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SPHERICAL CLASSES IN THE BORDISM OF THE TORSION MOLECULE

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To José Adem, in memoriam

We characterize the spherical classes in the bordism of the torsion molecule via MU-operations.

1. Introduction

In [1], J. Harper constructed a finite simply-connected H-space, localized at p, for each odd prime number p.

Let $K(p)$ denote such a space, and

$$H^*(K(p));\mathbb{Z}/p) = \wedge(x_3, x_{2p+1}) \otimes \mathbb{Z}/p[x_{2p+2}]/(x_{2p+2})^p$$

$$|x_i| = i,\ P^1(x_3) = x_{2p+1} \text{ and } \beta(x_{2p+1}) = x_{2p+2}.$$

$K(p)$ is an important object in the theory of finite mod p H-spaces because

a) $K(p)$ has p-torsion; that is, $K(p)$ is an example of a finite H-space with p-torsion for p as large as we wish;

b) For lower primes, $p = 3, 5$, $K(p)$ appears as a mod p factor of the exceptional Lie groups $\mathbb{F}_4$ and $\mathbb{E}_8$, respectively;

c) $K(p)$ is not of the mod p homotopy type of a Lie group.

In this paper we will extend the calculations, made in [3] by R. Kane and the author, of the spherical classes in the bordism of compact Lie groups to $K(p)$, Harper's torsion molecule.

The main result of this paper is

THEOREM. *The image of the Hurewicz map*

$$h^{MU} : \Pi_*(K(p))/\mathrm{Tor} \longrightarrow MU_*(K(p))/\mathrm{Tor}$$

agrees with the primitive classes in $MU_*(K(p))/\mathrm{Tor}$.

2. Spherical classes and spherical numbers

In this section X is a simply connected finite H-space. Thus $H_*(X;\mathbb{Q}) = \wedge(x_{2k_1+1}, \ldots, x_{2k_r+1})$ where r =rank of X and $k_1 \leq k_2 \leq \cdots \leq k_r$ are the exponents of X.

Suppose that $k_1 < k_2 < k_3 < \cdots < k_r$. Then the commutativity of the diagram

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$$\begin{tikzcd}
\Pi_*(X)\otimes\mathbb{Q} \arrow[r, "h\otimes Q", "\cong"'] & PH_*(X;\mathbb{Q}) \\
\Pi_*(X)/\mathrm{Tor} \arrow[u, hook] \arrow[r, "h"] & PH_*(X;\mathbb{Z})/\mathrm{Tor} \arrow[u, hook]
\end{tikzcd}$$

where h is the Hurewicz map, implies that for each k_i for $i = 1, 2, \ldots, r$,

$$\begin{tikzcd}
\Pi_{2k_i+1}(X)/\mathrm{Tor} \arrow[r, "h"] \arrow[d, "="] & PH_{2k_i+1}(X;\mathbb{Z})/\mathrm{Tor} \arrow[d, "="] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

is multiplication by a positive integer $S(X;k_i)$ which depends on X and k_i.

Here $PH_*(X;\mathbb{Q})$ denotes the diagonal-primitives of $H_*(X;\mathbb{Q})$. In other words, let $x_{2k+1} \in PH_{2k+1}(X;\mathbb{Z})/\mathrm{Tor}$ be the generator, then $S(X;k)$ is the least positive integer such that

$$S(X;k)x_{2k+1} \in \text{ Image of } h$$

Definition. A homology class is *spherical* if it belongs to the image of the Hurewicz map. $\{S(X;k_i) \in \mathbb{Z}^+ | i = 1, 2, \ldots, r\}$ are the *spherical numbers*, for X a finite H-space.

Remark. It is easy to see that $S(X;k_1) = 1$

Examples ([3], [4]).

