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THE INDEX OF A VECTOR FIELD UNDER BLOW UPS

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A la memoria de José Adem, con admiración y respeto

A useful technique when studying the behaviour of holomorphic vector fields around their isolated singularities is that of blowing up the singular points. (See for instance [5].) On the other hand, the most basic invariant of a vector field with isolated singularities is its local index, as defined by Poincaré and Hopf [16]. It is thus natural to ask how does the index of a vector field behave under blowing ups ? In this work we study and answer this question, by taking a rather general point of view and bearing in mind that complex manifolds have a powerful birational invariant, the Todd genus [9]. The method we use lies within the framework of algebraic topology. Since the Todd genus is invariant under blow ups on almost complex manifolds [10, p.51], our method actually applies to smooth vector fields on almost complex manifolds, so we work in this category.

The idea is the following: suppose we are given a continuous vector field X on a compact manifold W with boundary M, such that X has no singularities on M and only finitely many singularities in the interior of W. Morse [17] showed that the total index of X in W, $\mathrm{Ind}(X;W)$, is determined by the behaviour of X near M, regardless of what happens in the interior of W. If we keep our boundary data $(M, X|_M)$ but we replace the interior of W by some other manifold $\widetilde{W}$ with boundary M, what information about $\mathrm{Ind}(X;W)$ can we read out from $\widetilde{W}$? To answer this question, let us impose some restrictions: we assume W is even dimensional and it has trivial complex tangent bundle; we also assume $\widetilde{W}$ is almost complex and the U-structure on its boundary M is compatible with the one induced from W. If X is as above, its complex orthogonal complement in TW is not necessarily trivial [20], though TW is trivial. However, we can use the techniques of [12,14] to define (in §1 below) a trivialization $P(X)$ of $TW|_M$, whose degree is canonically determined by the index of X in W. Moreover, all the Chern classes of W relative to $P(X)$ vanish, except the top one which is given by the degree of $P(X)$, so by the index of X. Hence, up to a constant multiple, we can think of $\mathrm{Ind}(X;W)$ as being the Todd genus of W relative to $P(X)$. (See Lemma (2.2) below.) Thus we can use the theorems of [1,9,10] for the Todd genus to say something about the index . Using this we prove:

THEOREM (1). *With the above hypotheses and notation:*
a) *If the (real) dimension of* W *is* $4k$, *then*

$$\mathrm{Ind}(X;W) \equiv \mathrm{Ind}(X;\widetilde{W}) + \frac{(-1)^{k-1}\cdot(2k)!}{B_k}\cdot Td^*_{2k}[\widetilde{W}] \qquad \mathrm{mod}\,\frac{(2k)!}{B_k}$$

where $\mathcal{B}_k$ is the k^{th} Bernoulli number and $Td^*_{2k}[\widetilde{W}]$ is a fixed contribution of $\widetilde{W}$, independent of X: It is the part of the Todd polynomial Td_{2k} involving lower Chern classes c_i, $i = 1, \dots, 2k-1$, of $\widetilde{W}$ relative to a trivialization of $T\widetilde{W}|_M$, evaluated on the orientation cycle of $\widetilde{W}$.

b) *If $\pi : \widetilde{W} \to W$ is the composition of a finite number of blow ups away from M, then*

$$\operatorname{Ind}(X; W) = \operatorname{Ind}(X; \widetilde{W}) + \frac{(-1)^{k-1} \cdot (2k)!}{\mathcal{B}_k} \cdot Td^*_k[W].$$

We recall [9] that the Bernoulli numbers are the coefficients $\mathcal{B}_k$ in the power series

$$Q(z) = \sqrt{z}(\tan h\sqrt{z})^{-1} = 1 + \sum_{i=1} (-1)^{k-1} \cdot \frac{2^{2k}}{(2k!)} \mathcal{B}_k \quad z^k.$$

In Theorem (1), by blow ups we mean almost complex blow ups along submanifolds [10;p. 51]. In fact, the same conclusion holds if we perform blowing downs too, but the manifolds in question must be non-singular. When W has real dimension $4k + 2$ our method does not throw any information about the index. Instead, what the method gives in these dimensions is that the Todd genus is an obstruction for $\widetilde{W}$ to be "birationally equivalent" to a manifold with trivial tangent bundle (see §2). It is worth mentioning that the Chern numbers involved in the polynomial $Td^*_k[\widetilde{W}]$ coincide with the Chern numbers introduced by Looijenga in [15].

In §3 we specialize the discussion to the case of a single blow up at the origin in $\mathbb{C}^n$, where explicit computations are easy. Similar, but more complicated, formuli hold when we make several blow ups at points.

