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HOMOTOPY HOMOMORPHISMS AND THE HAMMOCK LOCALIZATION

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Dedicated to the memory of José Adem

1. Introduction

In the theory of A_∞ or E_∞ monoids, rings and modules, or in the study of diagrams of spaces homomorphisms as structure preserving maps are often too rigid: they are not homotopy invariant. For example, if one changes a homomorphism by a homotopy one obtains a homomorphism up to coherent homotopy, called h-morphism for short. If $f : X \to Y$ is a homotopy equivalence and X has an A_∞ or E_∞ monoid or ring structure, then Y admits an A_∞ or E_∞ monoid or ring structure making f into an h-morphism but not a homomorphism. In the E_∞ monoid case the addition is an h-morphism but again not a homomorphism.

Although h-morphisms seem to be the correct notion of morphisms in homotopy coherence theory, they have draw-backs: composition is defined only up to homotopy. Fortunately, composition is homotopy associative with canonical identities. Hence there is a perfectly good homotopy category. The description of naturality properties of constructions such as homotopy limits and colimits of diagrams or topological Hochschild homology or algebraic K-theory of A_∞ or E_∞ rings is rather involved unless one passes to the homotopy category.

This passage reduces the spaces of h-morphisms to their path components, thus depriving homotopy coherence algebra of its appropriate Hom-sets. In view of the work of Bökstedt [2], Waldhausen [17], and, in particular, Robinson [8], this is a real loss of information. Hence we are led to analyze the structure of the collection of all h-morphisms before passage to homotopy. It forms what we will call a Δ-category, a category-like structure which can be interpreted as a category up to coherent homotopy. To be precise, any small Δ-category can be rectified to a homotopy equivalent honest topological category with discrete space of objects, and the rectification is a "functor" of Δ-categories up to coherent homotopy.

In the first part of this paper we develop the necessary theory of Δ-categories, give some examples from homotopy coherence theory, and prove the rectification result.

The description of a homotopy invariant Hom-space by h-morphisms —though it arises naturally— is still unsatisfactory from the view point of formal homotopy theory. Diagrams, A_∞ or E_∞ monoids and rings and homomorphisms form closed simplicial model categories in the sense of Quillen,

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whose localizations with respect to the weak equivalences are isomorphic to the homotopy categories of h-morphisms. This indicates a relationship of the spaces of h-morphisms with the simplicial Hom-sets of the hammock localization of closed model categories introduced by Dwyer and Kan [3]. They associate with each closed model category a simplicial category such that the path components of the simplicial Hom-sets coincide with Quillen's localization and hence, in our examples, with the homotopy classes of h-morphisms.

The second part of the paper deals with the relationship between h-morphisms and hammocks. Up to homotopy they are two sides of the same coin: the space of h-morphisms is equivalent to the simplicial set of hammocks in the strongest sense one possibly can expect. There is a sequence of maps of Δ-categories and genuine simplicial functors of simplicial categories joining the Δ-category of h-morphisms with the simplicial category of hammocks, and each map is a weak homotopy equivalence.

2. Homotopy coherent diagrams

Throughout this paper we work in the category $\mathcal{T}op$ of compactly generated spaces in the sense of [16].

(2.1). A *topological category* $\mathcal{C}$ is a topologically enriched small category, i.e. its morphism spaces are topologized and composition is continuous. If its space of objects is discrete, we call it a *topological index category*. A $\mathcal{C}$*-diagram* is a continuous functor $D : \mathcal{C} \to \mathcal{T}op$, where $\mathcal{C}$ is a topological index category. A natural transformation $f : D_1 \to D_2$ is called a *homomorphism*. Its *underlying map* is the collection of maps $f(A) : D_1(A) \to D_2(A), A \in ob\ \mathcal{C}$. If all the $f(A)$ are h-equivalences (homotopy equivalences), we call f a *weak equivalence*. The category of $\mathcal{C}$-diagrams is denoted by $\mathcal{T}op^{\mathcal{C}}$.

In the theory of homotopy limits and colimits, but also in classical homotopy theory, one encounters situations where one has to substitute a given $\mathcal{C}$-diagram by a homotopy invariant modification, a homotopy coherent $\mathcal{C}$-diagram. Such diagrams are codified by a "homotopy resolution" of $\mathcal{C}$.

CONSTRUCTION (2.2). [1]. *Let $\mathcal{C}at$ denote the category of topological indexing categories and continuous functors. We construct a functor*

$$W : \mathcal{C}at \to \mathcal{C}at$$

and a natural transformation $\varepsilon : W \to Id$ as follows:

$$ob\, W\mathcal{C} = ob\ \mathcal{C}$$

$$W\mathcal{C}(A, B) = \left(\coprod_{n \geq 0} \mathcal{C}_{n+1}(A, B) \times [0,1]^n \right) / \sim$$

where $\mathcal{C}_{n+1}(A, B)$ is the space of composible morphisms

$$A \xrightarrow{f_0} \cdot \xrightarrow{f_1} \cdot \xrightarrow{f_2} \cdots \to \cdot \xrightarrow{f_n} B$$

with the relations

$$\begin{aligned}(f_n, t_n, \ldots, f_0) &= (f_n, t_n, \ldots, f_i \circ f_{i-1}, \ldots, f_0) & \textit{if } t_i = 0 \\ &= (f_n, \ldots, f_{i+1}, max(t_{i+1}, t_i), f_{i-1}, \ldots, f_0) & \textit{if } f_i = id\end{aligned}$$

($t_i \in [0,1]$, $f_i \in morC$; if $f_n = id$ or $f_0 = id$, delete f_n, t_n resp. t_1, f_0). Composition is

$$(g_m, u_m, \ldots, g_0) \circ (f_n, t_n, \ldots, f_0) = (g_m, u_m, \ldots, g_0, 1, f_n, t_n, \ldots, f_0).$$

The natural transformation ε is defined by

$$\varepsilon(f_n, t_n, \ldots, f_0) = f_n \circ \cdots \circ f_0.$$

It is an h-equivalence on morphism spaces: there is a non-functorial section

$$s : C \to WC, \quad f \mapsto (f)$$

and $H_t(f_n, t_n, \ldots, t_1, f_0) = (f_n, t \cdot t_n, \ldots, t \cdot t_1, f_0)$ deforms WC into this section.

The notion of homotopy resolution is justified by following result.

PROPOSITION (2.3). [1, (3.17)]. *Given a diagram of topological index categories*

$$\begin{array}{ccc} WC & & \mathcal{E} \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle G} \\ C & \xrightarrow{\quad F \quad} & \mathcal{D} \end{array}$$

and continuous functors such that G is an h-equivalence on morphism spaces, there exists a continuous functor $H : WC \to \mathcal{E}$ such that $F \circ \varepsilon \simeq G \circ H$ (homotopic through functors). Moreover H is unique up to homotopy (through functors).