(a) $X = SU(n+1)$, $n \geq 1$. Then

$$H_*(X;\mathbb{Q}) = \wedge(x_3, x_5, x_7, \ldots, x_{2n+1}),$$

rank of $X = n$ $k_1 = 1,\ k_2 = 2, \ldots, k_n = n,$

$$S(X, k_i) = k_i! \text{ (Bott periodicity).}$$

(b) $X = G_2$. Then X has only 2-torsion,

$$H_*(G_2;\mathbb{Q}) = \wedge(x_3, x_{11}) \text{ rank of } X = 2,\ k_1 = 1,\ k_2 = 5,$$

$$S(G_2; 5) = 5! = 2^3 \cdot 3 \cdot 5$$

(c) $X = \mathbb{F}_4$. Then X has only 2 and 3 torsion,

$$H_*(X;\mathbb{Q}) = \wedge(x_3, x_{11}, x_{15}, x_{23}),\ \text{rank } X = 4,$$

	i	1	2	3	4
exponent	k_i	1	5	7	11
dimension	$2k_i + 1$	3	11	15	23
spherical no.	$S(X; k_i)$	1	$2^3 \cdot 5$	$2^3 \cdot 3 \cdot 7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

(d) $X = \mathbb{E}_6$. Then X has only 2 and 3 torsion,

$$H_*(X;\mathbb{Q}) = \wedge(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}),\ \text{rank } X = 6,$$

	i	1	2	3	4	5	6
exponent	k_i	1	4	5	7	8	11
dimension	$2k_i + 1$	3	9	11	15	17	23
spherical no.	$S(X; k_i)$	1	2	$2^2 \cdot 5$	$2^3 \cdot 3 \cdot 7$	$2^5 \cdot 3 \cdot 5$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

3. The spherical number for $K(p)$

PROPOSITION (3.1). *$\mathbb{Z}_{(p)}$ denotes $\mathbb{Z}$ localized at p*

$$H^*(K(p); \mathbb{Z}_{(p)})/\mathrm{Tor} = \wedge(y_3, y_{2p^2+2p-1})$$

and the generator can be chosen such that

$$\rho(y_3) = x_3 \text{ and } \rho(y_{2p^2+2p-1}) = x_{2p+1} x_{2p+2}^{p-1},$$

where ρ is reduction modulo p.

Proof. Let $\{B_r, d^r\}$ be the Bockstein spectral sequence analyzing p-torsion in $H^*(K(p); \mathbb{Z}_{(p)})$. Then $B_2 = H(H^*(K(p); \mathbb{Z}/p); \beta) = \wedge(y_3, y_{2p^2+2p-1})$, where $y_3 = \{x_3\}$ and $y_{2p^2+2p-1} = \{x_{2p+1} \cdot x_{2p+2}^{p-1}\}$. The spectral sequence must now collapse, thus $B_\infty = H^*(K(p); \mathbb{Z}_{(p)})$ has the same description. Q.E.D.

COROLLARY (3.2). *$H_*(K(p); \mathbb{Q}) = \wedge(\bar{z}_3, \bar{z}_{2p^2+2p-1})$ and $H_*(K(p); \mathbb{Z}_{(p)})/\mathrm{Tor} = \wedge(z_3, z_{2p^2+2p-1})$. Therefore rank $K(p) = 2$, and $k_1 = 1$ and $k_2 = p^2 + p - 1$ are the exponents of $K(p)$, so we have to calculate $S(K(p), k_2)$. Consider the fibration*

$$F \xrightarrow{f} K(p) \xrightarrow{g} K(\mathbb{Z}_{(p)}; 3)$$

where $y_3 = [g] \in H^3(K(p);\mathbb{Z}_{(p)})$*; i.e., F is the third connected stage of* $K(p)$.

LEMMA (3.3). *In dimensions* $\leq 2p^2+2p-1$,

$$H^*(F;\mathbb{Z}/p) = \wedge(u_{2p^2+1}, u_{2p^2+2p-1}) \otimes \mathbb{Z}/p[u_{2p^2}],$$

$$\beta(u_{2p^2}) = u_{2p^2+1} \text{ and } P^1(u_{2p^2+1}) = u_{2p^2+2p-1}.$$

Proof. Consider the Serre spectral sequence

$$E_2 = H^*(F;\mathbb{Z}/p) \otimes H^*(K(\mathbb{Z}_{(p)};3);\mathbb{Z}/p) \Rightarrow H^*(K(p);\mathbb{Z}/p).$$