Finally, in §4 we prove (see the text for the definitions involved):

THEOREM (2). *Let W be a compact, almost complex $2n$-manifold, $n > 1$, with boundary M. Let X be a continuous vector field, defined and non-singular on a neighbourhood of M in W. Then :*

i) *If $\mathcal{D} = \mathcal{D}_X$ is a distribution on W by complex lines, with isolated singularities and extending the distribution defined by X near M, then*

$$\operatorname{Ind}(X; W) = \mu(\mathcal{D}; W) + (c_1(\mathcal{D}, X) \cdot c_{n-1}(N\mathcal{D}))[W],$$

independently of the choice of distribution $\mathcal{D}$, where $\mu(\mathcal{D}; \mathcal{W})$ is the sum of the local indices of $\mathcal{D}$ at the singular points, $c_1(\mathcal{D}, \mathcal{X})$ is the Chern class of $\mathcal{D}$ relative to X and $c_{n-1}(N\mathcal{D})$ is the top Chern class of the normal bundle of $\mathcal{D}$.

ii) *Let $\pi : \widetilde{W} \to W$ be as in Theorem (1.b), with $n = 2k$, then*

$$\operatorname{Ind}(X; W) = \mu(\mathcal{D}; \widetilde{W}) + (c_1(D, X) \cdot c_{n-1}(N\mathcal{D}))[\widetilde{W}] + Td^*_n[\widetilde{W}],$$

where D is a distribution on $\widetilde{W}$ *with isolated singularities and extending the distribution defined by* $\pi^*(X)$.

The above integer $\mu(\mathcal{D}; W)$ is essentially the $\mathbb{Z}$-index of a 2-distribution, introduced by E. Thomas in [21] and our Theorem (2) is indeed a refinement of his Theorem (1.3). The first part of Theorem (2) is an extension to manifolds with boundary of a theorem in [8]. If, in particular, the vector field X can be extended to the interior of W being contained in $\mathcal{D}$, then $c_1(\mathcal{D}, X) = 0$ and Theorem (1.i) becomes the Poincaré–Hopf index theorem for manifolds with boundary [16]. The second statement in Theorem (2) is a corollary of the first statement and Theorem (1).

Theorem (2) applies in particular to the case when W is a neighbourhood of 0 in $\mathbb{C}^{2n}$, X is the germ of a holomorphic vector field in $\mathbb{C}^{2n}$ with an isolated singularity at 0 and π is an analytic morphism. In this situation (c.f. [5].), if the strict transform $\mathcal{F}$ of the holomorphic foliation on $\widetilde{W}$ defined by $\pi^*(X)$ has isolated singularities, then Theorem (2) says ,

$$\mu(X) = \mu(\mathcal{F}; \widetilde{W}) + (c_1(\mathcal{F}) \cdot c_{2n-1}(N\mathcal{F}))[\widetilde{W}] + Td^*_{2n}[\widetilde{W}],$$

where $\mu(X)$ is the Milnor number of X, because that number equals the local index [4,8]. Explicit computations can be done using the techniques of §3. Examples of vector fields satisfying these hypotheses are provided by the absolutely isolated singularities of [5]. When n is 1, every holomorphic vector field with an isolated singularity satisfies that the strict transform of its induced foliation on $\widetilde{W}$ has isolated singularities.

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1. The trivialization $\mathcal{P}(X)$

Let W be a compact, $2k$-manifold with boundary M and trivial (complex) tangent bundle TW. We let X be a vector field, defined and non-singular on a neighbourhood of M in W. Since TW is trivial, there exists an immersion $\mathcal{I} : W \to \mathbb{R}^{2k}$, by the theorem of Hirsch-Poenarú [18]. Let

$$\mathcal{I}_* : TW \to T\mathbb{R}^{2k} = \mathbb{R}^{2k} \times \mathbb{R}^{2k},$$

be its derivative.

LEMMA (1.1). *The map* $\mathcal{G}(X) : M \to S^{2k-1}$ *defined by*

$$\mathcal{G}(X)(m) = \frac{\mathcal{I}_*(X(m))}{||\mathcal{I}_*(X(m))||}, \qquad m \in M,$$

has degree $r = \operatorname{Ind}(X; W)$.

Proof. We can assume X has only one singular point x_0 in the interior of W and we take $\mathcal{I}$ to be an embedding near x_0. So $\mathcal{G}$ extends to W minus a disc D_ε around x_0; The degree of $\mathcal{G}$ on the boundary $\partial(W - D_\varepsilon)$ is 0, because $\mathcal{G}$ is defined everywhere, but $\partial(W - D_\varepsilon)$ is M union a small sphere around x_0, with the orientation coming from W. Hence the lemma.

