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PRODUCT FORMULAS FOR QUADRATIC FORMS

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Dedicated to the memory of Professor José Adem

A product formula of size (r, s, t) over a field F is a formula of the type

$$(x_1^2 + x_2^2 + \ldots + x_r^2) \cdot (y_1^2 + y_2^2 + \ldots + y_s^2) = z_1^2 + z_2^2 + \ldots + z_t^2$$

where each z_k is a bilinear form in the sets of indeterminates X, Y with coefficients in F. Throughout this work we assume that the coefficient field F has characteristic not 2. It is a well-known open problem to determine the possible sizes (r, s, t) for such families. In 1898 Hurwitz [Hu1] formulated the problem in terms of matrices and proved that a formula of size (n, n, n) can exist over the complex field $\mathbb{C}$ if and only if $n = 1$, 2, 4 or 8. In the 1920's Hurwitz [Hu2] and Radon [Ra] settled the question for formulas over $\mathbb{C}$ of sizes (r, n, n). The matrix technique used by Hurwitz can be extended to establish the same results for any coefficient field F. See [Sh1], [Sh3] for further details.

The existence of a product formula can be restated in a more geometric framework. Let ρ, σ, τ be the sums-of-squares quadratic forms on the vector spaces F^r, F^s, F^t, respectively. Then there is a product formula of size (r, s, t) over F if and only if there is a bilinear map

$$f : F^r \times F^s \to F^t$$

satisfying the formula

$$\rho(x) \cdot \sigma(y) = \tau(f(x,y)) \qquad \text{for } x \in F^r,\ y \in F^s.$$

In 1940 Stiefel [St] and Hopf [Ho] used this formulation to analyze such formulas over the field $\mathbb{R}$ of real numbers.

THEOREM (Stiefel, Hopf). *If there exists a product formula of size (r, s, t) over $\mathbb{R}$ then the binomial coefficient $\binom{t}{i}$ is even whenever $t - s < i < r$.*

For example it follows that no formulas of sizes (3, 5, 6) or (5, 9, 12) or (6, 10, 13) can exist over $\mathbb{R}$. Since the present work is an attempt to generalize Hopf's proof, we present an outline of his ideas, suppressing the details. The given bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^t$ satisfies the norm condition $\|x\| \cdot \|y\| = \|f(x,y)\|$. (Here $\|x\|$ denotes the euclidean norm of $x \in \mathbb{R}^r$.) This map f induces a map on the real projective spaces

$$\tilde{f} : \mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{t-1},$$

and hence a map on the corresponding cohomology rings. Let $H(X)$ denote the cohomology ring of a topological space X, with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Then $H(X \times Y) \cong H(X) \otimes H(Y)$ and $H(\mathbb{P}^{t-1}) \cong \mathbb{Z}[T]/(2, T^t)$ where the class [T] represents the fundamental 1-cocycle. The induced cohomology ring homomorphism then becomes

$$f^* : \frac{\mathbb{Z}[T]}{(2, T^t)} \longrightarrow \frac{\mathbb{Z}[R]}{(2, R^r)} \otimes \frac{\mathbb{Z}[S]}{(2, S^s)}.$$

Since f^* arose from a normed, bilinear map it follows that $f^*([T]) = [R] \otimes 1 + 1 \otimes [S]$. Therefore

$$0 = f^*([T]^t) = ([R] \otimes 1 + 1 \otimes [S])^t = \sum \binom{t}{i} [R]^i \otimes [S]^{t-i},$$

and hence $\binom{t}{i} = 0$ in $\mathbb{Z}/2\mathbb{Z}$ whenever $i < r$ and $t - i < s$. This completes our sketch of the proof.