Definition. (2.4) A *homotopy coherent C-diagram* is a WC-diagram.

The homotopy invariance of homotopy coherent C-diagrams is a consequence of (2.3). For its formulation we need the notion of a homotopy homomorphism.

Let $\mathcal{L}_n$ denote the linear category

$$0 \to 1 \to 2 \to \cdots \to n.$$

An order preserving map $\alpha : [k] \to [n]$, where $[n] = \{0, 1, \ldots, n\}$, induces a functor $\mathcal{L}_k \to \mathcal{L}_n$ and hence a functor $\alpha : W(C \times \mathcal{L}_k) \to W(C \times \mathcal{L}_n)$. As usually, we denote the n order preserving injections $[n-1] \to [n]$ and surjections $[n+1] \to [n]$ by δ^i resp. σ^i.

Definition. (2.5). An *h-morphism (homotopy homomorphism) $D_0 \to D_1$* of homotopy coherent C-diagrams is a continuous functor

$$\alpha : W(C \times \mathcal{L}_1) \to \mathcal{T}op$$

such that $\alpha\circ\delta^0 = D_1$ and $\alpha\circ\delta^1 = D_0$. The collection of maps

$$\alpha((id_A, 0 \to 1)) : D_0(A) \to D_1(A), \qquad A \in ob\ \mathcal{C}$$

is called the *underlying map* of α.

The correspondence $((f_n, j_n), t_n, \dots, (f_0, j_0)) \mapsto ((f_n, t_n, \dots, f_0), j_n\circ\cdots\circ j_0)$ defines a functor $W(\mathcal{C} \times \mathcal{L}_1) \to W\mathcal{C} \times \mathcal{L}_1$. Hence any homomorphism of $W\mathcal{C}$-diagrams is an h-morphism in a canonial way. While it is clear how to compose an h-morphism with a homomorphism on either side it is far from clear how to compose h-morphisms. To define this composition consider the simplical class $\mathcal{K}\mathcal{T}op^{\mathcal{C}}$ whose n-simplices are continuous functors

$$W(\mathcal{C} \times \mathcal{L}_n) \to \mathcal{T}op.$$

PROPOSITION (2.6). [1, (4.9)]. *$\mathcal{K}\mathcal{T}op^{\mathcal{C}}$ satisfies the restricted Kan extension condition, i.e. a horn can be filled in provided the missing face is not the first or the last one.*

(2.7). Given h-morphisms $\alpha : D_0 \to D_1$ and $\beta : D_1 \to D_2$ there is a 2-simplex τ with $\tau\circ\delta^0 = \beta, \tau\circ\delta^2 = \alpha$. We take $\gamma = \tau\circ\delta^1$ as a composite of α and β. Then γ is uniquely determined up to homotopy [1, (4.12), (4.13)], and composition is homotopy associative with the identical homomorphisms as units. Hence we obtain a homotopy category $\pi\mathcal{K}\mathcal{T}op^{\mathcal{C}}$.

(2.8). Let $\mathcal{C}$ be a topological index category and $\{X_C; C \in ob\ \mathcal{C}\}$ a set of spaces. We topologize the set of continuous functors $F : \mathcal{C} \to \mathcal{T}op$ satisfying $F(C) = X_C$ with the subspace topology of

$$\prod_{A,B} \mathcal{T}op(\mathcal{C}(A, B), \mathcal{T}op(F(A), F(B))).$$

In particular, the set $\mathcal{K}\mathcal{T}op^{\mathcal{C}}(D_0, D_1)$ of h-morphisms $D_0 \to D_1$ obtains a topology. More generally, if S is a set of homotopy coherent $\mathcal{C}$-diagrams and $\mathcal{K}_S\mathcal{T}op^{\mathcal{C}}$ is the simplicial subset of $\mathcal{K}\mathcal{T}op^{\mathcal{C}}$ of all simplices having vertices in S, then $\mathcal{K}_S\mathcal{T}op^{\mathcal{C}}$ has the structure of a simplicial space.

Let $\mathcal{V}_k \subset W(\mathcal{C}\times\mathcal{L}_n)$ denote the subcategory generated by all faces except of the k-th face, $0 < k < n$. Then (2.6) is proved by constructing a deformation retraction functor $W(\mathcal{C} \times \mathcal{L}_n) \to \mathcal{V}_k$. Now let $\pi_i : [1] \to [n]$ be the order preserving map sending 0 to $i-1$ and 1 to i. Let $\mathcal{W} \subset W(\mathcal{C} \times \mathcal{L}_n)$ be the subcategory generated by all $\pi_i(W(\mathcal{C} \times \mathcal{L}_1)), i = 1, \dots, n, n \geq 2$. Applying the previous remark inductively, we obtain a deformation retraction functor

$$W(\mathcal{C} \times \mathcal{L}_n) \longrightarrow \mathcal{W}.$$

PROPOSITION (2.9). *$\pi_1, \dots, \pi_n$ induce an h-equivalence of $(\mathcal{K}_S\mathcal{T}op^{\mathcal{C}})_n$ with the space of strings of h-morphisms*

$$D_0 \to D_1 \to \cdots \to D_n$$

with $D_i \in S$.

This result exhibits $\mathcal{K}_S \mathcal{T}op^{\mathcal{C}}$ as an example of what we will call a Δ-category (see (3.3)).

(2.10). Homotopy invariance. [1, (4.18), (4.19)]. Let $\mathcal{I}_s$ denote the category

$$0 \underset{q}{\overset{p}{\rightleftarrows}} 1$$

consisting of two isomorphic objects. Let D_0 be a homotopy coherent $\mathcal{C}$-diagram.

(1) Given h-equivalences $\alpha(A) : D_0(A) \longrightarrow D_1(A)$, one for each $A \in ob\ \mathcal{C}$, then D_0 and the $\alpha(A)$ extend to a continuous functor

$$\alpha : W(\mathcal{C} \times \mathcal{I}_s) \longrightarrow \mathcal{T}op.$$

(2) Given an h-morphism $\beta : D_0 \to D_1$ of homotopy coherent C-diagrams, whose underlying map consists of h-equivalences, then β extends to a continuous functor

$$\alpha : W(\mathcal{C} \times \mathcal{I}_s) \longrightarrow \mathcal{T}op.$$

Let $u, v : \mathcal{L}_1 \to \mathcal{I}_s$ be given by $u(0) = 0, u(1) = 1, v(0) = 1, v(1) = 0$. Then $\beta = \alpha \circ u$. Clearly $\alpha \circ v$ is a homotopy inverse of β.