By Bott's computations [3] one has $\pi_{2k-1}(U(k)) \equiv \mathbb{Z}$, let j_k be a smooth representative of the usual generator. Then $j_k^0\mathcal{G}(X) : M \to U(k)$ determines a linear transformation on each fibre of $TW|_M$. If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a complex trivialization of TW, we can twist α over M using the map $j_k^0\mathcal{G}(X)$, as in [12]. The result is a new trivialization of $TW|_M$ that we denote by $\mathcal{P}(X) = (p_1, \dots, p_k)$. Each section $p_1, \dots, p_k$ can be regarded as a non-singular vector field on a neighbourhood of M in W; they all have the same total index in W, and this index is *the degree of* $\mathcal{P}(X)$.

Consider the long exact sequence ,

$$
\begin{array}{ccccccccc}
\dots \to & \pi_{2k-1}(U(k)) & \xrightarrow{p_*} & \pi_{2k-1}\left(S^{2k-1}\right) & \to & \pi_{2k-2}(U(k-1)) & \to & \pi_{2k-2}(U(k) & \to \dots \\
& \Big\downarrow \| & & \Big\downarrow \| & & \Big\downarrow \| & & \Big\downarrow \| & \\
& \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(k \cdot 1)! & & 0 &
\end{array}
$$

By Bott's computations, these groups are as stated above, and therefore the map p_* is multiplication by $(k-1)!$. Hence $\mathcal{P}(X)$ has degree $r \cdot (k-1)!$ in W.

We recall [11] that given an almost complex manifold N with boundary M and a trivialization β of $TN|_M$, we have Chern classes of N relative to β, $c_i(N;\beta) \in H^{2i}(N,M)$, $i = 1, \dots, k$. They map to the usual Chern classes, but as relative classes they depend on the choice of β, generally speaking. If the components of β are $\beta_1, \dots, \beta_k$ then the top relative Chern class is the obstruction for extending one of the β_i's to the interior of N. When N is the above manifold W, lemma 1.1 implies $c_k(W, \mathcal{P}(X))[W] = r \cdot (k-1)!$, where [W] is the orientation cycle. Also, since $\mathcal{G}$ extends to W minus a disc, the lower Chern classes of W relative to $\mathcal{P}(X)$ vanish. We summarize the previous discussion in the following proposition:

PROPOSITION (1.2). *Let W, M and X be as above. Then, for every trivialization* α *of TW and every immersion* $\mathcal{I}$ *of W in* $\mathbb{R}^{2k}$, *the vector field X determines (the homotopy class of) a trivialization* $\mathcal{P}(X)$ *of* $TW|_M$, *and the corresponding relative Chern classes* $c_i(W, \mathcal{P}(X))$ *are 0 for* $i = 1, \dots, k-1$, *while* $c_k(W, \mathcal{P}(X))[W]$ *is* $(k-1)! \cdot \mathrm{Ind}(X; W)$. *(Independently of the choices of* $\mathcal{I}$ *and* α.*)*

For example (c.f. [13]), when W is the Milnor fibre of a complex surface singularity , the trivialization $\mathcal{P}(X)$ is the canonical framing of [15], up to homotopy .

2. The Todd genus

Let N be a compact U-manifold of dimension $2k$, with a complex trivialization β of $TN|_{\partial N}$, its stable tangent bundle restricted to the boundary. Then β defines relative Chern classes $c_i(N;\beta) \in H^{2i}(N,\partial N;\mathbb{Z})$ as above. Let Td_k be the k^{th} Todd polynomial [9]. The *Todd genus of N relative to β* is:

$$Td[N;\beta] = Td_k((c_1(N;\beta),\ldots,c_k(N;\beta)) \in \mathbb{Q}.$$

So it is a rational linear combination of the relative Chern numbers of N. If ∂N is empty , this is the usual Todd genus [9] , otherwise the class of $Td[N;\beta]$ in $\mathbb{Q}/\mathbb{Z}$ is a framed cobordism invariant of $(\partial N,\beta)$, see [6].

LEMMA (2.1). *Let W be a compact, almost complex $2k$-manifold with boundary M and TW trivial. Let $\widetilde{W}$ be also almost complex with boundary M as U-manifolds. If X is a vector field as in §1 above and $\mathcal{P}(X)$ is the trivialization of §1 then:*

$$Td[W;\mathcal{P}(X)] \equiv Td[\widetilde{W};\mathcal{P}(X)] \quad \mod Z.$$

If, moreover, we can take W into $\widetilde{W}$ by a finite sequence of almost complex blow ups and blow downs over the interior of W, then

$$Td[W;\mathcal{P}(X)] = Td[\widetilde{W};\mathcal{P}(X)].$$

Proof. The first statement is well known [6] and it is a consequence of the integrality of the Todd genus for closed manifolds [9,1]. The second statement follows from the invariance of the Todd genus under blow ups and blow downs [9 and 10,p. 51].