By the classical work of Cartan, $H^*(K(\mathbb{Z}_{(p)};3);3)$ has no higher torsion and

$$H^*(K(\mathbb{Z}_{(p)};3);\mathbb{Z}/3) = \wedge(\xi_3, P^1(\xi_3),\ldots) \otimes \mathbb{Z}/p[\beta P^1(\xi_3),\ldots].$$

By the description of $H^*(K(p);\mathbb{Z}/p)$, the elements

$$P^p P^1(\xi_3),\ \beta P^p P^1(\xi_3) \text{ and } [\beta P^1(\xi_3)]^p$$

must be killed in the spectral sequence. Therefore there exist elements u_{2p^2}, u_{2p^2+1}, and u_{2p^2+2p-1} with

$$\begin{aligned} d_{2p^2}(u_{2p^2}) &= P^p P^1(\xi_3) \\ d_{2p^2+1}(u_{2p^2+1}) &= \beta P^p P^1(\xi_3) \qquad \text{and} \\ d_{2p^2+2p-1}(u_{2p^2+2p-1}) &= [\beta P^1(\xi_3)]^p. \end{aligned}$$

Since the differentials act transgressively (in this situation) and because of the action of the Steenrod algebra, the relations $\beta[P^p P^1(\xi_3)] = [\beta P^p P^1(\xi_3)]$ and

$$[P^p P^1(\xi_3)] = P^1 P^p \beta P^1(\xi_3) = P^p \beta P^1(\xi_3) = [\beta P^1(\xi_3)]^2$$

force $\beta(u_{2p^2}) = u_{2p^2+1}$ and $P^1(u_{2p^2+1}) = u_{2p^2+2p-1}$. Q.E.D.

COROLLARY (3.4). *In degrees* $\leq 2p^2+2p$

(i) $H_*(F;\mathbb{Z}/p) = \wedge(v_{2p^2+1}, v_{2p^2+2p-1}) \otimes \mathbb{Z}/p[v_{2p^2}]$

$$P^1(v_{2p^2+2p-1}) = v_{2p^2+1}$$

(ii) $H_*(F;\mathbb{Z}_{(p)})/\mathrm{Tor} = \wedge(w_{2p^2+1}, w_{2p^2+2p-1})/(pw_{2p^2+1} = 0)$

Proof. (i) Dualizing in Lemma 3.3.

(ii) Notice that the respective Bockstein spectral sequence in homology "almost" collapses i.e. $\beta v_{2p^2+1} = v_{2p^2}$ then $pw_{2p^2+1} = 0$.

Remark. As a by-product of Lemma 3.3 we see that the cellular decomposition of F looks as follows:

$$F \cong Y \cup e^{2p^2+2p-1} \cup \dots .$$

If $Y = S^{2p^2} \cup e^{2p^2+1}$ is a mod p Moore space, then the generator in dimension $2p^2$ in $H_*(K(p);\mathbb{Z})$ is spherical.

LEMMA (3.5).

$$f_* : PH_{2p^2+2p-1}(F;\mathbb{Z}_{(p)}) \longrightarrow PH_{2p^2+2p-1}(K(p);\mathbb{Z}_{(p)})$$

is multiplication by p.

Proof. Consider the Serre spectral sequence

$$E_2 = H^*(K(\mathbb{Z}_{(p)};3);\, H^*(F;\mathbb{Z}_{(p)})) \Rightarrow H^*(K(p);\mathbb{Z}_{(p)}).$$

Since $H^*(K(\mathbb{Z}_{(p)};3);\mathbb{Z}_{(p)})$ has no higher p-torsion, we can reduce mod p without losing any information. In particular, we can use our previous knowledge of the mod p Serre spectral sequence to obtain complete information in degrees $\le 2p^2 + 2p$ in this case. Notably we have,

$$d_{2p^2+2p-1}(a) = b$$

where a and b are integral representatives of u_{2p^2+2p-1} and $[\beta P^1(\xi_3)]^p$ respectively. Thus

$$d_{2p^2+2p-1}(pa) = pb = 0$$

Thus pa is a permanent cycle in the spectral sequence. This tells us that

$$f^* : QH^{2p^2+2p-1}(K(p);\mathbb{Z}_{(p)}) \longrightarrow QH^{2p^2+2p-1}(F;\mathbb{Z}_{(p)})$$

is a multiplication by p. We conclude the proof by dualization. Q.E.D.