(2.11). Rectification of h-morphisms: There is an endofunctor

$$U : \mathcal{T}op^{W\mathcal{C}} \longrightarrow \mathcal{T}op^{W\mathcal{C}}.$$

together with a natural weak equivalence $r : U \to Id$ and h-morphisms $\eta_D : D \to UD$ having the following properties

(1) η_D is natural with respect to homomorphisms $f : D \to E$ of $W\mathcal{C}$-diagrams, i.e. $Uf \circ \eta_D = \eta_E \circ f$

(2) $\eta_D(A) : D(A) \to UD(A)$ embeds $D(A)$ as a strong deformation retract

(3) $r_D \circ \eta_D = id$ as h-morphisms

(4) any h-morphism $\alpha : D \to E$ factors uniquely as $\alpha = \bar{\alpha} \circ \eta_D$ with $\bar{\alpha} : UD \to E$ a homomorphism of $W\mathcal{C}$-diagrams

(5) The map $\mathcal{K}\mathcal{T}op^{W\mathcal{C}}(D, E) \to \mathcal{T}op^{\mathcal{C}}(UD, E), \alpha \mapsto \bar{\alpha}$ is continuous.

The composites are always meant to be the canonical composites of an h-morphism with a homomorphism of WC-diagrams.

Proof: We confine ourselves to the definitions. For a detailed proof see [1, (4.43)].

$$UD(A) = \left(\coprod_B W(\mathcal{C} \times \mathcal{L}_1)\left((B,0),(A,1)\right) \times D(B) \right) / \sim$$

with the relations $(g \circ f, x) \sim (g, D(f)(x))$ for $f \in \delta^1 W\mathcal{C}$. The $W\mathcal{C}$-structure is defined by

$$UD(h)(g, x) = (h \circ g, x)$$

for $h : A \to A'$. The h-morphism η_D is given by the adjoints of the maps

$$W(\mathcal{C} \times \mathcal{L}_1)((B,0),(A,1)) \times D(B) \longrightarrow UD(A),$$

and if $\alpha : W(\mathcal{C} \times \mathcal{L}_1) \to \mathcal{T}op$ is an h-morphism $D \to E$, its adjoints

$$W(\mathcal{C} \times \mathcal{L}_1)((B,0),(A,1)) \times D(B) \to E(A)$$

define the $W\mathcal{C}$-homomorphisms $\bar{\alpha}$. If $\alpha = id_D$, then $\bar{\alpha} : UD \to D$ is the natural map r_D. □

Hence any h-morphism α can be "decomposed" canonically into a broken arrow diagram

$$D \xleftarrow{r_D} UD \xrightarrow{\bar{\alpha}} E$$

where the map into the wrong direction is a weak equivalence. This suggests that there should be a connection with the localization of $\mathcal{T}op^{W\mathcal{C}}$ given by inverting the weak equivalences:

PROPOSITION (2.12). [1, p. 140 ff]: *Let $\mathcal{W}$ be the class of weak equivalences in $\mathcal{T}op^{W\mathcal{C}}$, then*

$$\mathcal{T}op^{W\mathcal{C}}[\mathcal{W}^{-1}] \cong \pi\mathcal{K}\mathcal{T}op^{\mathcal{C}}$$

3. Δ-categories

Let Δ be the category of finite ordered sets $[n]$ and order preserving maps. Let X be a simplicial space, i.e. a functor $X : \Delta^{op} \to \mathcal{T}op$. We define

$$(3.1)\quad \begin{aligned}(RLX)_n &= \{(x_1, \ldots, x_n) \in (X_1)^n; d^1 x_{i+1} = d^0 x_i, i = 1, \ldots, n-1\} \quad n \geq 1\\ (RLX)_0 &= X_0\end{aligned}$$

i.e. $(RLX)_n$ is an iterated pullback $X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

Let $\pi_i : [1] \to [n]$ be given by $\pi_i(k) = i - 1 + k$. We have the following result:

LEMMA (3.2). *A simplicial space X is the nerve of a category object in $\mathcal{T}op$ iff*

$$(\pi_1, \ldots, \pi_n) : X_n \longrightarrow (RLX)_n$$

is a homeomorphism for all $n \geq 2$.

Proof. If X is the nerve of a topological category then $X_n = (RLX)_n$. Conversely,

$$d^1 \circ (\pi_1, \pi_2)^{-1} : (RLX)_2 \to X_2 \to X_1$$

defines an associative composition with identities $s^0 : X_0 \to X_1$. □

We weaken (3.2) up to homotopy and define

Definition (3.3). A *special* Δ*-space* is a simplicial space X such that $(\pi_1, \ldots, \pi_n) : X_n \to (RLX)_n$ is an h-equivalence for all $n \geq 2$. A Δ*-category* is a special Δ-space for which X_0 is discrete. A Δ*-functor* is a simplicial map between Δ-categories. Let $\Delta\mathcal{C}at$ denote the category of Δ-categories.

Example (3.4). Let Y be a topological space and SY its topologized singular complex, where $S_nY = \mathcal{T}op(\Delta^n, Y)$ has the function space topology. SY is a special Δ-space.

Remark (3.5). Special Δ-spaces with $X_0 \simeq *$ are of importance in loop space theory [13, (1.5)].

The passage from Δ-categories to the hammock localization, mentioned in the introduction, is given by Δ-functors up to coherent homotopies, which we will now introduce. Any simplicial space X and hence any Δ-category is canonically a homotopy coherent Δ^{op}-diagram via

$$W\Delta^{op} \xrightarrow{\varepsilon} \Delta^{op} \xrightarrow{X} \mathcal{T}op.$$

Definition (3.6). An $h\Delta$*-map* between simplicial spaces is an h-morphism of the associated $W\Delta^{op}$-diagrams.

As in § 2 Δ-categories and $h\Delta$-maps extend to a simplicial class $\mathcal{K}\Delta\mathcal{C}at$. Let S be a set of Δ-categories and $\mathcal{K}_S\Delta\mathcal{C}at$ the restriction of $\mathcal{K}\Delta\mathcal{C}at$ to simplices with vertices in S, then (2.6) and (2.9) give

PROPOSITION (3.7). *$\mathcal{K}\Delta\mathcal{C}at$ satisfies the restricted Kan extension condition and $\mathcal{K}_S\Delta\mathcal{C}at$ is a Δ-category.*

(3.8). Homotopy invariance: If X is a simplicial space, then

$$Gr(X) : X_1 \underset{d^0}{\overset{d^1}{\rightrightarrows}} X_0$$

is a directed topological graph. A simplicial map $f : X \to Y$ defines an *underlying map of graphs* $Gr(f) : Gr(X) \to Gr(Y)$. Similarly if $\alpha : X \to Y$ is an $h\Delta$-map we have underlying maps $\alpha_i : X_i \to Y_i, i = 0, 1$ which commute with d^i up to homotopy. We call (α_0, α_1) the *underlying h-map of graphs*. Given an $h\Delta$-map $\alpha : X \to Y$ of Δ-categories for which α_0, α_1 are h-equivalences, then α_0, α_1 extend to h-equivalences $(RLX)_n \to (RLY)_n$, because X_0 and Y_0 are discrete. Since X and Y are special, (2.10.2) implies:

PROPOSITION (3.9). *Given an $h\Delta$-map $\alpha : X \to Y$ of Δ-categories whose underlying h-map of graphs (α_0, α_1) consists of h-equivalences, then α extends to a continuous functor $W(\Delta^{op} \times \mathcal{I}s) \to \mathcal{T}op$.*

Remarks (3.10): (1) Given a Δ-category X and a map of graphs $Gr(f)$: $Gr(X) \to Y$ which consists of h-equivalences we cannot expect to extend $Gr(f)$

and X to an $h\Delta$-map of Δ-categories. The analogue of (2.10.1) extends Y to a $W\Delta^{op}$-space which need not factor through $\varepsilon : W\Delta^{op} \to \Delta^{op}$ and hence need not define a Δ-category.