Hopf actually proved a stronger result where the norm condition on f is replaced by the hypothesis that f be "nonsingular". That is, we assume only that if $f(x,y) = 0$ then either $x = 0$ or $y = 0$. (The bilinearity condition can also be weakened.) Since the 1940's a number of topological tools have been applied to obtain further results for formulas over $\mathbb{R}$, but only a few of these results are known over more general fields. In 1939 Behrend [Be] used ideas from real algebraic geometry to extend the Stiefel-Hopf result to general real closed fields. In 1984 T.-Y. Lam and K.-Y. Lam observed that the Stiefel-Hopf condition remains true for product formulas over any field F of characteristic 0. This was done by reducing to the case $F = \mathbb{C}$, noting that from a product formula of size (r, s, t) over $\mathbb{C}$ one can produce a nonsingular bilinear pairing of size (r, s, t) over $\mathbb{R}$, and applying Hopf's result. See [Sh1]§3 for the details.

Pfister's theory of multiplicative quadratic forms can also be applied to analyze product formulas over a field F. With this approach we can show that if there is a product formula of size (r, s, t) over F and if the quadratic form $t\langle 1\rangle = \langle 1, 1, \dots, 1\rangle$ is anisotropic over F, then the Stiefel-Hopf conditions must hold. See [Sh1]§4 for the precise statements. Unfortunately this argument is worthless over a field F of positive characteristic since $t\langle 1\rangle$ is isotropic over such F whenever $t > 2$.

There are three results about product formulas which are known to be valid for fields of any characteristic $\neq 2$. These are:

(1) The Hurwitz-Radon Theorem for sizes (r, n, n).

(2) Adem's Theorem for sizes $(r, n-1, n)$.

(3) Yuzvinsky's Theorem for sizes $(4, n-2, n)$.

For (1) see [Sh 1]; for (2) see [Ad1], [Ad3], [Sh2]; for (3) see [Yu], [Ad2]. It is tempting to conjecture as in [Sh 1] (3.8) that the existence of product formulas of size (r, s, t) is independent of the field F.

In the present work we apply the machinery of Chow rings to find another proof that the Stiefel-Hopf criterion is valid in characteristic 0 and to find a weakened version of that criterion which is valid in characteristic p. In fact, these results are valid for arbitrary quadratic forms, not just for the sum-of squares forms.

The Chow ring of a quasiprojective variety is a notion from classical algebraic geometry which codifies some of the properties of intersections of subvarieties. The Chow ring methods can be viewed as a generalization of combinatorial arguments dealing with degrees and roots of polynomials in one variable. The proof of our Theorem is modelled after Hopf's original proof, replacing the cohomology ring by the Chow ring. As one concrete consequence of this work we deduce that a product formula of size $(5, 9, 12)$ is not possible over any field.

1. Nonsingular pairings

We use here the simplest parts of the Chow ring technique described briefly in [Ha]§I.7 and Addendum A, although we refer to the monograph [Fu] as well.

For an arbitrary field F we denote by $F\mathbb{P}^{n-1}$ the set of lines in the vector space F^n. Then $F\mathbb{P}^{n-1}$ is the set of rational points of the projective space $\mathbb{P}_F^{n-1}$, which is the scheme

$$\text{Proj } F[x_1, x_2, \ldots, x_n].$$

In order to investigate a bihomogeneous polynomial map

$$f : F^r \times F^s \longrightarrow F^t$$

our basic strategy is to pass to the induced map

$$\tilde{f} : F\mathbb{P}^{r-1} \times F\mathbb{P}^{s-1} \longrightarrow F\mathbb{P}^{t-1},$$

and then lift this to a morphism of schemes

$$f^{\#} : \mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1} \longrightarrow \mathbb{P}_F^{t-1}$$

to obtain a homomorphism of the corresponding Chow rings. However these induced maps might fail to exist. In fact, $\tilde{f}$ exists if and only if f is a "nonsingular" map in the sense defined above. Similarly $f^{\#}$ exists if and only if f is nonsingular over every algebraic extension field of F.

Examples (1.1). (1) If $t = rs$ there is a standard nonsingular bilinear map f given by

$$f\left((x_1, \ldots, x_r),\ (y_1, \ldots, y_s)\right) = (x_1y_1, x_1y_2, \ldots, x_1y_s, \ldots, x_ry_s).$$

This f does yield well-defined maps $\widetilde{f}$ and $f^{\#}$, providing the classical Segre immersion [Ha] Exer. I.2.14.

