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THE LENGTH OF DYER-LASHOF OPERATIONS AND STABLE POSTNIKOV SYSTEMS

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§1. Introduction

Let X be a path connected infinite loop space. If p is any prime, the Dyer-Lashof algebra, $\mathcal{R}(p)$, [AK], [DL], [CLM] acts on the mod p homology of X. In the case that $p = 2$, if $I = (s_1, \ldots, s_k)$ is a sequence of integers for which $s_j \geq 0$, the length of I is defined as $\ell(I) = k$. The sequence I determines the operation $Q^I = Q^{s_1} \cdots Q^{s_k}$. There is a similar defintion for odd primes (see [CLM]). The following theorem is a result of May ([CLM], I.6.2) which is a generalization of a result of Moore and Smith [MS].

THEOREM (1.1)[CLM]. *If X is a stable n-stage Postnikov system ($n \geq 1$), then $Q^I x = 0$ if $x \in \tilde{H}_*(X; \mathbf{F}_p)$ and $\ell(I) \geq n$.*

A special case of this theorem, when $n = 1$, is well known. In this case a stable 1-stage Postnikov system is just a product of Eilenberg-MacLane spaces (with the standard infinite loop structure). An interesting fact is that the case $n = 1$ has a (partial) converse.

THEOREM (1.2)[Sl]. *Suppose X is a path connected infinite loop space of finite type, and suppose that $\mathcal{R}(p)$ acts trivially on $\tilde{H}_*(X; \mathbf{F}_p)$. Then there is a* (mod p) *homotopy equivalence*

$$f : X \to K, \tag{1.1}$$

where K is a product of Eilenberg-MacLane spaces.

It is not true in general in Theorem (1.2) that f is an infinite loop map if K is given the standard infinite loop structure. The reason for this is the existence of obstructions to delooping maps which are not associated to (primary) Dyer-Lashof operations (see [Sl] for a discussion of this). However, the map f does carry enough structure to insure the preservation of Dyer-Lashof operations.

Given Theorem (1.2), it is natural to ask whether there is a similar converse to Theorem (1.1) when $n \geq 2$. Thus we have the following question.

QUESTION (1.3). *Suppose X is a path connected infinite loop space, and suppose that each (mod p) Dyer-Lashof operation, Q^I, for which $\ell(I) \geq n$, acts trivially on $\tilde{H}_*(X; \mathbf{F}_p)$. Then is there a* (mod p) *homotopy equivalence*

$$f : X \to E, \tag{1.2}$$

where E is a stable n-stage Postnikov system, such that f_* is an isomorphism over $\mathcal{R}(p)$?

The main result of this paper is that the answer to Question (1.3) is no. For simplicity, only the case $p = 2$ is addressed here. The counterexample given has obvious analogues when p is odd.

THEOREM (1.4) (Main Theorem). *Let X be the zero space of the following 3-stage stable Postnikov system.*

$$(1.3)\qquad \begin{array}{ccccc} \Sigma^8\mathbf{HZ}/2 & \xrightarrow{j_2} & \mathbf{X} & & \\ & & \big\downarrow p_2 & & \\ \Sigma^2\mathbf{HZ}/2 & \xrightarrow{j_1} & \mathbf{Y} & \xrightarrow{\mathrm{Sq}^5\phi_{11}} & \Sigma^9\mathbf{HZ}/2 \\ & & \big\downarrow p_1 & & \\ & & \Sigma\mathbf{HZ} & \xrightarrow{\mathrm{Sq}^2} & \Sigma^3\mathbf{HZ}/2 \end{array}$$

Following Adams [Ad], ϕ_{11} is the stable secondary operation associated to the relation $\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ *(on integral classes). Then*

(i) *(mod 2)Dyer-Lashof operations of length 1 act nontrivially, and operations of length ≥ 2 act trivially on $\tilde{H}_*(X)$;*
(ii) *there is no stable 2-stage Postnikov system E with a* (mod 2) *homotopy equivalence $f : X \to E$ preserving Dyer-Lashof operations.*

Before giving the proof, there is a comment that should be made. The counterexample of this theorem is somewhat (and deliberately) artificial. Most readers will quickly realize that there are in fact many counterexamples to Question (1.3). The one given is just the simplest that I could find.

§2. Proof of the Main Theorem

Throughout the proof, the coefficients for H^* are understood to be $\mathbf{F}_2$. In order to prove (i), it suffices to show that there is at least one nontrivial operation on $\tilde{H}_*(X)$, and that operations of length equal to 2 act trivially.