(2) For special Δ-spaces (3.9) holds, provided d^0 or d^1 is an h-fibration.

(3.11). Rectification of $h\Delta$-maps: There is an endofunctor

$$M : \Delta Cat \longrightarrow \Delta Cat$$

together with a natural transformation $r : M \to Id$ and $h\Delta$-maps $\eta_X : X \to MX$ such that

(1) η_X is natural with respect to Δ-functors
(2) $(\eta_X)_n : X_n \to (MX)_n$ embeds X_n as a strong deformation retract
(3) $r_X \circ \eta_X = id_X$ as $h\Delta$-maps
(4) any $h\Delta$-map $\alpha : X \to Y$ of Δ-categories factors uniquely as $\alpha = \bar{\alpha} \circ \eta_X$ with $\bar{\alpha} : MX \to Y$ a Δ-functor.
(5) The map $\mathcal{K}\Delta Cat(X, Y) \to \Delta Cat(X, Y), \alpha \mapsto \bar{\alpha}$ is continuous.

The functor U of (2.11) will not do since it produces a $W\Delta^{op}$-diagram from a Δ-category. We need a modification.

Define a special Δ-space NX by

$$(NX)_n = \left(\coprod_k W(\Delta^{op} \times \mathcal{L}_1)(([k], 0), ([n], 1)) \times X_k \right) / \sim$$

with the relations $(h \circ g \circ f, x) \sim (s\varepsilon(h) \circ g, X(\varepsilon(f))(x))$ in the notation of (2.2) for $h \in \delta^0 W\Delta^{op}$ and $f \in \delta^1 W\Delta^{op}$. Now proceed as in the proof of (2.11) to show that NX has all the required properties apart from the fact that $(NX)_0$ is not discrete (but $(NX)_0 \simeq X_0$). For the proof of (2) consult [10, (4.5)].

Let N_0X denote the constant simplicial space on $(NX)_0$. Define MX to be the pushout

$$\begin{array}{ccc} N_0X & \longrightarrow & NX \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & MX \end{array}$$

Then $NX \to MX$ is a weak equivalence in the sense of (2.1) because $N_0X \to NX$ is a cofibration. Since Δ-categories have a discrete space of 0-simplices the passage from NX to MX preserves properties (1),...,(5). □

As in section 2 we conclude

(3.12). An $h\Delta$-map $\alpha : X \to Y$ of Δ-categories "decomposes" canonically into a broken arrow diagram of Δ-functors

$$X \xleftarrow{\;r_X\;} MX \xrightarrow{\;\bar{\alpha}\;} Y$$

where the Δ-functor into the wrong direction is a weak equivalence.

4. Rectification of Δ-categories

The following is an adaptation of methods of May [6] to our situation. Let $J \subset \Delta$ denote the subcategory of all maps $f : [m] \to [n]$ of the form $f(i) = f(0) + i$ for all $i \in [m]$. If $J_1 \subset J$ is the full subcategory with objects [0] and [1], then $\mathcal{T}op^{J_1^{op}}$ is the category $\mathcal{G}raph$ of directed topological graphs. The obvious forgetful functor

$$L : \mathcal{T}op^{J^{op}} \longrightarrow \mathcal{G}raph$$

has a right adjoint

$$R : \mathcal{G}raph \longrightarrow \mathcal{T}op^{J^{op}}$$

given by the iterated pullback (3.1). The following is obvious.

LEMMA (4.1). *$L \circ R = Id$ and the adjunction unit $\delta : Id \to R \circ L$ is given by the maps of* (3.2)

$$\delta_n = (\pi_1, \ldots, \pi_n) : Y_n \to (RLY)_n$$

The forgetful functor $R_1 : \mathcal{T}op^{\Delta^{op}} \to \mathcal{T}op^{J^{op}}$ has a left adjoint

$$L_1 : \mathcal{T}op^{J^{op}} \to \mathcal{T}op^{\Delta^{op}}, \qquad Y \mapsto Y \otimes_J \Delta = \left(\coprod_k Y_k \times \Delta([n],[k]) \right) / \sim$$

with the relations $(Y(g)(y), f) \sim (y, g \circ f)$ for $y \in Y_k, g \in J([m],[k])$, and $f \in \Delta([n],[m])$. The simplicial structure is given by composition on the right. The composite $D = R_1 \circ L_1$ is a monad on $\mathcal{T}op^{J^{op}}$.

(4.2). Analysis of DY: Let $\Delta_* \subset \Delta$ be the subcategory of all morphisms $g : [m] \to [n]$ with $g(0) = 0$ and $g(m) = n$. A morphism $f : [n] \to [k]$ in Δ factors uniquely as

$$f : [n] \xrightarrow{f_1} [q] \xrightarrow{f_2} [k]$$

with $f_2 \in J$ and $f_1 \in \Delta_*$. Note that $q = f(n) - f(0)$. Hence

$$(DY)_n = \coprod_k Y_k \times \Delta_*([n],[k]).$$

In particular,

$$(DY)_0 = Y_0 \quad , \quad (DY)_1 = \coprod_k Y_k.$$

The unit of D is

$$(\eta_Y)_n : Y_n \longrightarrow \coprod_k Y_k \times \Delta_*([n],[k]), \quad y \mapsto (y, id).$$

(4.3) Δ_* is a strictly monoidal category with $[k] \oplus [l] = [k+l]$ and $f \oplus g : [k] \oplus [l] \to [m] \oplus [n]$ defined by

$$(f \oplus g)(i) = \begin{cases} f(i) & i \leq k \\ m + g(i-k) & i > k \end{cases}$$

Any $f \in \Delta_*([n],[k])$ is uniquely of the form $f = f_1 \oplus \cdots \oplus f_n$ with $f_i : [1] \to [f(i) - f(i-1)]$.

