive integers  $i$ , and  $M_n$  is spanned by those with  $\alpha(i) \leq n$ .
- For  $0 \leq b \leq 3$ , let  $\rho(4a + b) = 8a + 2^b$ . Thus  $\rho(n)$  is the grading of the  $n$ th nonzero homotopy group of  $BSO$ . Then  $X_n = \Sigma^{-1}QH^*(BSO[2, \rho(n)])$ , and  $H^*(SO[1, \rho(n) - 1]) \approx U(X_n)$ .
- $F_n = \begin{cases} F(\rho(n)) & \text{if } n \equiv 0, 1 \pmod{4} \\ F'(\rho(n)) & \text{if } n \equiv 2, 3 \pmod{4} \end{cases}$
- $i_n f_n : F_{n+1} \rightarrow \Omega F_n$  is  $\cdot Sq^d$ , where

$$d = \rho(n+1) - \rho(n) + 1 = \begin{cases} 2 & n \equiv 0, 3 \pmod{4} \\ 3 & n \equiv 1 \pmod{4} \\ 5 & n \equiv 2 \pmod{4} \end{cases}$$

and, as before,  $K_{n+1} = \ker(i_n f_n) = \ker(f_n)$ ,  $J_n = \text{im}(i_n f_n) \approx \text{im}(f_n)$ , and  $M_n = \text{im}(X_n \rightarrow X_{n+1})$ .

- Let  $h(i) = \begin{cases} 1 & i \equiv 0, 1 \pmod{4} \\ \infty & i \equiv 2, 3 \pmod{4} \end{cases}$ . Then

$$D^{s,t} = \begin{cases} Z_2 & \text{if } t - s = \rho(i) - 1 \text{ and } 0 \leq i - 1 \leq s < i - 1 + h(i) \\ 0 & \text{otherwise} \end{cases}$$

and

$$D_n^{s,t} = \begin{cases} D^{s,t} & \text{if } t - s \leq \rho(n+1) - 1 \\ \text{Ext}^{s-n-1, t-n}(J_{n+1}) & \text{otherwise} \end{cases}$$

- Proposition (2.2) is replaced by

PROPOSITION (3.1).

i) if  $n \leq 3$ , then  $\text{Ext}^{s,t}(K_n) = \begin{cases} Z_2 & t - s = \rho(n), 0 \leq s < h(n) \\ 0 & \text{otherwise;} \end{cases}$

ii) if  $n \geq 4$ , then  $\text{Ext}^{s,t}(J_{n-1}) \approx \text{Ext}^{s-1,t}(K_n)$  unless  $t - s = \rho(n)$  and  $0 \leq s < h(n)$ , in which case  $\text{Ext}^{s,t}(J_{n-1}) = Z_2$  and  $\text{Ext}^{s-1,t}(K_n) = 0$ .

With the new meaning of terms, Theorem (2.6), Proposition (1.4), and Corollary (1.5) are valid. Also, there is a SS (1.6), which we conjecture agrees with the UASS of  $SO$ . Conjecture (1.2) would be a corollary of this agreement, since it relates two ways of expressing the  $E_2$ -term.

Analogous to the second remark at the end of Section 2 is that the following sequences on the left have homology as indicated on the right.



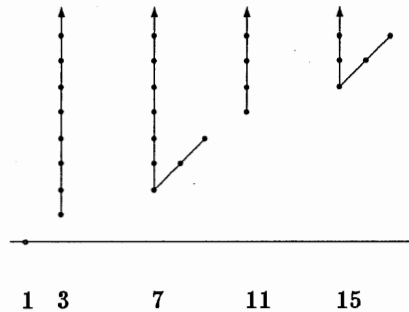
sequence	homology	
$F(8t + 1) \rightarrow F'(8t - 1) \rightarrow F'(8t - 6)$	$M_{4t-1}/M_{4t-2}$	(3.2)
$F(8t + 2) \rightarrow F(8t) \rightarrow F'(8t - 2)$	$M_{4t}/M_{4t-1}$	
$F'(8t + 4) \rightarrow F(8t + 1) \rightarrow F(8t - 1)$	$M_{4t+1}/M_{4t}$	
$F'(8t + 8) \rightarrow F'(8t + 3) \rightarrow F(8t)$	$M_{4t+2}/M_{4t+1}$	

There is a curious difference between this result and the unitary case discussed in the second remark at the end of Section 2. In that case, the sequence (2.14) is what comes up in our work, and its homology differs from that of the natural destabilization of (2.13). However, in the  $SO$ -case, using the second of the four cases of (3.2) to illustrate, the sequence

$$F(8t + 2) \rightarrow F(8t) \rightarrow F'(8t - 1),$$

where the second morphism is  $Sq^2$  but increases degree by 1, is the one which comes up in our work, and its homology equals that of the sequence listed in (3.2), which is the natural desuspension of a stable exact sequence. These are not obviously equal; their equality follows from a comparison of our results and those of [5]. This difference between the  $U$  and  $SO$  cases seems to beg an explanation, but as it is peripheral to this project, we shall not pursue it here.