THEOREM (3.6). *The spherical number of $K(p)$ at the exponent (p^2+p-1) is p^2; i.e.,*

$$S(K(p);p^2+p-1) = p^2.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}_{(p)} \cong \Pi_{2p^2+2p-1}(K(p);\mathbb{Z}_{(p)}) & \xrightarrow[h]{} & PH_{2p^2+2p-1}(K(p);\mathbb{Z}_{(p)})) \cong \mathbb{Z}_{(p)} \\
\Big\uparrow \cong & & \Big\uparrow f_* \\
Z_{(p)} \cong \Pi_{2p^2+2p-1}(F;\mathbb{Z}(p)) & \xrightarrow{h} & PH_{2p^2+2p-1}(F,\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}
\end{array}$$

By Corollary (3.4), $P^1(v_{2p^2+2p-1}) = v_{2p^2+1} \in H_*(F;\mathbb{Z}/p)$. Therefore the lower Hurewicz map is a multiplication by p. By Lemma (3.5), f_* is a multiplication by p, so

$$S(K(p);p^2+p-1) = p^2 \qquad \text{Q.E.D.}$$

Remarks. Looking at the cases $p = 3$ and $p = 5$ for $K(p)$ and using the results of Harper and Wilkerson about the splitting mod p of the exceptional Lie groups $\mathbb{F}_4$ and $\mathbb{E}_8$, we see that

$$\begin{aligned}
S(\mathbb{F}_4, 11)_{(3)} &= 3^2 \text{ (See example (c))};\\
S(\mathbb{E}_8, 29)_{(5)} &= 5^2;
\end{aligned}$$

i.e., the Hurewicz map of $\mathbb{E}_8$ is divisible exactly by 5^2 in dimension 59.

4. Primitivity classes and primitivity number

We recall that the bordism ring $\Pi_*(MU)$ rationally looks like

$$\begin{aligned}
\Pi_*(MU)\otimes\mathbb{Q} &\cong \mathbb{Q}[[\mathbb{C}P^1],\ldots,[\mathbb{C}P^n],\ldots].\\
&\cong \mathbb{Q}[m_1,m_2,\ldots,m_n,\ldots].\\
&\cong \mathbb{Q}[b_1,b_2,\ldots,b_n,\ldots]
\end{aligned}$$

where $[\mathbb{C}P^n] = (n+1)m_n$, $|m_n| = |[\mathbb{C}P^n]| = 2n$, and if $\log(t) = \sum_{i=0}^{\infty} m_i t^{i+1}$, then $\exp(t) = \sum_{i=0}^{\infty} b_i t^{i+1}$. For $E = (e_1,\ldots,e_n,\ldots)$, a sequence of non-negative integers almost all zero, let $|E| = 2\sum_{i\geq 0} ie_i$

$$\begin{aligned}
S_E &: MU^*(X) \longrightarrow MU^{*+|E|}(X) \text{ and}\\
S_E &: MU_*(X) \longrightarrow MU_{*-|E|}(X)
\end{aligned}$$

be the Landweber-Novikov operations acting on cohomology and homology, respectively, for X a finite complex.

For each $E = (e_1, \dots, e_n, \dots)$ as above, define the monomial

$$m^E = m_1^{e_1} \cdot m_2^{e_2} \cdots m_n^{e_n} \cdots \in \Pi_*(MU) \otimes \mathbb{Q}$$

and the total operations

$$\begin{aligned} \mathcal{P} &= \sum_E m^E S_E : MU^*(X) \otimes \mathbb{Q} \longrightarrow MU^*(X) \otimes \mathbb{Q}, \\ \mathcal{P}_* &= \sum_E m^E S_E : MU_*(X) \otimes \mathbb{Q} \longrightarrow MU_*(X) \otimes \mathbb{Q}, \end{aligned}$$

for which the following properties hold (see [3]):

1. $\mathcal{P}$ and $\mathcal{P}_*$ preserved degree;
2. for $y \in MU^*(X) \otimes \mathbb{Q}$ and $\alpha \in MU_*(X) \otimes \mathbb{Q}$, $\langle \mathcal{P}(y), \mathcal{P}_*(\alpha) \rangle = \langle y, \alpha \rangle$;
3. $\mathcal{P}$ is multiplicative, i.e., $\mathcal{P}(xy) = \mathcal{P}(x)\mathcal{P}(y)$ for x and y in $MU^*(X) \otimes \mathbb{Q}$;
4. $\mathcal{P}$ is idemponent, i.e., $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$.