LEMMA (4.4). *$\delta DR : DR \to RLDR$ is a natural isomorphism of functors $\mathcal{G}raph \to \mathcal{T}op^{\mathcal{J}^{op}}$.*

Proof. Let $Y = RZ, Z \in \mathcal{G}raph$. Think of $(y_1, \ldots, y_k) \in Y_k$ as a diagram

$$x_0 \xrightarrow{y_1} x_1 \xrightarrow{y_2} \cdots \xrightarrow{y_n} x_k$$

with $x_i \in Y_0$ and $y_i \in Y_1$. For $((y_1, \ldots, y_k), f_1 \oplus \cdots \oplus f_n) \in Y_k \times \Delta_*([n],[k]) \subset DY_n$ we have

$$\pi_i((y_1, \ldots, y_k), f_1 \oplus \cdots \oplus f_n) = (x_{f(i-1)} \to x_{f(i-1)+1} \to \cdots \to x_{f(i)}, f_i).$$

The statement follows. □

Let $\mathcal{J}\mathcal{T}op \subset \mathcal{T}op^{\mathcal{J}^{op}}$ denote the full subcategory of all Y for which Y_0 is discrete and $\delta_Y : Y \to RLY$ is a weak equivalence in the sense of (2.1). Note that $R_1X \in \mathcal{J}\mathcal{T}op$ for a Δ-category X. From (4.2) we obtain

LEMMA (4.5). *D preserves weak equivalences in $\mathcal{T}op^{\mathcal{J}^{op}}$ and RL weak equivalences in $\mathcal{J}\mathcal{T}op$.*

Consider the commutative diagram in $\mathcal{T}op^{\mathcal{J}^{op}}$

$$\text{(4.6)} \qquad \begin{array}{ccc} DY & \xrightarrow{\delta_{DY}} & RLDY \\ \downarrow{\scriptstyle D\delta_Y} & & \downarrow{\scriptstyle RLD\delta_Y} \\ DRLY & \xrightarrow{\delta_{DRLY}} & RLDRLY \end{array}$$

If $Y \in \mathcal{J}\mathcal{T}op$ then δ_{DRLY} is an isomorphism (4.4) and $D\delta_Y, RLD\delta_Y$ are weak equivalences (4.5). Hence δ_{DY} is a weak equivalence. Since $(DY)_0 = Y_0$, we obtain

LEMMA (4.7). *D defines a monad on $\mathcal{J}\mathcal{T}op$.*

By [6, (5.2), (5.3), (5.5)] these observations imply

PROPOSITION (4.8). *Let $\mathcal{G}raph_0 \subset \mathcal{G}raph$ be the full subcategory of all graphs X with discrete X_0. Then*
(1) *$C = LDR$ is a monad on $\mathcal{G}raph_0$*
(2) *$\bar{\delta} : D \xrightarrow{\delta D} RLD \xrightarrow{RLD\delta} RLDRL = RCL$ is a morphism of monads*

(3) *If $Y \in \mathcal{JT}op$ is a D-object, there is a natural diagram*

$$Y_* \xleftarrow{\varepsilon_*} B_*(D, D, Y) \xrightarrow{\bar{\delta}_*} RB_*(CL, D, Y)$$

of simplicial D-objects. $B_*(\ )$ is the two-sided bar construction, Y_* the constant simplicial D-object on Y, and ε_* a simplicial h-equivalence in $\mathcal{JT}op$.

We apply this result to our problem: the D-objects in $\mathcal{JT}op$ are exactly the Δ-categories, the D-objects in $\mathcal{JT}op$ of the form RZ are exactly the topological categories. Since $(DY)_0 = Y_0$, the simplicial object $(B_*(D, D, Y))_0$ is constant on Y_0. Since $(RLY)_0 = Y_0$, the same holds for $RB_*(CL, D, Y)$. Since the topological realization functor preserves small limits, R commutes with topological realization, and we obtain a diagram of D-objects in $\mathcal{JT}op$

$$Y \xleftarrow{|\varepsilon_*|} |B_*(D, D, Y)| \xrightarrow{|\bar{\delta}_*|} R|B_*(CL, D, Y)|.$$

Since the degeneracies in $B_*(D, D, Y)$ and $RB_*(CL, D, Y)$ are inclusions of topological summands and hence cofibrations, (4.6) implies that $\bar{\delta}$ of (4.8.2) is a weak equivalence. Consequently the D-maps $|\varepsilon_*|$ and $|\bar{\delta}_*|$ are still weak equivalences in the sense of (2.1).

We summarize:

THEOREM (4.9). *There are functors $\mathcal{M}, \mathcal{C} : \Delta\mathcal{C}at \to \Delta\mathcal{C}at$ together with natural transformations $\varepsilon : \mathcal{M} \to Id, \delta : \mathcal{M} \to \mathcal{C}$ which are weak equivalences, such that $\mathcal{C}Y$ is a topological index category:*

$$Y \underset{\simeq}{\xleftarrow{\varepsilon_Y}} \mathcal{M}Y \underset{\simeq}{\xrightarrow{\delta_Y}} \mathcal{C}Y.$$

Inverting ε_Y according to (3.9) *we obtain an $h\Delta$-map $Y \to \mathcal{C}Y$ which is a homotopy equivalence.* (All maps are the identity on 0-simplices).

Remark (4.10). There is another rectification process which passes from Δ-categories to A_∞ categories, i.e. A_∞ monoids in $\mathcal{G}raph_O$, and then to topological categories. A theory of A_∞ categories can be developed along the lines of Sections 2 and 3 using [1].

5. The Δ-categories of A_∞ or E_∞ monoids and rings

The structures of A_∞ and E_∞ monoids and rings are codified by appropriate topologized algebraic theories (see [12, §2]). Among these theories are universal ones obtained as follows: we start with a canonical theory Θ_U which is the theory Θ_m of monoids in the A_∞ monoid case, the theory $\Theta(\mathcal{Q}_\infty)$ associated with the little cubes *PROP* $\mathcal{Q}_\infty$ of [1, (2.49)] in the E_∞ monoid case, and the theories $\Theta(\mathcal{H}_\infty, \mathcal{L})$ associated with the CW-approximation of Steiner's canonical operad pair $(\mathcal{H}_\infty, \mathcal{L})$ [14], [15] in the ring cases. In the A_∞ ring case we forget the action of the symmetric groups on $\mathcal{L}$, in the E_∞ ring case this action is part of the structure. We then apply the theory version of Construction 2.2 [1, p. 72 ff] to Θ_U to arrive at a universal theory: each A_∞ or E_∞ monoid or ring can be structured by $W\Theta_U$ [1, (3.17), (6.31)].

We now proceed as in Section 2 with $W\mathcal{C}$ replaced by $W\Theta_U$: A_∞, E_∞ monoids or rings are product preserving continuous functors $X : W\Theta_U \to \mathcal{T}op$, $X(1)$ is the *underlying space* of X, a *homomorphism* is a natural transformation of such functors. A *weak equivalence* is a homomorphism $f : X \to Y$ whose *underlying map* $f(1)$ is a homotopy equivalence. Denote the resulting categories by $\mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ and $\mathcal{R}\mathcal{E}_\infty$.