(2) There is an ancient formula for a product of sums of two squares:

$$(x_0^2+x_1^2)\cdot(y_0^2+y_1^2)=(x_0y_0-x_1y_1)^2+(x_0y_1+x_1y_0)^2.$$

Let $f: F^2\times F^2 \longrightarrow F^2$ be the corresponding map:

$$f((x_0,x_1),(y_0,y_1))=(x_0y_0-x_1y_1,\ x_0y_1+x_1y_0).$$

The induced map $\widetilde{f}$ exists only if the form $\langle 1,1\rangle$ is anisotropic over F. For if F contains a square root i of -1, then $\widetilde{f}$ is not defined at the two points: $((1:i),(1:-i))$ and $((1:-i),(1:i))$. Even if the form $\langle 1,1\rangle$ is anisotropic over F this map f does not define a morphism $f^{\#}$ on the projective spaces. In this case the closed subscheme

$$x_0y_0-x_1y_1=x_0y_1+x_1y_0=0 \text{ of } \mathbb{P}^1_F\times\mathbb{P}^1_F$$

consists of one closed degree two point with residue class field $F(i)$. (To compute the residue field of this point one may restrict to the product of affine lines $x_0\neq 0$ and $y_0\neq 0$ and put $t=x_1/x_0$ and $u=y_1/y_0$. The point under consideration has the equations $1+tu=t+u=0$, and its residue field is $F[t,u]/(1-tu,t+u)\cong F[t]/(1+t^2)\cong F[i]$.)

Our first application of the Chow ring technique is to determine when a morphism like $f^{\#}$ can exist.

PROPOSITION (1.2). *Suppose $g:\mathbb{P}^{r-1}_F\times\mathbb{P}^{s-1}_F\longrightarrow\mathbb{P}^{t-1}_F$ is a polynomial morphism of schemes. If $t<r+s-1$ then g is constant with respect to at least one of the variables $x\in\mathbb{P}^{r-1}_F$ or $y\in\mathbb{P}^{s-1}_F$.*

Proof. The morphism g defines a graded ring homomorphism of the Chow rings

$$g^*: A^*(\mathbb{P}^{t-1}_F)\longrightarrow A^*(\mathbb{P}^{r-1}_F\times\mathbb{P}^{s-1}_F),$$

([Ha] Add. A §1, or [Fu] §8.3). This ring $A^*(\mathbb{P}^n_F)$ is isomorphic to $\mathbb{Z}[u]/(u^{n+1})$ as a graded ring, where the coset $[u]$ has degree 1 and corresponds to the class of a hyperplane. Furthermore the coset of u^k corresponds to an intersection of k hyperplanes in general position, so it is the class of a linear subvariety of dimension $n-k$ ([Ha] Add. A, Example 2.0.1 or [Fu] Example 1.9.3). We introduce indeterminates R,S,T such that

$$A^*(\mathbb{P}^{r-1}_F)\cong\mathbb{Z}[R]/(R^r)\qquad A^*(\mathbb{P}^{s-1}_F)\cong\mathbb{Z}[S]/(S^s)\qquad A^*(\mathbb{P}^{t-1}_F)\cong\mathbb{Z}[T]/(T^t).$$

Taking products of cycles induces a graded ring isomorphism

$$A^*(\mathbb{P}^{r-1}_F)\otimes_{\mathbb{Z}}A^*(\mathbb{P}^{s-1}_F)\cong A^*(\mathbb{P}^{r-1}_F\times\mathbb{P}^{s-1}_F)$$

as in [Fu] Example 8.3.7 (see also [Ha] Add. A, §2 A11). Under this isomorphism $[R] \otimes 1$ corresponds to the class of

$$(\text{hyperplane in } \mathbb{P}_F^{r-1}) \times \mathbb{P}_F^{s-1}$$

while $1 \otimes [S]$ corresponds to the class of

$$\mathbb{P}_F^{s-1} \times (\text{hyperplane in } \mathbb{P}_F^{s-1})$$

and both are homogeneous of degree one. Since g^* preserves degrees of homogeneous elements of Chow rings, there are integers a, b such that

$$g^*([T]) = a \cdot ([R] \otimes 1) + b \cdot (1 \otimes [S]).$$

CLAIM. a is the degree of g as a homogeneous function of $x \in \mathbb{P}_F^{r-1}$ and b is the degree of g as a homogeneous function of $y \in \mathbb{P}_F^{s-1}$.