Example. $X = \mathbb{C}P^\infty$, $\mathcal{P}(\tau) = \sum m_i \tau^{i+t}$ when $MU^*(\mathbb{C}P^\infty) = \Pi_{-*}(MU)[[t]]$.

Definition. $\alpha \in MU_*(X) \otimes \mathbb{Q}$ is *primitive* if all non-trivial MU-operations act trivially on α, i.e., $S_E(\alpha) = 0 \; \forall E \neq (0, 0, \dots)$.

$\mathcal{P}_*$ characterizes the (rational) primitive classes in the following way: $\alpha \in MU_*(X) \otimes \mathbb{Q}$ is primitive if and only if $\mathcal{P}_*(\alpha) = \alpha$, or equivalently, $\alpha \in \mathrm{Im}\mathcal{P}_*$. (See [3]).

Suppose from now on that X is a finite H-space. Then

$$MU_*(X) \otimes \mathbb{Q} \cong \wedge(X_{k_1}, X_{k_2}, \dots, X_{k_r}),$$

where the exterior algebra has $\Pi_*(MU) \otimes \mathbb{Q}$ as coefficients, and $|X_{k_i}| = 2k_i + 1$, and if

$$T : MU_*(X) \otimes \mathbb{Q} \longrightarrow H_*(X; \mathbb{Q})$$

denotes the Thom reduction map, then $T(X_{k_i}) = x_{2k_i+1}$ as in §2. Therefore, to calculate primitives in

$$MU_*(X)/\mathrm{Tor} \subset MU_*(X) \otimes \mathbb{Q}$$

for X a finite H-space, we have to solve a two phase-problem:

(1) Calculate $\mathrm{Im}\mathcal{P}$ and $\mathrm{Im}\mathcal{P}_*$;

(2) Integrality problem: find the smallest positive integer

$$N(X;k_i) \in \mathbb{Z}^+ \text{ such that } N(X;k_i)\mathcal{P}_*(X_{k_i}) \in MU_{2k_i+1}(X)/\mathrm{Tor}.$$

Definition. $N(X;k_i) \in \mathbb{Z}^+$ is the "primitivity number" for X at the exponent k_i.

The spherical and the primitivity numbers are closely related. In fact, we have that for all X and all k_i's $N(X;k_i)$ divides $S(X;k_i)$. This follows from the fact that the operations act trivially on $MU^*(S^n)$ for all $n > 0$, and the diagram

$$\begin{tikzcd} \Pi_*(X)/\mathrm{Tor} \arrow[rr, "h^{MU}"] \arrow[dr, "h"'] & & MU_*(X)/\mathrm{Tor} \arrow[dl, "T"] \\ & H_*(X;\mathbb{Z})/\mathrm{Tor} & \end{tikzcd}$$

commutes. Therefore the point is to answer the following question: given a finite H- space X, for what k_i's do we have $N(X;k_i) = S(X;k_i)$?

It is easy to see that for every X, $N(X;k_1) = S(X;k_1) = 1$.

Examples. (See [3])

(a) $X = SU(n+1)\ n \geq 1;\ \ N(X;k_i) = S(X;k_i) = k_i!$;

(b) $X = G_2;\ \ N(X;5) = 5!/2$ but $S(X;5) = 5!$;

(c) $X = \mathbb{F}_4$. $N(X;5) = 2^2{\cdot}5$ and $S(X;5) = 2^3{\cdot}5; N(X;7) = S(X;7) = 2^3{\cdot}3{\cdot}7$;

$$N(X;11) = S(X;11) = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11.$$

This last example gives us information about the situation of the torsion molecule at $p = 3$. Localizing at $p = 3$, $\mathbb{F}_4 \underset{3}{\simeq} B_5(3) \times K(3)$. Using this we can say that, at $p = 3$, $N(K(p);p^2+p-1)_{(p)} = S(K(p);p^2+p-1)_{(p)} = p^2$.