LEMMA (2.2). *Let W, $\widetilde{W}$ and $\mathcal{P}(X)$ be as above. Then :*

i) $Td[W;\mathcal{P}(X)] = q_k \cdot (k-1)! \cdot \mathrm{Ind}(X;W)$, *where q_k is the coefficient of c_k in the Todd polynomial Td_k.*

ii) *The Chern number $c_k(\widetilde{W};\mathcal{P}(X))[\widetilde{W}]$ is*

$$c_k(\widetilde{W};\mathcal{P}(X))[\widetilde{W}] = [(k-1)!-1]\cdot(\chi(W)-\chi(\widetilde{W})) + (k-1)!\mathrm{Ind}(X;\widetilde{W}),$$

where χ is the (Topological) Euler-Poincaré characteristic.

iii) *We can express $Td[\widetilde{W};P(X)]$ as*

$$Td[\widetilde{W};\mathcal{P}(X)] = q_k c_k(\widetilde{W};\mathcal{P}(X))[\widetilde{W}] + Td_k^*[\widetilde{W}],$$

where $Td_k^*[\widetilde{W}]$ *is the part of the Todd polynomial involving the Chern classes* $c_i(\widetilde{W};\mathcal{P}(X))$, $i=1,\dots,k-1$. $Td_k^*[\widetilde{W}]$ *is independent of* $\mathcal{P}(X)$. *In particular* $Td_k^*[\widetilde{W}]=0$ *if* $T\widetilde{W}$ *is trivial.*

Proof. The first statement follows from (1.2) above and the definition of the Todd genus. Also [14,15], the relative Chern numbers involving classes $c_1,\dots,c_{k-1}$ are indeed independent of the choice of the trivialization on the boundary, hence statement (iii). Thus, the only new point in (2.2) is statement (ii). To prove this we need the following lemma.

LEMMA (2.3). *Let* W *and* W' *be arbitrary compact, smooth manifolds with diffeomorphic boundary* M, *let* X *and* Y *be non-singular vector fields defined on a neighbourhood of* M *in* W *and denote also by* X, Y *the corresponding vector fields on a neighbourhood of* M *in* W'. *Then*

$$\mathrm{Ind}(X;W)-\mathrm{Ind}(Y;W)=\mathrm{Ind}(X;W')-\mathrm{Ind}(Y;W').$$

This lemma is well known. We include an elementary proof for completeness.

Proof. Let us attach a "collar" $M\times[0,\varepsilon]$, $\varepsilon>0$, to W and we denote by W_ε the resulting manifold, which is diffeomorphic to W, with boundary $M\times\varepsilon$. We put on $M\times 0$ the unit outwards normal field of M in W, we put the vector field X on the boundary $M\times\varepsilon$ and we extend this to a vector field on W_ε with finite singularities. Obstruction theory and the theorem of Poincaré-Hopf for manifolds with boundary [16] tell us

$$\mathrm{Ind}(X;W)=\chi(W)+I(X;M^*)$$

where the last term in this formula is the number of singularities, counted with multiplicities, that we have on the collar $M\times[0,\varepsilon]$. Similar arguments apply to the vector field Y. The same arguments applied on W' yield

$$\mathrm{Ind}(X;W')=\chi(W')+I(X;M^*),$$

and similarly for Y. Hence

$$\begin{aligned}\mathrm{Ind}(X;W)-\chi(W)&=\mathrm{Ind}(X;W')-\chi(W'),\\ \mathrm{Ind}(Y;W)-\chi(W)&=\mathrm{Ind}(Y;W')-\chi(W'),\end{aligned}$$

and the result follows.