The identity $[T]^t = 0$ in $A^*(\mathbb{P}^{t-1})$ yields

$$0 = g^*([T]^t) = (a \cdot ([R] \otimes 1) + b \cdot (1 \otimes [S]))^t = \sum \binom{t}{i} a^i b^{t-i} [R^i] \otimes [S^{t-i}]$$

in $\mathbb{Z}[R]/(R^r) \otimes \mathbb{Z}[S]/(S^s)$. Since $t < r + s - 1$ at least one of the terms $[R^i] \otimes [S^{t-i}]$ must be nonzero, so that either $a = 0$ or $b = 0$. Using the Claim we conclude that g must be constant in one of the variables.

To prove this Claim consider an arbitrary rational point P of $\mathbb{P}_F^{s-1}$ and the regular imbedding (c.f. [Fu] §B.7.1):

$$i : \mathbb{P}_F^{r-1} \xrightarrow{\times P} \mathbb{P}_F^{r-1} \times P \xrightarrow{\subseteq} \mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1}.$$

Then $(g \circ i)^*([T]) = i^* \circ g^*([T])$ is the intersection of $g^*([T])$ with $\mathbb{P}_F^{r-1} \times P$ regarded as a class of cycles on $\mathbb{P}_F^{r-1}$ ([Fu] §6.1). Since $[P] = [S^{s-1}]$, we have $[\mathbb{P}_F^{r-1} \times P] = 1 \otimes [S^{s-1}]$ and hence

$$(g \circ i)^*([T]) = (a[R] \otimes 1 + b \otimes [S]) \cdot (1 \otimes [S^{s-1}]) = a[R].$$

On the other hand, denote by $(z_1 : z_2 : \ldots : z_t)$ the homogeneous coordinates in $\mathbb{P}_F^{t-1}$ and choose a hyperplane H defined by the equation

$$a_1 z_1 + a_2 z_2 + \ldots + a_t z_t = 0$$

as a representative of the class [T]. If $(g_1 : g_2 : \ldots : g_t)$ denotes the homogeneous coordinates of g, then $(g \circ i)^*([T])$ is the class of the Weil divisor

$$a_1 g_1(x, P) + a_2 g_2(x, P) + \cdots a_t g_t(x, P) = 0,$$

unless $g(\mathbb{P}_F^{r-1} \times P)$ lies inside H ([Fu] §2.2). Note that Weil divisors and Cartier divisors are the same thing in this case ([Ha] Prop. II.6.11). Of course the variety $g(\mathbb{P}_f^{r-1} \times P)$ cannot be inside all such hyperplanes H because their (set-theoretical) intersection is empty, so one may choose H to be proper. Therefore the number a equals the degree of that divisor ([Ha] Exer. II.6.2), which is simply the common degree of the g_i as homogeneous functions of x. The value of b follows analogously and the Claim is proved. □

From Proposition (1.2) we can determine when nonsingular bilinear maps exist over an algebraically closed field F. Recall that $f : F^r \times F^s \to F^t$ is *nonsingular* if $f(x,y) = 0$ implies $x = 0$ or $y = 0$. One easy example of a nonsingular bilinear map $f_0 : F^r \times F^s \to F^{r+s-1}$ is given by the expansion of the product

$$(x_0 + x_1T + \cdots + x_{r-1}T^{r-1}) \cdot (y_0 + y_1T + \cdots + y_{s-1}T^{s-1})$$

in the polynomial ring $F[T]$. Therefore there certainly exist nonsingular bilinear maps of size (r, s, t) whenever $t \geq r + s - 1$.