Let $\mathcal{M}$ denote any of these categories. h-morphisms between objects in $\mathcal{M}$ can be defined as in (2.5) with $\mathcal{C} \times \mathcal{L}_1$ replaced by $\Theta_U \diamond \mathcal{L}_1$, where $\Theta_U \diamond \mathcal{L}_n$ is the quotient of $\Theta_U \times \mathcal{L}_n$ obtained by identifying the objects $(0, i), i \in \mathcal{L}_n$, to a single terminal object. As in Section 2 the h-morphisms extend to a simplicial class $\mathcal{K}\mathcal{M}$, and we have

PROPOSITION (5.1). *$\mathcal{K}\mathcal{M}$ satisfies the restricted Kan extension condition for $\mathcal{M} = \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ and $\mathcal{R}\mathcal{E}_\infty$.*

The proof is a refinement of [1, (4.9)] and [9, (3.2)], which also provides

PROPOSITION (5.2). *If S is a set of objects in $\mathcal{M} = \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ or $\mathcal{R}\mathcal{R}_\infty$, then $\mathcal{K}_S\mathcal{M}$ is a Δ-category.*

(5.1) allows the construction of the homotopy category $\pi\mathcal{K}\mathcal{M}$ of homotopy classes of h-morphisms.

Given an h-morphism $\alpha : X \to Y$ of objects in $\mathcal{M}$, we call

$$\alpha((id_1, 0 \to 1)) : X(1) \longrightarrow Y(1)$$

the *underlying map* of α.

(5.3). Homotopy invariance [1, (4.18),(4.19)], [9, (4.4),(4.6)]: Let $\mathcal{M} = \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ or $\mathcal{R}\mathcal{E}_\infty$

(1) Let X be in $\mathcal{M}$ and $f : X(1) \to Y$ be an h-equivalence in $\mathcal{T}op$. Then there is a $\bar{Y} \in \mathcal{M}$ with underlying space Y and a continuous product preserving functor

$$\alpha : W(\Theta_U \diamond \mathcal{I}s) \to \mathcal{T}op$$

extending $X, \bar{Y}$ and f.

(2) Any h-morphism $\alpha : X \to Y$ of objects in $\mathcal{M}$, whose underlying map is an h-equivalence in $\mathcal{T}op$, extends to a continuous product preserving functor

$$W(\Theta_U \diamond \mathcal{I}s) \to \mathcal{T}op.$$

PROPOSITION (5.4). *The rectification result (2.11) holds in $\mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ and $\mathcal{R}\mathcal{E}_\infty$.*

Proofs are given in [1, (4.43)] and [9, (4.12)]. The rectification process in [9] is a variant of our functor N of (3.11). The necessary changes are easily made.

COROLLARY (5.5). *Let $\mathcal{W} \subset \mathcal{M}$ be the class of weak equivalences. Then* $\mathcal{M}[\mathcal{W}^{-1}] \cong \pi\mathcal{K}\mathcal{M}$.

6. Comparison with the hammock localization

Let $\mathcal{M}$ be a model category in the sense of Quillen [7] and $\mathcal{W}$ its subcategory of weak equivalences. To avoid confusion we call them model equivalences. Dwyer and Kan associated with $\mathcal{M}$ its hammock localization $L^H\mathcal{M}$ with respect to $\mathcal{W}$ [3].

(6.1). The hammock localization: *ob* $L^H\mathcal{M}$ = *ob* $\mathcal{M}$ and $L^H\mathcal{M}(X,Y)$ is a simplicial class whose k-simplices are hammocks of arbitrary length and width k. A *hammock* in $L^H\mathcal{M}(X,Y)$ of length n and width k is a commutative diagram in $\mathcal{M}$

$$
\begin{array}{ccccccccc}
 & Z_{0,1} & \text{———} & Z_{0,2} & \text{—}\cdots\text{—} & Z_{0,n} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & Z_{1,1} & \text{———} & Z_{1,2} & \text{—}\cdots\text{—} & Z_{1,n} & \\
X & \downarrow & & \downarrow & & \downarrow & Y \\
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & Z_{k,1} & \text{———} & Z_{k,2} & \text{—}\cdots\text{—} & Z_{k,n} &
\end{array}
$$

such that

(1) all vertical maps are model equivalences

(2) in each column all horizontal maps go in the same direction. If they go to the left, they are model equivalences

(3) the maps in two adjacent columns of horizontal maps point in opposite directions

(4) no column of horizontal maps contains only identities.

The i-th face operator omits the i-th row and the i-th degeneracy repeats it. Composition is the obvious one. Possibly one has to reduce the resulting diagram to a hammock by composing arrows and deleting identities.

Although $L^H\mathcal{M}(X,Y)$ is a simplicial class it is homotopically small in the following sense [3, (2.2), (4.1)]: $L^H\mathcal{M}(X,Y)$ contains a simplicial subset U such that for each simplicial subset V containing U there is a simplicial subset W containing V for which the inclusion $U \to W$ is a homotopy equivalence after realization. In particular, $L^H\mathcal{M}(X,Y)$ has a well-defined homotopy type, and one has [3, (4.2),(4.7)]:

PROPOSITION (6.2). (1) *If $\mathcal{M}$ is a model category, then* $\pi_0 L^H\mathcal{M} \cong \mathcal{M}[\mathcal{W}^{-1}]$.

(2) *If $\mathcal{M}_*$ is a closed simplicial model category, $X \in \mathcal{M}_*$ is cofibrant and $Y \in \mathcal{M}_*$ is fibrant, then $\mathcal{M}_*(X,Y)$ has the same homotopy type as $L^H\mathcal{M}(X,Y)$.*

In (6.2.2), $L^H\mathcal{M}(X,Y)$ extends to a bisimplicial structure $L^H\mathcal{M}_*(X,Y)$ with one direction from $\mathcal{M}_*$, and the homotopy equivalence is given by the obvious functors [3, (4.8)]

$$\mathcal{M}_* \longrightarrow \operatorname{diag} L^H\mathcal{M}_* \longleftarrow L^H\mathcal{M} \tag{6.3}$$

By [4] and [11], the categories $\mathcal{T}op^{\mathcal{C}}, \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty, \mathcal{R}\mathcal{E}_\infty$ of Section 2 and 5 form closed simplicial model categories. Their model equivalences respectively fibrations are homomorphisms whose underlying maps are weak h-equivalences in $\mathcal{T}op$ resp. Serre fibrations. In particular, all objects are fibrant. By (2.8) these categories have topologized morphism sets and the simplicial structure is obtained by passing to their singular complexes.

Constructions such as homotopy limits and colimits, classifying spaces, algebraic K-theory [12], [17] or topological Hochschild homology [2] of homotopy ring spaces obviously define functors on $L^H\mathcal{M}$. A comparison of the Δ-categories of Sections 2 and 5 with the corresponding categories of hammocks hence describes the functoriality of these constructions with respect to h-morphisms. This comparison is established in the following two theorems.