What we will prove now is that this equality is true for any odd prime p.

5. Primitivity number of $K(p)$.

The main goal of this section is to prove that

$$N(K(p);p^2+p-1)_{(p)} = p^2.$$

We are working localized at p, thus we will use BP (co)homology instead of MU (co)homology. Recall that $\Pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, $|v_n| = 2p^n - 2$ and

$$v_n = pm_n - \sum_{\substack{i+j=n \\ i,j<n}} v_i^{p^j} m_j$$

$$\Pi_*(BP) \otimes \mathbb{Q} \cong H_*(BP;\mathbb{Q}) \cong \mathbb{Q}[m_1, m_2, \ldots].$$

The m_i's in BP-theory are identified with the m_i's in MU-theory under the Quillen idempotent in dimensions $2p-2$, $2p^2-2, \ldots, 2p^4-2, \ldots$. Therefore we redefine the total operation $\mathcal{P}_* : BP_*(X) \otimes \mathbb{Q} \longrightarrow BP_*(X) \otimes \mathbb{Q}$ as $\mathcal{P}_* = \sum m^E S_E$, where $E = (e_1, \ldots, e_n, \ldots)$ as above, $m^E = m_1^{e_1} \cdot m_2^{e_2} \ldots$ and S_E is the conjugate of r_E, the Quillen operation.

An extra feature of this total operation is that it ties in nicely with the Steenrod module structure of $H_*(X;\mathbb{Z}/p)$; more precisely,

$$\begin{array}{ccc} BP_*(X) & \longrightarrow & BP_*(X) \\ \Big\downarrow{\scriptstyle \rho\circ T} & & \Big\downarrow{\scriptstyle \rho\circ T} \\ H_*(X,\mathbb{Z}/p) & \longrightarrow & H_*(X;\mathbb{Z}/p) \end{array}$$

commutes, where $T : BP_*(X) \to H_*(X;\mathbb{Z}_{(p)})$ is the Thom reduction map, ρ is reduction modulo p, and P^E is the Milnor Steenrod operation of E.

PROPOSITION (5.1). $H_*(\Omega K(p);\mathbb{Z}/p) \cong \mathbb{Z}/p\,[a_2, a_{2p}, a_{2p^2+2p-2}]\,/\,(a_2^p)$, $P^1(a_{2p}) = a_2$.

Proof. Consider the Eilenberg–Moore spectral sequence

$$E_2 = \mathrm{Tor}_{H^*(K(p);\mathbb{Z}/p)}(\mathbb{Z}/p, \mathbb{Z}/p) \Rightarrow H^*(\Omega K(p);\mathbb{Z}/p).$$

Calculating, $E_2 = \Gamma(sx_3) \otimes \Gamma(sx_{2p+1}) \otimes \Lambda(sx_{2p+2}) \otimes \Gamma[tx_{2p+2}]$, where bideg $(sx_i) = (-1, i)$ $i = 3,\ 2p+1,\ 2p+2$, we have

$$\text{bideg } (tx_{2p+2}) = (-2, 2p^2+2).$$

Now $d_2 = \beta P^1$, so $d_2(\gamma_2(sx_3)) = sx_{2p+2}$ and d_2 is trivial in all other cases. Thus

$$E_3 \cong \Gamma[sx_3]/(\gamma(sx_3) = 0; i \geq 3) \otimes \Gamma[sx_{2p+1}] \otimes \Gamma[tx_{2p+2}].$$

Clearly $dr = 0$ for $r \geq 3$ and $E_3 = E_\infty$. By duality

$$H_*(\Omega K(p);\mathbb{Z}/p) = \mathbb{Z}/p[a_2, a_{2p}, a_{2p^2+2p-2}]/(a_2^p),$$

where duality goes as follows: sx_3 ——— a_2, sx_{2p} ——— a_{2p} and tx_{2p+2} —— a_{2p^2+2p-2} and finally $P^1(x_3) = x_{2p+1}$ implies $P^1(a_{2p}) = a_2$.
Q.E.D.