COROLLARY (1.3). *Assume that F is algebraically closed. There exists a nonsingular bilinear map $f : F^r \times F^s \to F^t$ if and only if $t \geq r + s - 1$.*

Proof. Assume $t < r + s - 1$. The nonsingularity implies that f defines the map $\tilde{f}$ mentioned at the start of §1. Suppose f induces a morphism $f^\# : \mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1} \longrightarrow \mathbb{P}_F^{t-1}$. Then Proposition 1.2 implies that f must be constant with respect to one variable. Since f is bilinear it must equal zero, contrary to the nonsingularity. Therefore f cannot define a morphism of schemes, and the subscheme of $\mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1}$ defined by putting all homogeneous coordinates of f equal zero must be nonempty. Since F is algebraically closed Hilbert's Nullstellensatz implies that this subscheme contains a rational point. But then $\tilde{f}$ is not defined at this point of $F\mathbb{P}^{r-1} \times F\mathbb{P}^{s-1}$, a contradiction. □

This result was proved over $\mathbb{C}$ in [Sm] using the cohomology of complex projective spaces. We can also deduce the Corollary (for any algebraically closed F) from a dimension calculation given in [We] Theorem 2.1: Suppose $s \leq t$ and let $\mathcal{S}(t, s)$ be the set of singular $t \times s$ matrices over F (here a matrix is singular if its rank is $< s$). Westwick showed that $\mathcal{S}(t, s)$ is an irreducible subvariety of $\mathbb{M}_{t\times s}(F)$ of codimension $t - s + 1$. For a map f as in Corollary 1.3 the induced map $\hat{f} : F^r \longrightarrow \mathbb{M}_{t\times s}(F)$ carries F^r injectively to a subspace which meets $\mathcal{S}(t, s)$ only in 0. Therefore $r \leq \operatorname{codim} \mathcal{S}(t, s) = t - s + 1$, as claimed.

2. Product formulas for anisotropic forms

We shall investigate another type of condition on the map $f : F^r \times F^s \to F^t$. No confusion should arise between the notion of "nonsingular" maps defined above and the usual notion of nonsingular quadratic forms.

Consider three nonsingular quadratic forms ρ, σ, τ over F of dimensions r, s, t, respectively. We now consider bihomogeneous polynomial (hence bilinear) maps $f : F^r \times F^s \longrightarrow F^t$ satisfying the following product formula:

$$\rho(x) \cdot \sigma(y) = \tau(f(x,y)) \quad \text{for } x \in F^r,\ y \in F^s.$$

Such a map certainly exists when $\tau \cong \rho \otimes \sigma$. As in Example (1.1)(2) such a map f might not define a morphism on the product of projective spaces. However it does induce a morphism on the open subscheme of $\mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1}$ defined by the condition that at least one homogeneous coordinate of $f(x,y)$ does not vanish. To fix notations let X, Y, Z be the quadrics in $\mathbb{P}_F^{r-1}$, $\mathbb{P}_F^{s-1}$ and $\mathbb{P}_F^{t-1}$ defined by the forms ρ, σ, τ respectively, and denote by A, B, C the open complements to X, Y or Z in the corresponding projective spaces. The product formula implies that points of irregularity (i.e. where all the coordinates vanish) of the rational map $f^\#$ must lie inside the subvariety $X \times \mathbb{P}_F^{s-1} \cup \mathbb{P}_F^{r-1} \times Y$ of the product $\mathbb{P}_F^{r-1} \times \mathbb{P}_F^{s-1}$. We shall derive information on f from the induced morphism of schemes $f^\# : A \times B \longrightarrow C$, and its action on Chow rings.

The following computation of the Chow ring $A^*(A)$ is easily deduced from Lemma 13.4 of [Sw], but we give a more elementary proof here.

PROPOSITION (2.1). *Let A be the open complement of the quadric X defined by the equation $\rho = 0$ in $\mathbb{P}_F^{r-1}$. If the quadratic form ρ is anisotropic, then* $A^*(A) \cong \mathbb{Z}[R]/(R^r, 2R)$.