Now it is easy to see that the zero space $X = \Omega^\infty\mathbf{X}$ has the homotopy type of a product of Eilenberg-MacLane spaces; specifically

$$X \simeq K(\mathbf{Z},1) \times K(\mathbf{Z}/2,2) \times K(\mathbf{Z}/2,8). \tag{2.1}$$

Consider the adjoint action of the Dyer-Lashof algebra on $H^*(X)$. This is given by the formula ($x \in H^*(X)$, $y \in H_*(X)$)

$$\langle Q^I_* x, y\rangle = \langle x, Q^I y\rangle. \tag{2.2}$$

May's result (Theorem (1.1)) implies that operations of length 2 act trivially on the elements of $\tilde{H}^*(X)$ which come from $K(\mathbf{Z},1)$ and $K(\mathbf{Z}/2,2)$. The Nishida relations [Ni], [CLM] imply that the value of operations applied to elements of $H^*(X)$ which come from $K(\mathbf{Z}/2,8)$ are completely determined by their action on the fundamental class ι_8. Specifically, if we show that operations of length 2 act trivially on ι_8, then the Nishida relations imply that they act trivially on $\tilde{H}^*(K(\mathbf{Z}/2,8))$, and hence on all of $\tilde{H}^*(X)$.

Recall that the quadratic construction on X is defined as

$$D_2X = E\Sigma_2^+ \wedge_{\Sigma_2} (X \wedge X), \tag{2.3}$$

and that Dyer-Lashof operations are determined by a structure map

$$\theta : \Sigma^\infty D_2X \to \Sigma^\infty X. \tag{2.4}$$

$H^*(D_2X)$ is generated as a vector space by classes $e_i \otimes x^2$ (for $i \geq 0$) and $e_0 \otimes \langle x, y\rangle$, where e_i corresponds to the generator of $H^i(B\Sigma_2)$, $x, y \in H^*(X)$, and $\langle x, y\rangle = x \otimes y + y \otimes x$.

Now it follows from standard arguments (e.g. using the Eilenberg-Moore spectral sequence) that

$$\begin{aligned} \theta^*(\iota_2) &= e_0 \otimes \iota_1^2 \\ \theta^*(\iota_8) &= e_0 \otimes (\mathrm{Sq}^2\iota_2)^2. \end{aligned} \tag{2.5}$$

The Nishida relations and naturality imply

$$\begin{aligned} \theta^*(\mathrm{Sq}^2\iota_2) &= \mathrm{Sq}^2\theta^*(\iota_2) \\ &= \mathrm{Sq}^2(e_0 \otimes \iota_1^2) \\ &= e_0 \otimes (\mathrm{Sq}^1\iota_1)^2 \\ &= 0. \end{aligned} \tag{2.6}$$

Therefore the only length 1 Dyer-Lashof operation which is nontrivial on ι_8 is

$$Q_*^4(\iota_8) = \mathrm{Sq}^2\iota_2, \tag{2.7}$$

and any Dyer-Lashof operation applied to $\mathrm{Sq}^2\iota_2$ must give zero. It follows that operations of length 2 are trivial on ι_8, and hence (i) holds.

To prove (ii), suppose that there is a stable 2-stage Postnikov system E for which there is a (mod 2) homotopy equivalence $f : X \to E$ preserving Dyer-Lashof operations. We show that this leads to a contradiction.

Since f is a homotopy equivalence, $E \simeq K(\mathbf{Z},1) \times K(\mathbf{Z}/2,2) \times K(\mathbf{Z}/2,8)$. Because E is a 2-stage stable Postnikov system, there is a fibration sequence of infinite loop spaces and infinite loop maps

$$K_1 \xrightarrow{j} E \xrightarrow{p} K_0, \tag{2.8}$$

where K_0 and K_1 are products of Eilenberg-MacLane spaces with the standard infinite loop structures.

Now formula (2.5) implies that in $H^*(X)$

$$Q^1_*(\iota_2) = \iota_1;\qquad (2.9)$$
$$Q^4_*(\iota_8) = \mathrm{Sq}^2\iota_2.$$

Because f^* preserves Dyer-Lashof operations, the same formulas hold in $H^*(E)$. If $j^*(\iota_1) \neq 0$, then $Q^1_* j^*(\iota_2) = j^*(\iota_1) \neq 0$, which is impossible since Dyer-Lashof operations must be trivial on $\tilde{H}^*(K_1)$. Therefore $\iota_1 \in im\, p^*$. Similarly, if either ι_2 or ι_8 were in the image of p^*, this would force a nontrivial Dyer-Lashof operation on $\tilde{H}^*(K_0)$, which would be a contradiction.

Since E has only 3 nonvanishing homotopy groups, and the fundamental classes (in dimensions 1, 2 and 8) of $H_*(E)$ are in the image of the mod 2 Hurewicz homomorphism, the long exact homotopy sequence coming from (2.8) implies that $j^*(\iota_2)$ and $j^*(\iota_8)$ must be nontrivial, and $K_0 = K(\mathbf{Z}, 1)$ and $K_1 = K(\mathbf{Z}/2, 2) \times K(\mathbf{Z}/2, 8)$. But then in $H^*(K_1)$ we must have $Q^4_*(\iota_8) = \mathrm{Sq}^2\iota_2$, which is a contradiction (since K_1 has the standard infinite loop structure). This completes the proof of (ii).

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