Let us return to the proof of (2.2) ii) If we denote by p_1 one of the k sections that define $\mathcal{P}(X)$, then lemma (2.3) implies:

$$\mathrm{Ind}(p_1;\widetilde{W})=\mathrm{Ind}(p_1;W)+\chi(\widetilde{W})-\chi(W).$$

Hence by §1,

$$\begin{aligned}\mathrm{Ind}(p_1;\widetilde{W}) &= (k-1)!\cdot\mathrm{Ind}(X;W)+\chi(\widetilde{W})-\chi(W)\\ &= (k-1)!\cdot[\mathrm{Ind}(X;\widetilde{W})+\chi(W)-\chi(\widetilde{W})]+\chi(\widetilde{W})-\chi(W)\\ &= (1-(k-1)!)\cdot(\chi(\widetilde{W})-\chi(W))+(k-1)!\cdot\mathrm{Ind}(X;\widetilde{W}).\end{aligned}$$

Statement (ii) now follows because by definition,

$$\mathrm{Ind}(p_1;\widetilde{W}) = c_k(\widetilde{W};\mathcal{P}(X))[\widetilde{W}].$$

We are now ready to prove Theorem (1), stated in the introduction. From lemmas (2.1) and (2.2) we have

$$\begin{aligned}q_k\cdot(k-1)!\cdot\mathrm{Ind}(X;W) \equiv\ & q_k\Big\{[(k-1)!-1]\cdot(\chi(W)-\chi(\widetilde{W}))\\ &+(k-1)!\cdot\mathrm{Ind}(X;\widetilde{W})\Big\}+Td_k^*[\widetilde{W}] \bmod \mathbb{Z}.\end{aligned}$$

Suppose first that restricted to M the vector field X is everywhere transversal to M, then

$$\mathrm{Ind}(X;W)=\chi(W) \quad \text{and} \quad \mathrm{Ind}(X;\widetilde{W})=\chi(\widetilde{W}),$$

by the theorem of Poincaré-Hopf for manifolds with boundary [16]. The above formula becomes $q_k\chi(W)\equiv q_k\chi(\widetilde{W})+Td_k^*[\widetilde{W}] \bmod \mathbb{Z}$. The first claim in Theorem (1) follows from this equation together with lemma (2.3) and the fact that for $k=2r$ even, the coefficient q_k of c_k in Td_k is [2],

$$q_k=\frac{(-1)^{k-1}\mathcal{B}_r}{2r!}.$$

To prove the second statement in Theorem (1) we observe that in this situation lemmas (2.1) and (2.2) imply

$$\begin{aligned}q_k\cdot(k-1)!\cdot\mathrm{Ind}(X;W) \equiv\ & q_k\Big\{[(k-1)!-1]\cdot(\chi(W)-\chi(\widetilde{W}))\\ &+(k-1)!\cdot\mathrm{Ind}(X;\widetilde{W})\Big\}+Td_k^*[\widetilde{W}].\end{aligned}$$

For k even this means

$$\mathrm{Ind}(X;W)=\mathrm{Ind}(X;\widetilde{W})+q_k^{-1}\cdot Td_k^*[\widetilde{W}],$$

proving Theorem (1). For $k>1$ odd, the coefficient of c_k in Td_k is 0, by [9], thus we get that the Todd genus $Td[\widetilde{W},\beta]$ is 0 for every complex trivialization of $T\widetilde{W}|_M$, as mentioned in the introduction.

Example (2.4). Let $(\mathcal{V}, P)$ denote a hypersurface germ in $\mathbb{C}^{2k+1}$, $k > 0$, with an isolated singularity at P. Let F be a Milnor fibre of P, let $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ be a resolution of P and let $\widetilde{V} \subset \widetilde{\mathcal{V}}$ be a compact tubular neighbourhood of the exceptional set $\pi^{-1}(P)$. Then Theorem (1) says

$$\chi(F) \equiv \chi(\widetilde{V}) + \frac{(-1)^{k-1}2k!}{\mathcal{B}_k}(Td^*_{2k}[\widetilde{V}]) \qquad \text{mod } \frac{2k!}{\mathcal{B}_k},$$

which is a weak version of Looijenga's result [15]. For $k = 1$ this is Durfee's formula [7],

$$\mu + 1 \equiv \chi(\widetilde{V}) + K^2 \qquad \text{mod } 12,$$

where K is the canonical class and μ is the Milnor number. For $k = 2$ the formula reads

$$\mu + 1 \equiv \chi(\widetilde{V}) - (c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4)[\widetilde{V}] \quad \text{mod } 720,$$

where the c_i's are the Chern classes of $\widetilde{V}$ relative to a trivialization of the tangent bundle of $\widetilde{V}$ restricted to its boundary. For $k = 3$ the formula gives a congruence modulo 30240 , for $k = 4$ a congruence modulo 1,209,600, etc.

Example (2.5). Let $(\mathcal{V}, \mathcal{P})$ be a smoothable normal Gorenstein surface singularity (See [15] for the definitions involved.), and let $\widetilde{V}$ be a resolution of P. Then the Euler-Poincaré characteristic of $\widetilde{V}$ depends on the choice of resolution. The same happens with the self-intersection number K^2. However, any two resolutions of a surface singularity can be taken into each other by blow ups and blow downs, hence Theorem (1) implies that $\chi(\widetilde{V}) + K^2$ is independent of the resolution. (This is well known, see [15].)

