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EXPLICIT BRAUER INDUCTION AND SHINTANI DESCENT

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1. Explicit Brauer Induction

In this section we shall recall the Explicit Brauer Induction homomorphism, a_G. The first Explicit Brauer Induction map appeared in [9]. There are now several Explicit Brauer Induction maps ([1], [9], [15]), which are all related [3] and may be constructed either algebraically or topologically. The topological formula is given in (1.11) and the algebraic formula in (1.20).

For related literature the reader is referred to ([1], [3], [9], [10], [11], [12], [13], [14], [15], [16]).

We must begin with some definitions.

Definition (1.1). Let G be a finite group. Let $R_+(G)$ denote the free abelian group on G-conjugacy classes of characters, $\varphi : H \to C^*$, where $H \leq G$. We shall denote this character by (H, φ) and its G-conjugacy class by $(H, \varphi)^G \in R_+(G)$.

If $J \leq G$ we define a *restriction homomorphism*

$$\mathrm{Res}_J^G : R_+(G) \to R_+(J) \tag{1.2}$$

by the double coset formula

$$\mathrm{Res}_J^G((H, \varphi)^G) = \Sigma_{z \in J \backslash G/H} (J \cap zHz^{-1}, (z^{-1})^*(\varphi))^J \tag{1.3}$$

where $(z^{-1})^*(\varphi)(u) = \varphi(z^{-1})uz) \in C^*$.

If $\pi : J \to G$ is a surjection then we define

$$\pi^* : R_+(G) \to R_+(J) \tag{1.4}$$

by $\pi^*((H, \varphi)^G) = (\pi^{-1}(H), \varphi\pi)^J$.

By means of (1.3)–(1.4) we may define $f^* : R_+(G) \to R_+(J)$ for any $f : J \to G$ by factorizing f as $f : J \to \mathrm{im}(f) \subset G$ and setting

$$f^* = \pi^* \mathrm{Res}_{\mathrm{im}(f)}^G : R_+(G) \to R_+(\mathrm{im}(f)) \to R_+(J). \tag{1.5}$$

One may also define an induction map, $\mathrm{Ind}_J^G : R_+(J) \to R_+(G)$, and a product which makes $R_+(G)$ into a ring-valued functor satisfying Frobenius reciprocity.

Define a (surjective) homomorphism

$$b_G : R_+(G) \to R(G) \quad \text{by} \tag{1.6}$$

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$$b_G((H,\varphi)^G) = \mathrm{Ind}_H^G(\varphi).$$

This is a natural ring homorphism.

(1.7). The objective of this section is to describe the construction of a natural homomorphism

$$a_G : R(G) \to R_+(G)$$

such that $b_G a_G = 1$. This amounts to a canonical form for Brauer induction.

The first canonical form was given in [9]. The method was based upon the group action of G on $U(n)/NT^n$ via a unitary representation, $\rho : G \to U(n)$. The details, together with several applications, are given at length in [10] and further elaborated upon in [11]. The topological procedure of [9] automatically gives a functional association of an element of $R_+(G)$ to a representation. Furthermore, the simplicity of the group action in the one-dimensional case ensures that a one-dimensional representation, $\varphi : G \to C^*$, is associated with $(G,\varphi)^G \in R_+(G)$. In this section we will follow the method, due to R. Boltje [1], which starts by taking these two properties as axioms.

(1.8) Axioms for a_G

(i) For $H \leq G$ the following diagram commutes.

$$\begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ \Big\downarrow{\scriptstyle \mathrm{Res}_H^G} & & \Big\downarrow{\scriptstyle \mathrm{Res}_H^G} \\ R(H) & \xrightarrow{a_H} & R_+(H) \end{array}$$

(ii) Let $\rho : G \to GL_n(C)$ be a representation and suppose that

$$a_G(\rho) = \sum \alpha_{(H,\varphi)^G}(H,\varphi)^G \in R_+(G)$$

then $\alpha_{(G,\varphi)^G} = \langle \rho, \varphi \rangle_G$ (the Schur inner product) for each $(H,\varphi)^G$ such that $H = G$.

THEOREM (1.9). *There is at most one family of homomorphims, a_G, satisfying the axioms of (1.8).*

(1.10). Symond's description of a_G [16].

In [16] one finds the following topological construction, which is similar in flavour to the construction of [9]. Given $v : G \to GL(V)$ we may let G act, via v on $P(V)$, the projective space of V. Triangulate $P(V)$ so that G acts simplicially on $P(V)$. For each simplex, σ, of $G\backslash P(V)$ let $H(\sigma)$ denote the stabiliser of a simplex, $\widetilde{\sigma}$, chosen above σ. The points of $\widetilde{\sigma}$ correspond to lines in V which are preserved by $H(\sigma)$. Let $\varphi_\sigma : H(\sigma) \to C^*$ be the resulting one-dimensional representation, given by $H(\sigma)$ acting on one of these lines.

Symonds defines [16]

$$(1.11) \qquad L_G(\nu) = \Sigma_{\sigma \in G\backslash P(V)}(-1)^{\dim(\sigma)}(H(\sigma), \varphi_\sigma)^G \in R_+(G).$$

THEOREM (1.12). *In the notation of (1.8),*

$$L_G(\nu) = a_G(\nu)$$

for all representations, $\nu \in R(G)$.