THEOREM (6.4). *Let $\mathcal{M} = \mathcal{T}op^{\mathcal{C}}, \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty$ or $\mathcal{R}\mathcal{E}_\infty$ and let $S \subset ob\ \mathcal{M}$ be a subset. Let $T = \{UX; X \in S\} \subset ob\ \mathcal{M}$ and let $\mathcal{M}_T \subset \mathcal{M}$ be the full subcategory of objects in T. Then there is an $h\Delta$-map*

$$\mathcal{K}_S\mathcal{M} \to \mathcal{M}_T$$

whose underlying map consists of h-equivalences.

(6.5). Let us call two Δ-categories $\mathcal{A}$ and $\mathcal{B}$ *equivalent* if there is an $h\Delta$-map $\mathcal{A} \to \mathcal{B}$ whose underlying map consists of h-equivalences. Then $\mathcal{A}$ and $\mathcal{B}$ are equivalent iff there are Δ-functors

$$\mathcal{A} \longleftarrow \mathcal{C} \longrightarrow \mathcal{B}$$

which are weak equivalences (3.9),(3.11).

$L^H\mathcal{M}(X,Y)$ is a set for small $\mathcal{M}$ but only homotopically small in general. Hence only the notion of equivalence of the second part of (6.5) makes sense: Two simplicial categories $\mathcal{A}, \mathcal{B}$ with possibly only homotopically small simplicial morphism classes are called *equivalent* if there is a string of simplicial functors

$$\mathcal{A} \longrightarrow \bullet \longleftarrow \quad \cdots \quad \bullet \longrightarrow \mathcal{B}$$

preserving the homotopy types of the morphism classes.

THEOREM (6.6). *Let $\mathcal{M} = \mathcal{M}\mathcal{A}_\infty, \mathcal{M}\mathcal{E}_\infty, \mathcal{R}\mathcal{A}_\infty, \mathcal{R}\mathcal{E}_\infty$ or $\mathcal{T}op^{\mathcal{C}}$ with mor $\mathcal{C}$ of the homotopy type of a CW-complex. Let $S \subset ob\ \mathcal{M}$ be a set of objects of the* *homotopy type of a CW-complex and* $T = \{UX; X \in S\}$. *Then the simplicial categories* $\mathcal{M}_T$ *and* $L^H_S\mathcal{M}$ *are equivalent, where* $\mathcal{M}_T$ *and* $L^H_S\mathcal{M}$ *are the full subcategories of* $\mathcal{M}$ *and* $L^H\mathcal{M}$ *of objects in* T *respectively in* S.

The model equivalences in $\mathcal{M}$ are homomorphisms whose underlying maps are weak h-equivalences in $\mathcal{T}op$. Equivalences in $\mathcal{K}\mathcal{M}$ come from h-morphisms whose underlying maps are honest h-equivalences. Hence the comparison (6.6) can only hold on subcategories where weak h-equivalences and honest ones of the underlying spaces agree ($\mathcal{M} = \mathcal{T}op^{\mathcal{L}_0}$, $\mathcal{L}_0$ the trivial category, is a counter example).

Proof of (6.4). We construct a sequence of Δ-categories and Δ-functors which are weak equivalences

$$\mathcal{K}_S\mathcal{M} \xleftarrow{f_0} \mathcal{N} \xrightarrow{f_1} \mathcal{K}_T\mathcal{M} \xleftarrow{f_2} \mathcal{M}_T.$$

The Δ-category $\mathcal{N}$ has product preserving continuous functors

$$\rho_X : W(\Theta_U \diamond I_s) \to \mathcal{T}op$$

as vertices, where $\rho_X|\delta^0 W\Theta_U = X$ and $\rho_X|\delta^1 W\Theta_U = UX$, one for each $X \in S$. The existence of ρ_X is guaranteed by the rectification process U. The n-simplices of $\mathcal{N}$ are product preserving continuous functors

$$W(\Theta_U \diamond I_s \diamond \mathcal{L}_n) \to \mathcal{T}op$$

extending the given functors on vertices. As in the other examples $\mathcal{N}$ is a Δ-category.

The Δ-functors f_0 and f_1 are induced by the two inclusions $\mathcal{L}_n \to I_s \times \mathcal{L}_n$.

LEMMA (6.7). *f_0 and f_1 are weak equivalences.*

Proof. By symmetry it suffices to prove this for f_0. Since f_0 is a Δ-functor of Δ-categories it suffices to show that it is an h-equivalence on 1-simplices. Let $\mathcal{V} \subset W(\Theta_U \diamond I_s \diamond \mathcal{L}_1)$ be generated under composition and taking products by $\delta^0 W(\Theta_U \diamond \mathcal{L}_1)$ and the subcategories $W(\Theta_U \diamond I_s \diamond \{i\}), i = 0, 1$. An h-morphism $\sigma : W(\Theta_U \diamond \mathcal{L}_1) \to \mathcal{T}op$ from X to Y with $X, Y \in S$ defines a continuous product preserving functor $G(\sigma) : \mathcal{V} \to \mathcal{T}op$

$$\begin{array}{ccc} UX & & UY \\ \downarrow\uparrow \rho_X & & \downarrow\uparrow \rho_Y \\ X & \xrightarrow{\sigma} & Y \end{array}$$

There is a deformation retraction $F : W(\Theta_U \diamond I_s \diamond \mathcal{L}_1) \to \mathcal{V}$ through product preserving functors [1, (4.18)], [9, (4.8)]. Hence we obtain a continuous section of f_0

$$s_0 : (\mathcal{K}_S\mathcal{M})_1 \longrightarrow \mathcal{N}_1 \quad , \quad \sigma \mapsto G(\sigma) \circ F.$$

Since F is a deformation retraction, $s_0 \circ f_0 \simeq id$. □

LEMMA (6.8). *The canonical Δ-functor $f = f_2 : \mathcal{M}_T \to \mathcal{K}_T\mathcal{M}$ is a weak equivalence.*

For the proof we need

LEMMA (6.9). *A weak equivalence $h : UX \to UY$ in $\mathcal{M}$,* i.e. *a homomorphism with underlying map an h-equivalence, has a homotopy inverse in $\mathcal{M}$.*

Proof: Let $\pi\mathcal{M}$ denote the category of homotopy classes of homomorphisms in $\mathcal{M}$. The rectification construction U defines a functor

$$\bar{U} : \pi\mathcal{K}\mathcal{M} \longrightarrow \pi\mathcal{M}$$

For a class $[\alpha] \in \pi\mathcal{K}\mathcal{M}(X,Y)$, $\bar{U}[\alpha]$ is defined as follows: Take any composite β

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow \eta_X & \beta & \downarrow \eta_Y \\ UX & \xrightarrow{\bar{\beta}} & UY \end{array}$$

of α and η_Y, then $\bar{U}[\alpha]$ is represented by $\bar{\beta}$, induced from β by the universal property of η_X. Since β is unique up to homotopy, so is $\bar{\beta}$.