3. Blow ups

Let us consider the germ at $0 \in \mathbb{C}^k$, $k > 1$, of a continuous vector field X with an isolated zero at 0 . We let $\pi : \widetilde{\Delta} \to \Delta$ be the blow up at 0, where $\Delta \subset \mathbb{C}^k$ is a small disc around 0 . The exceptional divisor is $E = \pi^{-1}(0) \equiv \mathbb{C}P^{k-1}$. We take Δ to be the manifold W of the previous sections. The boundary of Δ, and of $\widetilde{\Delta}$, is the $(2k-1)$-sphere $\mathbb{S} = \mathbb{S}^{2k-1}$. Since $H^i(\mathbb{S}) = 0$ for $i = 1, \ldots, 2k-2$, given any complex trivialization α of $T\widetilde{\Delta}|_{\mathbb{S}}$, the corresponding relative Chern classes $c_i(\widetilde{\Delta}; \alpha)$, $i = 1, \ldots, k-1$, are the unique relative classes that map to the usual Chern classes of $\widetilde{\Delta}$. The manifold $\widetilde{\Delta}$ retracts strongly to $E \equiv \mathbb{C}P^{k-1}$, hence its Chern classes are essentially the Chern classes of $TE \oplus \nu(E)$, where $\nu(E)$ is the normal bundle. We recall that $H^{2i}(\mathbb{C}P^{k-1}; \mathbb{Z}) \equiv \mathbb{Z}$ for $i = 1, \ldots, k-1$, generated by t^i, where $t \in H^2(\mathbb{C}P^{k-1})$ is the generator. The Chern class of $\nu(E)$ is $C_1(\nu(E)) = -t$, while the Chern classes of E are given [9] by the polynomial $(1+t)^k$, with $t^k = 0$. That is, $C_i(\mathbb{C}P^{k-1})$ is b_it^i, where b_i is the coefficient of t^i in $(1+t)^k$. By the axioms for Chern classes we have, for $i = i, \dots, k-1$,

$$c_i(T\widetilde{\Delta}|_E) = c_i(E) + c_{i-1}(E) \cdot C_1(\nu(E)) = (b_i - b_{i-1})t^i,$$

where $b_0 = 1$. When k is even Theorem (1) implies

$$\mathrm{Ind}(X;\Delta) = \mathrm{Ind}(X, \widetilde{\Delta}) + q_k^{-1} Td_k^*[\widetilde{\Delta}]$$

where

$$Td_k^*[\widetilde{\Delta}] = Td_k(c_1(T\widetilde{\Delta}), \dots, c_{k-1}(T\widetilde{\Delta}), 0)[\widetilde{\Delta}]$$

by definition. Thus we arrive to the following corollary of Theorem (1):

COROLLARY (3.1). *Let X be (the germ of) a continuous vector field in $\mathbb{C}^{2k}$, with an isolated singularity at 0, let $\mathrm{Ind}(X)$ be its local index at 0, and let $\pi : \widetilde{\Delta} \to \Delta$ be the blow up at 0, where Δ is a small disc around 0. If t^i is the usual generator of $H^{2i}(\mathbb{C}P^{k-1};\mathbb{Z}) \equiv H^{2i}(\widetilde{\Delta};\mathbb{Z})$, and b_i is the coefficient of t^i in $(1+t)^{2k}$, then*

$$\mathrm{Ind}(X) = \mathrm{Ind}(X, \widetilde{\Delta}) + \frac{(-1)^{k-1} \cdot 2k!}{\mathcal{B}_k} Td_{2k}(c_1, \dots, c_{2k-1}, 0)[\widetilde{\Delta}],$$

where the c_i's, $i = 1, \dots, 2k-1$, are the unique classes in $H^{2i}(\widetilde{\Delta}, \partial\widetilde{\Delta};\mathbb{Z})$ mapping to $(b_i - b_{i-1})t^i \in H^{2i}(\mathbb{C}P^{k-1};\mathbb{Z})$, $i = 1, \dots, 2k-1$.