Proof. By [Fu] Proposition 1.8 or [Ha] Add. A, §2 A.10 the following sequence of Chow groups is exact:

$$A^*(X) \longrightarrow A^*(\mathbb{P}_F^{r-1}) \xrightarrow{j^*} A^*(A) \longrightarrow 0.$$

The middle arrow here is the graded ring homomorphism j^* induced by the inclusion morphism $j : A \longrightarrow \mathbb{P}_F^{r-1}$, so that $A^*(A)$ is a factor ring of $A^*(\mathbb{P}_F^{r-1}) \cong \mathbb{Z}[R]/(R^r)$. The left arrow is a group homomorphism which shifts the grading up by one. It carries the element $1 \in A^*(X)$ to the class $[X]$ of the subvariety X in $\mathbb{P}_F^{r-1}$. Here $[X]$ is in $A^1(\mathbb{P}_F^{r-1})$, which is, by definition, the group of classes of Weil divisors ([Fu] §2.1). By [Ha] Proposition II.6.4 we find that $[X] = 2[R]$ where $[R]$ is the class of a hyperplane. Therefore $2[R]$ is in the kernel of j^* (compare [Fu] Example 1.9.4). To show that this kernel is the principal ideal generated by $2[R]$ it is enough to show that

$$j^* \ (\text{class of a rational point}) = j^*([R^{r-1}]) \ \text{ is nonzero.}$$

To do this let us consider the exact sequence

$$A^{r-2}(X) \longrightarrow A^{r-1}(\mathbb{P}_F^{r-1}) \longrightarrow A^{r-1}(A) \longrightarrow 0.$$

By definition $A^{r-2}(X)$ and $A^{r-1}(\mathbb{P}_F^{r-1})$ are factor groups of the free abelian groups generated by the closed points of X and $\mathbb{P}_F^{r-1}$, respectively ([Fu] §1.3 or [Ha] Add. A, §1). The left arrow in that sequence is induced by the inclusion map of sets of closed points ([Fu] §1.4, a closed immersion is a proper morphism, see [Ha] Corollary II.4.8). The degree of a closed point is the degree of its residue class field over the field F:

$$\deg(P) = (F(P) : F)$$

(see [Fu] Definition 1.4 and [Ha] §I.7). Two closed points P and Q in $\mathbb{P}_F^{r-1}$ have equal images in $A^*(\mathbb{P}_F^{r-1})$ if and only if they have equal degrees ([Fu] §8.4). The essential step is that under the assumption that the form ρ is anisotropic, all closed points inside X have even degrees. In fact the coordinates of such a point P define a nontrivial zero of ρ, so ρ becomes isotropic over the residue class field $F(P)$, and Springer's Theorem ([Sch] Ch. 2 Theorem 5.3) implies that $F(P)$ has even degree over F. Therefore the class $[R^{r-1}]$ of a rational point is outside the image of the map $A^*(X) \longrightarrow A^*(\mathbb{P}_F^{r-1})$. □

To continue with the analog of the argument of Hopf we must compute $A^*(A \times B)$.

PROPOSITION (2.2). *Continue the notations for ρ, σ, X, Y, A, B as above. If the forms ρ and σ are anisotropic then*

$$A^*(A \times B) \cong A^*(A) \otimes_{\mathbb{Z}} A^*(B).$$

Proof. If is known that for an arbitrary variety S, the product map

$$\times : A^*(S) \otimes_{\mathbb{Z}} A^*(\mathbb{P}^n) \longrightarrow A^*(S \times \mathbb{P}^n)$$

is an isomorphism of graded rings ([Fu] Examples 8.3.4 and 8.3.7, see also Theorem 3.3(b); or [Ha] Add. A, §2 A11). We have again the exact sequence of [Fu] (1.8):

$$A^*(A \times Y) \longrightarrow A^*(A \times \mathbb{P}_F^{s-1}) \longrightarrow A^*(A \times B) \longrightarrow 0$$

which is built from the following exact sequences in each degree i:

$$A^{i-1}(A \times Y) \longrightarrow A^i(A \times \mathbb{P}_F^{s-1}) \longrightarrow A^i(A \times B) \longrightarrow 0. \qquad (*)$$