We remark that a chart for  $D^{s,t}$  with coordinates  $(t - s, s)$  is the chart for the connective  $so$  spectrum, which begins as follows.



This can be seen since (see e.g., [3])

$$H^*(so) \approx \Sigma A/ASq^3 = \Sigma A \otimes_{A_1} A_1/A_1Sq^3,$$

where  $A_1$  is the subalgebra of  $A$  generated by  $Sq^1$  and  $Sq^2$ , and hence in the ASS for  $so$

$$E_\infty^{s,t} = E_2^{s,t} = Ext_{A_1}(\Sigma A_1/A_1Sq^3).$$

### 4. Topological and algebraic destabilizations

Our construction of the conjectured UASS for  $U$  (resp.  $SO$ ) can be viewed as a consequence of destabilizing the ASS for  $u$  (resp.  $so$ ). Let  $H_n = \Sigma^n HZ_{(2)}$  denote the Eilenberg-MacLane spectrum. An Adams-Postnikov tower for  $u$  has the simple form

$$\begin{array}{ccccc}
 & & u & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 H_5 & \rightarrow & u[1, 5] & \rightarrow & H_8 \\
 & & \downarrow & & \\
 H_3 & \rightarrow & u[1, 3] & \rightarrow & H_6 \\
 & & \downarrow & & \\
 & & H_1 & \rightarrow & H_4.
 \end{array}$$

Diagram (2.4) is just the destabilization of this. Whereas the sequence

$$H^*(H_{2n+4}) \rightarrow H^*(H_{2n+1}) \xrightarrow{\tau} H^*(H_{2n-1})$$

is exact, it destabilizes to (2.14), which has homology  $M_{n+1}/M_n$ . Our SS deals with the way in which the groups  $Ext(M_{n+1}/M_n)$  build  $\pi_*(U) \approx \pi_*(u)$ .

As pointed out by Paul Goerss, there is an algebraic analogue of this. Let  $\Omega^\infty : Mod_A \rightarrow \mathcal{U}$  be the left adjoint to the inclusion functor. Thus, if  $M$  is an  $A$ -module, then  $\Omega^\infty M$  is the quotient of  $M$  mod relations  $Sq^i x = 0$  if  $i > |x|$ . Let  $\Omega_s^\infty$  denote the  $s$ th derived functor of  $\Omega^\infty$ . Then we have

PROPOSITION (4.1). (Goerss) *If  $M_i$  is as in Section 1, then*

$$\Omega_n^\infty(\Sigma^{-n+1}A//\Lambda_1) \approx M_{n+1}/M_n \oplus \Sigma Z/2,$$

*while if  $M_i$  is as in Section 3, then*

$$\Omega_{n+1}^\infty(\Sigma^{-n-2}A//A_1) \approx M_n/M_{n-1}.$$

*Proof.* Define an acyclic chain complex  $C$  by

$$\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A//\Lambda_1 \rightarrow 0$$

with  $C_i = \Sigma^{3i}A/ASq^1$  and boundary morphisms  $\cdot Sq^3$ . Apply  $\Omega^\infty$  to the modules in an  $A$ -resolution of each  $\Sigma^{-n+1}C_i$  to obtain a spectral sequence

$$E_{p,q}^1 = \Omega_p^\infty \Sigma^{-n+1}C_q \implies \Omega_{p+q}^\infty \Sigma^{-n+1}A//\Lambda_1, \tag{4.2}$$

with  $d^r : E_{p,q}^r \rightarrow E_{p+r-1,q-r}^r$ . Noting that the only homology of the sequence

$$\dots \rightarrow F(3) \rightarrow F(2) \rightarrow F(1) \rightarrow F(0) \rightarrow 0,$$

with morphisms  $\cdot Sq^1$ , is  $F(0) = Z_2$  and  $\Sigma Z_2$  in  $F(1)$ , we deduce

$$\Omega_p^\infty \Sigma^m A / ASq^1 = \begin{cases} \Sigma^{m+p} Z_2 & \text{if } p > 0 \text{ and } m + p = 0 \text{ or } 1 \\ F'(m) & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the only nonzero groups  $E_{p,q}^1$  in (4.2) satisfying  $p + q = n + \epsilon$  with  $-1 \leq \epsilon \leq 1$  are  $E_{0,n+\epsilon}^1 = F'(2n + 1 + 3\epsilon)$ ,  $E_{n,0}^1 = \Sigma Z_2$ , and  $E_{n-1,0}^1 = Z_2$ . By (2.14),  $E_{0,n}^2 = M_{n+1}/M_n$ . The only possible remaining differential involving groups with  $p + q = n$  is  $d^n : E_{0,n}^n \rightarrow E_{n-1,0}^n$ , but this must be zero since these differentials preserve internal grading.