Also, (2.3) above implies,

$$\mathrm{Ind}(X;\Delta) - \mathrm{Ind}(X, \widetilde{\Delta}) = \chi(\Delta) - \chi(\widetilde{\Delta}) = q_k^{-1} Td_k^*[\widetilde{\Delta}].$$

Since Δ is a disc we have $\chi(\Delta) = 1$, and $\widetilde{\Delta}$ retracts to $\mathbb{C}P^{k-1}$, so $\chi(\widetilde{\Delta}) = k$ and one has,

$$\frac{(-1)^{k-1} \cdot 2k!}{\mathcal{B}_k} \cdot Td_{2k}(c_1(T\widetilde{\Delta}), \dots, c_{2k-1}(T\widetilde{\Delta}), 0)[\widetilde{\Delta}] = 1 - 2k.$$

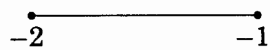
Example (3.2). Let us consider the germ at $0 \in \mathbb{C}^2$ of the linear vector field

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is the radial vector field, so it has index 1 at 0, which is the Euler-Poincaré characteristic of a small disc Δ around 0. Let us now perform a blow up at 0, obtaining a space $\widetilde{\Delta}$ and a map $\pi : \widetilde{\Delta} \to \Delta$ with exceptional set $E = \pi^{-1}(0) = \mathbb{C}P^1$ embedded in $\widetilde{\Delta}$ with self intersection -1. The adjunction formula [7] tells us that E represents the canonical class K, dual to minus the first Chern class of $\widetilde{\Delta}$ relative to a trivialization of $T\widetilde{\Delta}|_{\partial\widetilde{\Delta}}$. Hence $K^2 = -1$ and

$$1 = \mathrm{Ind}(Z;\Delta) = \chi(\widetilde{\Delta}) + K^2 = \mathrm{Ind}(Z;\widetilde{\Delta}) + q_2^{-1}Td_2^*[\widetilde{\Delta}],$$

as it should be by Theorem (2). Now pick a point x_0 in $E \subset \widetilde{\Delta}$ and blow it up obtaining $\tilde{\pi} : \tilde{\tilde{\Delta}} \to \Delta$, then $\tilde{\tilde{\Delta}}$ has graph



each vertex having genus 0. The exceptional set E has two irreducible components, corresponding to the two vertices, so $\chi(\tilde{\tilde{\Delta}}) = 3$, and the canonical class is $K = E_0 + 2E_1$, by the adjunction formula. Hence $K^2 = -2$ and we have

$$1 = \mathrm{Ind}(Z) = \chi(\tilde{\tilde{\Delta}}) + K^2 = \mathrm{Ind}(Z;\tilde{\tilde{\Delta}}) + q_2^{-1}Td_2^*[\tilde{\tilde{\Delta}}],$$

etc.

Example (3.3). More generally, consider the linear vector field

$$Z = \begin{pmatrix} 1 & 0 & \ldots\ldots & & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

in $\mathbb{C}^n$, $n > 1$. This has index 1 at $0 \in \mathbb{C}^n$. As above, consider a small disc Δ around 0 , and perform the blow up at 0, obtaining $\pi : \widetilde{\Delta} \to \Delta$ with exceptional set $E = \pi^{-1}(0) = \mathbb{C}P^{n-1}$. The index of Z in $\widetilde{\Delta}$ is the Euler-Poincaré characteristic of E, so it equals n. To evaluate the Chern classes of $\widetilde{\Delta}$, relative to some trivialization of its tangent bundle over the boundary, we recall that the coefficients of the t^i's in $(t+1)^n$ are given by Pascal's triangle . The corresponding Chern classes are determined by Theorem (2). When $n = 2$ this is the example above. For $n = 3$ we have:

$$b_0 = 1, \qquad b_1 = 3, \qquad b_2 = 3,$$

hence $c_1 = 1$ and $c_2 = 0$. The Todd polynomial Td_3^* is [9]:

$$Td_3^* = Td_3 = \frac{1}{24}c_1c_2 = 0,$$

as it should be since 3 is odd. For $n = 4$:

$$b_0 = 1, \qquad b_1 = 4, \qquad b_2 = 6, \qquad b_3 = 4.$$

Hence,

$$c_1 = 3, \qquad c_2 = 2, \qquad c_3 = -2,$$

the Todd polynomial Td is :

$$Td_4^* = \frac{1}{720}(c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4) = \frac{-3}{720}$$

so that $Td_4^*[\widetilde{\Delta}] = 3/720$, because $c_1(\nu(E)) = -t$. Since the coefficient q_4 of c_4 in Td_4 is $-1/720$ we get :

$$1 = \mathrm{Ind}(Z;\Delta) = \chi(\mathbb{C}P^3) + q_4^{-1}Td_4^*[\widetilde{\Delta}],$$

as claimed in Theorem (2), and so on.