Therefore $A^*(A \times B)$ is a factor ring of $\mathbb{Z}[R]/(R^r, 2R) \otimes_{\mathbb{Z}} \mathbb{Z}[S]/(S^s)$ by some ideal which contains $2 \otimes [S] = 1 \otimes 2[S]$, the class of $A \times Y$ in $A^*(A \times \mathbb{P}_F^{s-1})$. To complete the proof we need only show that

$$1 \otimes [S]^i,\ [R] \otimes [S]^{i-1},\ [R]^2 \otimes [S]^{i-2}, \ldots, [R]^i \otimes 1$$

are linearly independent over $\mathbb{Z}/2\mathbb{Z}$ for each $i > 0$. If there is a dependence relation

$$[R]^k \otimes [S]^{i-k} + [R]^m \otimes [S]^{i-m} + \cdots = 0,$$

we may multiply it by $[R]^{r-k-1} \otimes [S]^{s-i+k-1}$ to deduce that the class of the rational point vanishes:

$$[R]^{r-1} \otimes [S]^{s-1} = 0$$

in the group $A^{r+s-2}(A \times B)$. To show that this is impossible, put $i = r+s-2$ in the sequence (*) above to obtain:

$$A^{r+s-3}(A \times Y) \longrightarrow A^{r+s-2}(A \times \mathbb{P}_F^{s-1}) \longrightarrow A^{r+s-2}(A \times B) \longrightarrow 0.$$

The middle group is $\mathbb{Z}/2\mathbb{Z}$ with generator $[R]^{r-1} \otimes [S]^{s-1}$. This generator corresponds to the class of a rational point on $A \times \mathbb{P}_F^{s-1}$, since $r+s-2 = \dim(A \times \mathbb{P}_F^{s-1})$. So we have to show that no closed point of $A \times Y$ is a rational point of $A \times \mathbb{P}_F^{s-1}$, or equivalently that $A \times Y$ has no rational points. This is so since the image $\pi(P)$ of a rational point $P \in A \times Y$ under the projection $\pi : A \times Y \longrightarrow Y$ must be a rational point of Y, while Y has no rational points (the form σ is anisotropic). □

Now we can deduce the Stiefel-Hopf condition whenever there is a product formula for anisotropic forms.

THEOREM (2.3). *Suppose ρ, σ, τ are anisotropic quadratic forms over F with dimensions r, s, t respectively. If there exists a bilinear map $f : F^r \times F^s \longrightarrow F^t$ satisfying the product formula:*

$$\rho(x) \cdot \sigma(y) = \tau(f(x,y)) \quad \textit{for } x \in F^r,\ y \in F^s,$$

then the binomial coefficient $\binom{t}{i}$ is even whenever $t - s < i < r$.

Proof. Using the notations above, this f induces a morphism of schemes $f^{\#} : A \times B \longrightarrow C$, with its associated graded ring homomorphism $f^* : A^*(C) \longrightarrow A^*(A \times B)$. For the Weil divisor $[T]$ in $A^1(C)$ we have $f^*([T]) = [R] \otimes 1 + 1 \otimes [S]$ as in the proof of Proposition (1.2) (with coefficients in $\mathbb{Z}/2\mathbb{Z}$ as follows from Propositions (2.1) and (2.2)). Applying f^* to the identity $[T]^t = 0$ we obtain $\sum \binom{t}{i} [R]^i \otimes [S]^{t-i} = 0$ in the ring $A^*(A \times B) \cong \mathbb{Z}[R]/(R^r, 2R) \otimes_{\mathbb{Z}} \mathbb{Z}[S]/(S^s, 2S)$. This immediately yields the assertion. □

3. The isotropic case

When the quadratic forms ρ, σ, τ are isotropic these Chow rings are more difficult to compute. We find the answer by quoting some results of Swan.

If ρ is a (nonsingular) quadratic form over F let $w(\rho)$ denote the Witt index of ρ. That is, $w(\rho)$ is the dimension of a maximal totally isotropic subspace of ρ.