The  $SO$ -case is proved similarly, using the acyclic complex over  $\Sigma^{-n-2}A//A_1$  with  $C_{4t} = \Sigma^{12t-n-2}A/ASq^1$ ,  $C_{4t+1} = \Sigma^{12t-n}A$ ,  $C_{4t+2} = \Sigma^{12t-n+2}A$ , and  $C_{4t+3} = \Sigma^{12t-n+5}A/ASq^1$ . For  $\epsilon = 0, 1, 2$ , and  $3$ , let  $\delta(\epsilon) = 0, 2, 4$ , and  $7$ , respectively. Then in the SS analogous to (4.2),

$$\begin{aligned} E_{p,4t+\epsilon}^1 &= \begin{cases} \Omega_p^\infty \Sigma^{12t-n-2+\delta(\epsilon)} A / ASq^1 & \epsilon = 0, 3 \\ \Omega_p^\infty \Sigma^{12t-n-2+\delta(\epsilon)} A & \epsilon = 1, 2 \end{cases} \\ &= \begin{cases} F'(12t - n - 2 + \delta(\epsilon)) & \epsilon = 0, 3 \text{ and } p = 0 \\ F(12t - n - 2 + \delta(\epsilon)) & \epsilon = 1, 2 \text{ and } p = 0 \\ Z_2 & \epsilon = 0, 3 \text{ and } 12t - n - 2 + \delta(\epsilon) + p = 0 \\ \Sigma Z_2 & \epsilon = 0, 3 \text{ and } 12t - n - 2 + \delta(\epsilon) + p = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$E_{0,n+1}^2$  is the homology of  $E_{0,n+2}^1 \rightarrow E_{0,n+1}^1 \rightarrow E_{0,n}^1$ , which by (3.2) is  $M_n/M_{n+1}$ . Higher differentials must be 0 for dimensional reasons as in the previous case. Since for  $\epsilon = 0, 3$  we have  $\delta(\epsilon) \equiv \epsilon \pmod{4}$ , the only  $Z_2$ 's or  $\Sigma Z_2$ 's occur in  $E_{p,q}^1$  with

$$p + q \equiv p + 4t + \epsilon \equiv p + 12t + \delta(\epsilon) \equiv n + 2 \text{ or } 3 \pmod{4},$$

and so there are none when  $p + q = n + 1$ .

## 5. A generalization

In this section, we propose conditions on an algebraic resolution which are satisfied by (2.5) and its  $SO$  analogue. We show that the analogue of Theorem (1.5) is true in this generality, and conjecture that the analogue of Conjecture (1.1) is true.

Suppose the diagram of unstable  $A$ -modules

$$\begin{array}{ccccccc}
 & F_1 & & F_2 & & F_3 & \\
 & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \\
 X_0 & \xrightarrow{p_0} & X_1 & \xrightarrow{p_1} & X_2 & \xrightarrow{p_2} & \dots \rightarrow X \\
 & & \downarrow i_1 & & \downarrow i_2 & & \\
 & & \Omega F_1 & & \Omega F_2 & & 
 \end{array}$$

satisfies

- $F_n \rightarrow X_{n-1} \rightarrow X_n \rightarrow \Omega \ker(F_n \rightarrow X_{n-1}) \rightarrow 0$  is exact.
- $F_n$  is a direct sum of  $F(m)$ 's and/or  $F'(k)$ 's
- $(i_n f_n)^* = 0 : \text{Ext}(\Omega F_n) \rightarrow \text{Ext}(F_{n+1})$
- $\ker(X_n \rightarrow X) = \ker(X_n \rightarrow X_{n+1})$
- $X = \text{dirlim}(X_n)$ .

Let  $M_n = \text{im}(X_n \rightarrow X)$ .

CONJECTURE (5.1).  $\text{Ext}(X) = \bigoplus \text{Ext}(M_{n+1}/M_n)$

The following proposition generalizes Theorem (1.5).

PROPOSITION (5.2) *Let*

$$D^{s,t} = \bigoplus_n \text{Ext}^{s-n,t-n}(\Omega F_n).$$

*There is a spectral sequence with*

$$E_2^{n,s,t} = \text{Ext}^{s-n,t-n}(M_{n+1}/M_n)$$

*and  $E_\infty^{n,s,t}$  the  $n$ th subquotient of a filtration of  $D^{s,t}$ .*

The proof is identical to the proof for  $X = \tilde{H}^*(\Sigma CP_+^\infty)$  in Sections 1 and 2. One derives a SES

$$0 \rightarrow \text{im}(f_{n+1}) \rightarrow \Omega \ker(f_n) \rightarrow M_{n+1}/M_n \rightarrow 0,$$

and modifies its exact Ext sequence using that  $\text{Ext}^{s,t}(\text{im } f_n)$  and  $\text{Ext}^{s-1,t}(\ker f_n)$  differ only by  $\text{Ext}(F_{n+1})$ . This allows one to splice the exact sequences, yielding the desired spectral sequence.

One could modify the spectral sequence of the proposition to obtain a SS with  $E_2^{s,t} = \bigoplus \text{Ext}^{s,t}(M_{n+1}/M_n)$ . This  $E_2$ -term is the same as the  $E_2$ -term of the SS converging to  $\text{Ext}(X)$  whose collapsing we would like to prove. Somehow the fact that the SS of the proposition allowed us to consider the filtrations as being much higher is supposed to yield a proof of Conjecture (5.1).

If the conjecture isn't true in this generality, then what extra conditions are required to make it true?

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