Now R. Kane in [2] (page 354) gives a method to calculate, for X a finite H-space, $BP_*(\Omega X)$ from $H_*(\Omega X;\mathbb{Z}/p)$. Using this we have $BP_*(\Omega X) \cong \Pi_*(BP) = [\alpha_2, \alpha_{2p}, \alpha_{2p^2+2p-2}]/J$ where J is the ideal generated by the relation $R = \alpha_2^p - p\alpha_{2p} + v_1\alpha_2$, where $v_1 = pm_1$.

Now if Ω denotes the loop map, that is, Ω is induced from the identity map under the identifications $[X,X] \cong [\Omega X, \Omega X] \cong [\Sigma\Omega X, X]$, then $\Omega_* Q\, BP_*(\Omega X) \longrightarrow P\, BP_*(X)$ has degree $+1$ and $\operatorname{Im}\Omega_*$ generates the diagonal primitives of $BP_*(X)$. Thus

$$P\,BP_*(K(p)) = \wedge(X_3, X_{2p+1}, X_{2p^2+2p-1})/(pX_{2p+1} - v_1X_3).$$

Remark. The image of $h^{BP} : \Pi_*(K(p)) \longrightarrow BP_*(K(p))$ lies inside the diagonal primitives. It is enough for our purposes to calculate the diagonal primitives.

We use the indirect aproach of Kane, instead of the direct one, i.e., calculate $BP_*(K(p))$ and then identify the diagonal primitive generators there, in order to make this paper less unreadable. In any case, performing such calculation, our result is:

$$\begin{aligned} BP_*(K(p)) &\cong \Pi_*(BP)\{1, X_{2p^2+2p-1}, X_{2p^2+2p+2}\} \otimes \\ &\otimes \Pi_*(BP)\{X_3, X_{2p+1}\}/(v_1X_3 = pX_{2p+1}) \otimes \\ &\otimes \Pi_*(BP)/(p, v_1)\{X_{2p+4}, X_{4p+6}, \ldots, X_{2p^2}\}, \end{aligned}$$

where, following Yagita's approach as in [6], $\{\ldots\}$ means the free-module generated by.... We mention this as it was included in an early preprint of this paper. Now we return to our main topic.

Note that if we define the BP primitivity number in the same way, this agrees with the localization at p of the MU-version of the primitivity number.

Notation. $N = N(K(p); p^2 + p - 1)_{(p)}$.

PROPOSITION (5.2). *p divides N and N divides p^2.*

Proof. The second assertion follows from the fact that every spherical class is a primitive class. Now P^1 relates $H_{2p^2+2p-1}(K(p);\mathbb{Z}/p)$ with $H_{2p^2+1}(K(p);\mathbb{Z}/p)$. From the diagram above we deduce that for $E = (1, 0, \ldots)$, $S_E(X_{2p^2+2p-1}) = X_{2p^2+1} \in BP_*(K(p))$.

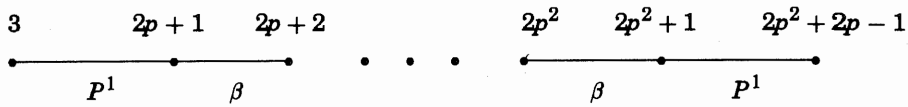
Hence $\mathcal{P}_* = \sum_E m^E S_E$, $\mathcal{P}_*(X_{2p^2+2p-1})$ has a non-trivial summand

$$m^E S_E(X_{2p^2+2p-1}) = m_1 X_{2p^2+1}.$$

Thus, $v_1 = pm_1$, and by definition of primitivity number, p divides N. Q.E.D.

Now we come to the most subtle and delicate part of the argument. Before we go into the formal proof, we will say something about what is going on behind this argument.