4. On complex distributions

In this section we prove Theorem (2). Let W be a compact almost complex $2n$-manifold, $n > 1$, with or without boundary. Let $\mathcal{D}$ be a continuous 1-dimensional complex distribution on W, with isolated singularities, all in the interior of W. By [8,4], if P is a singularity of $\mathcal{D}$, then $\mathcal{D}$ has a well defined *local index* $\mu = \mu(\mathcal{D},P)$ at P: this is the index at P of a continuous vector field on a neighbourhood of P in W, singular only at P and tangent to $\mathcal{D}$. It is shown in [8,4] that this local index is well defined, i.e. it does not depend on the choice of the vector field. We let $\mu(\mathcal{D},W)$ be the *total index* of $\mathcal{D}$ in W, i.e. the sum of all its local indices. Let $T\mathcal{D}$ and $N\mathcal{D}$ be, respectively, the tangent bundle and the normal bundle of $\mathcal{D}$ away from the singular points. The bundle $T\mathcal{D}$ has a Chern class $c_1(\mathcal{D}) \in H^2(W^*;\mathbb{Z})$, where W^* is W minus the singularities of $\mathcal{D}$. The group $H^2(W^*;\mathbb{Z})$ is isomorphic to $H^2(W;\mathbb{Z})$, because $\mathcal{D}$ has isolated singularities. Hence $c_1(\mathcal{D})$ can be regarded as a class in $H^2(W;\mathbb{Z})$. When W has no boundary, this is *the Chern class of* $\mathcal{D}$; when W has non-empty boundary M and we are given a vector field X tangent to $\mathcal{D}$ on M, then the Chern class of $\mathcal{D}$ has a representative in $H^2(W^*,M;\mathbb{Z})$, which is isomorphic to $H^2(W,M;\mathbb{Z})$. The class $c_1(\mathcal{D},X) \in H^2(W,M;\mathbb{Z})$ so obtained is *the Chern class of* $\mathcal{D}$ *relative to the vector field* X *on* M. Similarly, the normal bundle $N\mathcal{D}$ has well defined Chern classes $c_i(N\mathcal{D})$ in $H^{2i}(W;\mathbb{Z})$, $i = 1,\ldots,n-1$. If $c_{n-1}(N\mathcal{D})$ is the top Chern class of $N\mathcal{D}$, then the product $c_1(\mathcal{D},X)\cdot c_{n-1}(N\mathcal{D})$ lies in $H^{2n}(W,M;\mathbb{Z}) \equiv \mathbb{Z}$. The first statement in Theorem (2) says :

$$\mathrm{Ind}(X;W) = \sum_{i=1}^{r} \mu_i + (c_1(\mathcal{D},X)\cdot c_{n-1}(N\mathcal{D}))[W],$$

where the sum runs over the singular points $P_1,\ldots,P_r$ of $\mathcal{D}$. To prove this, let $B_1,\ldots B_r$ be pairwise disjoint balls around $P_1,\ldots P$, respectively, and let $W^* = W - \mathrm{Int}(B_1 \cup .. \cup B_r)$. Let Y be a vector field on W^*, tangent to $\mathcal{D}$, transversal to the zero section of $\mathcal{D}$, and such that Y extends X and it is non-zero on the boundary of the B_i's. The zero-locus Σ_Y of Y in W^* is a smooth submanifold of W^*, representing $c_1(\mathcal{D},X)$. Similarly, let Y^+ be a vector field on W^*, orthogonal to $\mathcal{D}$ and transversal to the zero-section of $N\mathcal{D}$; the zero-locus Σ_Y+ of Y^+ is a smooth submanifold of W^* representing $c_{n-1}(N\mathcal{D})$. We may assume that Σ_Y and Σ_Y+ intersect transversally, hence

$$\Sigma_Y \cdot \Sigma_Y+ = c_1(\mathcal{D}) \cdot c_{n-1}(N\mathcal{D})[W],$$

by duality. Let us extend Y to the interior of the B_i's tangent to $\mathcal{D}$. The local index of Y at each P_i is μ_i. Similarly, extend Y^+ to the interior of the B_i's being normal to $\mathcal{D}$; multiplying Y^+ by an appropiate continuous function $\phi : W \to [0,1]$, we obtain a vector field Y_o^+, which is the same as Y^+ away from a neighbourhood of ∂W^*, but Y_o^+ vanishes on M and near the P_i's. Then define $\Psi(x) = Y(x) + Y_o^+(x)$. So Ψ agrees with X on M, hence $\mathrm{Ind}(X;W) = \mathrm{Ind}(\Psi;W)$. The local indices of Ψ at the P_i's are the μ_i's, and the remaining singularities of Ψ on W are the points where both vector fields Y and Y^+ vanish . Hence

$$\mathrm{Ind}(\Psi;W) = \sum_{i=1}^{r} \mu_i + \Sigma_Y \cdot \Sigma_Y+,$$

and we arrive to Theorem (2).

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