Let $\pi f : \pi\mathcal{M} \to \pi\mathcal{K}\mathcal{M}$ be induced by the Δ-functor of (6.8) and let

$$g \quad = \quad r_Y \circ h \circ \eta_X \quad : \quad X \to UX \to UY \to Y$$

be the canonical composite of the h-morphism η_X with the homomorphism $r_Y \circ h$. By homotopy invariance, $[g]$ is an isomorphism in $\pi\mathcal{K}\mathcal{M}$. Hence $\bar{U}[g]$ is an isomorphism in $\pi\mathcal{M}$. Since $\pi f[r_Y]$ is inverse to $[\eta_Y]$, $h \circ \eta_X$ is a composite of η_Y and g. Hence $\bar{U}[g]$ is represented by $\overline{h \circ \eta_X} = h$. □

Proof of (6.8): The composite

$$\mathcal{M}(UX,UY) \xrightarrow{\ f\ } \mathcal{K}\mathcal{M}(UX,UY) \cong \mathcal{M}(UUX,UY)$$

sends $h : UX \to UY$ to $\bar{h} : UUX \to UY$ defined by $\bar{h} \circ \eta_{UX} = h$. Since h is a homomorphism, (2.11) implies that $\bar{h} = r_{UY} \circ Uh = h \circ r_{UX}$. By (6.9) r_{UX} is a homotopy equivalence in $\mathcal{M}$. Hence $f : \mathcal{M}(UX,UY) \to \mathcal{K}\mathcal{M}(UX,UY)$ is a homotopy equivalence. □

Proof of (6.6). We work in the ambient categories $\mathcal{M}$ and $L^H\mathcal{M}$. We have simplicial functors

$$\mathcal{M}_{*T} \longrightarrow \operatorname{diag} L^H_T\mathcal{M}_* \longleftarrow L^H_T\mathcal{M} \xleftarrow{\ f\ } L^H_S\mathcal{M}$$

where the left two arrows are given by (6.3) and f is induced by the functor U, which preserves model equivalences. By [5, (3.3), (3.5)]

$$f : L^H\mathcal{M}(X,Y) \longrightarrow L^H\mathcal{M}(UX,UY)$$

is a weak homotopy equivalence. Since all objects in $\mathcal{M}$ are fibrant the equivalence of $\mathcal{M}_{*T}$ and $L_T^H\mathcal{M}$ follows from (6.2) and

LEMMA (6.10). *If $X \in \mathcal{M}$ has the homotopy type of a CW-complex UX is homotopically cofibrant, i.e. UX is homotopy equivalent in $\mathcal{M}$ to a cofibrant object.*

Proof. Let $\emptyset$ be the initial object of $\mathcal{M}$. Decompose $\emptyset \to X$ into a cofibration and a trivial model fibration $q : QX \to X$. Then QX is cofibrant. Let R be the CW-approximation and $\alpha_X : RX \to X$ the associated h-morphism described in the appendix (for simplicity we use RX for F^*RX). Consider

$$RX \xleftarrow{Rq} RQX \xrightarrow{\alpha_{QX}} QX.$$

Since q is a model equivalence Rq is a weak equivalence in the sense of (2.1) or Section 5. Hence there is an h-morphism $RX \to RQX$ inverse to Rq up to homotopy, and we can choose a composite $\beta : RX \to QX$. Rectification gives a homomorphism $\bar{\beta} : URX \to QX$. Since the underlying map of β consists of weak h-equivalences, $\bar{\beta}$ is a model equivalence. If $\gamma : \mathcal{M} \to Ho\ \mathcal{M}$ denotes Quillen's localization functor [7, (I.1.11)], $\gamma(\bar{\beta})$ is an isomorpism. But QX is cofibrant and URX fibrant. Hence [7, (I.1.16)]

$$Ho\ \mathcal{M}(QX, URX) \cong \pi\mathcal{M}(QX, URX).$$

We can lift the inverse of $\gamma(\bar{\beta})$ to a homomorphism $g : QX \to URX$. Since $\gamma(\bar{\beta}\circ g) = id_{QX}$ and $Ho\ \mathcal{M}(QX, QX) \cong \pi\mathcal{M}(QX, QX)$, $\bar{\beta}\circ g$ is strictly simplicially homotopic to id in the sense of [7, (II.2.5)]. Since URX is of the homotopy type of a CW-complex $g\circ\bar{\beta}$ is a weak equivalence in $\mathcal{M}$ and hence an h-equivalence by (6.9). Consequently $\bar{\beta}$ is an h-equivalence in $\mathcal{M}$. The h-morphism α_X induces a homomorphism $URX \to UX$ whose underlying map is an h-equivalence. So URX and hence QX are homotopy equivalent to UX in $\mathcal{M}$ by (6.9). □

Appendix: CW-approximations

Let $R : \mathcal{T}op \to \mathcal{T}op$ be the composite of the singular functor and the topological realization functor, and let $r : R \to Id$ denote the associated adjunction counit. Let Θ be a theory or a topological index category $\mathcal{C}$ and $X : W\Theta \to \mathcal{T}op$ a $W\Theta$-space. Since R preserves products we obtain a $RW\Theta$-space RX defined by the adjoints of

$$RW\Theta(n,k) \times (RX)^n \cong R(W\Theta(n,k) \times X^n) \to R(X^k) \cong (RX)^k,$$

if Θ is a theory and correspondingly for $\mathcal{C}$. Note that $RW\Theta$ is an A_∞ resp. E_∞ monoid or ring theory if Θ is one and that $r_\Theta : RW\Theta \to W\Theta$ is a theory functor. Moreover, r induces a homomorphism

$$r_X : RX \to r_\Theta^* X = X \circ r_\Theta$$

of $RW\Theta$-spaces whose underlying map is a weak h-equivalence in $\mathcal{T}op$. If Θ is one of the universal theories Θ_U of Section 5, there is a theory functor

$$F : W\Theta \longrightarrow RW\Theta,$$

unique up to homotopy through functors, such that $r_\Theta \circ F \simeq Id$ through functors. If $mor\ \mathcal{C}$ has the homotopy type of a CW-complex the same holds for $\Theta = \mathcal{C}$. In particular, we have a homomorphism of $W\Theta$-spaces

$$r_X : F^* RX \longrightarrow F^* r_\Theta^* X$$

whose underlying map is a weak h-equivalence.

By [1, (6.23)] (in the ring cases this result also holds by application of [9, (2.17)]) the homotopy $r_\Theta \circ F \simeq Id$ provides an h-morphism $\beta : F^* r_\Theta^* X \to X$ with id_X as underlying map. Consequently, we have an h-morphism $\alpha_X : F^* RX \to X$ of $W\Theta$-spaces, whose underlying map is r_X.

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