The torsion molecule has a cell decomposition as follows:



The Steenrod algebra module structure of $H^*(X;\mathbb{Z}/p)$ does *not* tell us if the lower part (left in the diagram) is attached essentially to the upper part (right in the diagram). This is one of the reasons that we go to a richer (co)homology theory (*BP* in this case) but up to this point of the argument we are not able to answer this question. Therefore we use the important fact that the torsion molecule is indeed an H-space.

Let $\Sigma K(p)$ denote the suspension of $K(p)$. Suppose that $\Sigma K(p) \cong A \cup J$ where A is the part of $K(p)$ represented in the diagram suspended once, and $A = A_1 \cup A_2$; i.e., $A_1 = S^4 \cup e^{2p+2} \cup e^{2p+3}$, $A_2 = e^{2p^2+1} \cup e^{2p^2+2} \cup e^{2p^2+2p}$, $J =$ junk.

LEMMA (5.3). (J. Harper). *In $A = A_1 \cup A_2$, A_1 and A_2 are attached non-trivially.*

Proof. Suppose that $A = A_1 \vee A_2$ i.e. the attaching is trivial. Let $r = 2p^2 + 2p - 1$. Hence $K(p)$ rationally looks like a product of two spheres,

$$K(p) \underset{0}{\simeq} S^3 \times S^r \ \text{ then } \ \Sigma K(p) \underset{0}{\simeq} S^4 \vee S^{r+1} \vee S^{r+4}.$$

Thus $A_2 \underset{0}{\simeq} S^{r+1}$, and if $pt : \Sigma K(p) \longrightarrow A_2$ denotes the pinching map, then

$$pt_* : \Pi_{r+1}(\Sigma K(p)) \otimes \mathbb{Q} \longrightarrow \Pi_{r+1}(A_2) \otimes \mathbb{Q}$$

is an isomorphism.

On the other hand, the suspension map

$$\sigma_* : H_r(K(p))/\mathrm{Tor} \longrightarrow H_{r+1}(\Sigma K(p))/\mathrm{Tor}$$

is mono and splits ($K(p)$ is an H-space). Therefore the Hurewicz map

$$h : \Pi_{r+1}(\Sigma K(p)) \longrightarrow H_{r+1}(\Sigma K(p))$$

is multiplication by p^2 and consequently the Hurewicz map

$$h : \Pi_{r+1}(A_2)/\mathrm{Tor} \longrightarrow H_{r+1}(A_2)/\mathrm{Tor}$$

is multiplication by p, which is a contradiction. Q.E.D.

By the calculations in §3 on the (co)homology of F, the third connective cover of $K(p)$, we can now deduce that the generator in $H_{2p^2}(K(p);\mathbb{Z})$ is spherical.

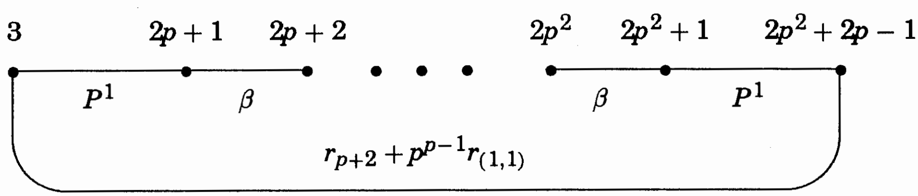
By duality ($K(p)$ is self-dual) and the calculation of $PBP_*(K(p))$, we conclude that the cell in dimension $2p^2+2p-1$ is linked to the 3-dimensional one; i.e., there exists a non-trivial map in $\Pi^S_{2p^2+2p-5}$ that represents this homotopy class.

But α_t for $t = p+2$ generates $\Pi^S_{2p^2+2p-5}$ and by Smith-Zahler [5], α_t is detected by a primary BP-operation, namely $r_{p+2} + p^{p-1}r_{(1,1)}$.

THEOREM (5.4). *The BP primitivity number for $K(p)$ in dimension $2p^2+2p-1$ is p^2.*

Proof. We know that $p|N$ and $N|p^2$. The possibility $p = N$ cannot occur, since there exists a BP-primary operation linking the 3 and the $2p^2+2p-1$ generators in $PBP_*(K(p))$. Therefore $N = p^2$. Q.E.D.

The complete picture of $K(p)$ is



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