PROPOSITION (3.1). *Let A be the open complement of the quadric X defined by the equation $\rho = 0$ in $\mathbb{P}_F^{r-1}$. Then $A^*(A) \cong \mathbb{Z}[R]/(R^{r-w(\rho)}, 2R)$.*

Proof. We continue with the ideas used to prove Proposition (2.1). Since $\dim X = r - 2$ we apply Lemma 13.4 of [Sw] to conclude that, up to torsion, $A^i(X)$ is generated by one cycle of degree two for $i = 0, 1, \dots, r - w(\rho) - 2$, and $A^i(X)$ is generated by (one or two) cycles of degree one for $i = r - w(\rho) - 1, \dots, r - 2$. Recall that for projective space the degree map furnishes an isomorphism $A^i(\mathbb{P}_F^{r-1}) \cong \mathbb{Z}$. Therefore in the exact sequence

$$A^{i-1}(X) \longrightarrow A^i(\mathbb{P}_F^{r-1}) \longrightarrow A^i(A) \longrightarrow 0,$$

the left arrow is an epimorphism for $i \geq r - w(\rho)$ and is an injection onto the subgroup of index 2 for $i < r - w(\rho)$. Therefore

$$A^i(A) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for} \quad i = 0, 1, \dots, r - w(\rho) - 1 \\ 0 & \text{for} \qquad\qquad i \geq r - w(\rho) \end{cases}$$

The stated ring structure quickly follows. □

PROPOSITION (3.2). *Continue the notations for ρ, σ, X, Y, Z, A, B as above. Then*

$$A^*(A \times B) \cong A^*(A) \otimes_{\mathbb{Z}} A^*(B).$$

Proof. The argument proving Proposition (2.2) applies the same way here. □

THEOREM (3.3). *Suppose rho, σ, τ are quadratic forms over F with dimensions r, s, t, respectively. Let $r_0 = r - w(\rho)$ where $w(\rho)$ is the Witt index of ρ and similarly define s_0 and t_0. If there exists a bilinear map f: $F^r \times F^s \longrightarrow F^t$ satisfying the product formula:*

$$\rho(x) \cdot \sigma(y) = \tau(f(x,y)) \quad \text{for } x \in F^r,\ y \in F^s,$$

then the binomial coefficient $\binom{t_0}{i}$ is even whenever $t_0 - s_0 < i < r_0$.

Proof. Imitate the proof of Theorem (2.3). □

As one consequence we see that a product formula of size $(r, s, t) = (5, 9, 12)$ is impossible over any field (with characteristic $\neq 2$). On the other hand product formulas of size $(6, 10, 13)$ are impossible in characteristic 0, but it is unknown whether they are possible over fields of positive characteristic. The reader is invited to investigate further small examples of these types. Further numerical properties of the original Stiefel-Hopf criterion and this weaker version for isotropic forms are studied in [Sh3].

Remark (3.4). The results above can be immediately extended to a "composed product formula" $\{f^1,\ldots,f^n\}$. Here each f^i is a formula defined on an open subscheme U_i, where $\{U_i\}$ forms a cover of $A \times B$, and the maps f^i are compatible in the sense that

$$\left(f_1^i(x,y) : \ldots : f_t^i(x,y)\right) = \left(f_1^j(x,y) : \ldots : f_t^j(x,y)\right)$$

whenever both f^i and f^j are defined at the point (x,y). This extension is done by the usual method of defining morphisms of quasi-projective varieties, and all we need for the proof is a morphism of such a variety. The results can be extended further to the case when the components of the f^i's are rational functions of degree 1 defined in the whole open subset U_i. (The degree of a rational function is the degree of the numerator minus the degree of the denominator.) Only some technical tricks with degrees of divisors are needed to show that $f^*([T]) = [R] \otimes 1 + 1 \otimes [S]$.

However using these methods we cannot say anything about composed product formulas with independent (not necessarily compatible) parts. These sorts of formulas arise from identities

$$\sigma(x) \cdot \rho(y) = \tau\left(f(x,y)\right)$$

where $f(x,y)$ is a rational function in the indeterminates in x and y. See [Sh1] §4 for a discussion of such "rational compositions".

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