Definition (1.13) Let $\mathcal{M}$ be a finite partially ordered set (a poset). The *Möbius function* of $\mathcal{M}$ is an integer-valued function, μ, on $\mathcal{M} \times \mathcal{M}$ which is defined in the following manner. A *chain of length i* in $\mathcal{M}$ is a totally ordered subset of elements of $\mathcal{M}$,

$$(1.14) \qquad M_0 \lneqq M_1 \lneqq \ldots \lneqq M_i.$$

We define $\mu_{A,B}$, for $A, B \in \mathcal{M}$, by

$$(1.15) \quad \begin{cases} \mu_{A,B} \\ = \Sigma_i(-1)^i \#\{\text{chains of length } i \text{ with } (M_0 = A, M_i = B \text{ in } (1.14)\}. \end{cases}$$

(1.16). The Poset $\mathcal{M}_G$

Let G be any finite group and denote by $\mathcal{M}_G$ the set of characters on subgroups, (H, φ), where $H \leq G$ and $\varphi : H \to C^*$. $\mathcal{M}_G$ is a *poset* if we define the partial ordering by

$$(1.17) \qquad \begin{cases} (H, \varphi) \leq (H', \varphi') \\ \text{if and only if} \\ H \leq H' \text{ and } \operatorname{Res}_H^{H'}(\varphi') = \varphi. \end{cases}$$

In addition, G acts on $\mathcal{M}_G$ by the formula

$$(1.18) \qquad g(H, \varphi) = (gHg^{-1}, (g^{-1})^*(\varphi))(g \in G)$$

where $(g^{-1})^*(\varphi)(u) = \varphi(g^{-1}ug)$.

Note that $R_+(G)$ is the free abelian group on the elements of the orbit space, $\mathcal{M}_G/G$.

(1.19). The formula for a_G in terms of Möbius functions.

Let $\mu^{\mathcal{M}_G}$ denote the Möbius function for the poset, $\mathcal{M}_G$, of pairs, (H, φ), of (1.16).

THEOREM (1.20). *The homomorphism*

$$a_G : R(G) \to R_+(G)$$

of (1.9) is given by the formula

$$a_G(\rho) = \#(G)^{-1} \sum_{\substack{(H,\varphi) \leq (H',\varphi') \\ \text{in } \mathcal{M}_G}} \#(H)\mu^{\mathcal{M}_G}_{(H,\varphi),(H',\varphi')}\langle \varphi', \mathrm{Res}^G_{H'}(\rho)\rangle (H,\varphi)^G.$$

2. Irreducible representation of GL_2F_q

In this section we shall recall the description of all the irreducible representations of GL_2F_q. All such irreducible representations are well-known. In fact, all the irreducible representations of GL_nF_q were described in ([5]; see also [7] Chapter IV) for all values of n and q. However, for completeness and convenience, we shall recall here the explicit construction of the irreducible representations of GL_2F_q. We shall begin with the *cuspidal* or *Weil representations* which are the most difficult ones to construct. These representations are originally due to A. Weil. The construction works in greater generality than we will need. For example, in ([6], p.122) the Weil representation is described for the case in which F_q is replaced by a local field.

(2.1). Let F be any field. Define te *Borel subgroup*, $B \leq GL_2F$, to be

$$B = \{X \in GL_2F | x = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\}.$$

Define the *unitriangular subgroup*, $U \leq B$, to be

$$U = Y \in B | Y = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\}.$$

Define $w \in SL_2F$ to be given by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The *Bruhat decomposition* of GL_2F takes the form

$$GL_2F = B \sqcup BwU.$$

This is elementary in the case of 2×2 matrices.

PROPOSITION (2.2) *Let F be any field then GL_2F is generated by matrices of the form* $(\alpha, \delta \in F^*, u \in F)$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

subject to the follwing relations:

(i) $$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha u \delta^{-1} \\ 0 & 1 \end{pmatrix},$$

(ii) $$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix},$$

(iii) $$w\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}w^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

(iv) $$w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}w = \begin{pmatrix} -u^{-1} & 0 \\ 0 & -u \end{pmatrix}\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}w\begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix},$$

and (v) $w^4 = 1$.

(2.3). We will now discuss the $(q-1)$-dimensional complex vector space upon which we are going to inflict the Weil representation of GL_2F_q. We will require some preliminary notation.

Let F_{q^2} denote the field of order q^2 so that the Galois group, $G(F_{q^2}/F_q)$, is cyclic of order two generated by the Frobenius automorphism

$$F : F_{q^2} \to F_{q^2}$$

given by $F(z) = z^q$ for all $z \in F_{q^2}$.

In order to construct a Weil representation we shall need a character of the form

$$\Theta : F_{q^2}^* \to C^*$$

which we shall generally assume to be distinct from its conjugate by the Frobenius, $\Theta \neq F^*(\Theta)$ where $F^*(\Theta)(z) = \Theta(F(z))$.

Let $\mathcal{H}$ denote the following complex vector space

$$\mathcal{H} = \left\{f : F_{q^2}^* \to C | f(t^{-1}x) = \Theta(t)f(x) \text{ if } N(t) = 1\right\}$$

where $N = N_{F_{q^2}/F_q} : F_{q^2}^* \to F_q^*$ is the norm.

Let H denote the abelian subgroup which consists of matrices whose diagonal entries are equal. Hence there is an isomorphism of the form

$$\gamma : H \xrightarrow{\cong} F_q^* \times F_q$$

given by

$$\gamma\begin{pmatrix} z & y \\ 0 & z \end{pmatrix} = (z, yz^{-1}).$$

Define the *additive character* of F_q to be the homomorphism

$$\Psi = \Psi_{F_q} : F_q \to C^* \tag{2.4}$$

by the formula ($\mathrm{char}(F_q) = p$)

$$\Psi_{F_q}(y) = \exp(2\pi i(\mathrm{Trace}_{F_q/F_p}(y))/p).$$

Hence we may obtain a one-dimensional representation of H, denoted by $\Theta \otimes \Psi$, by the composition

$$\Theta \otimes \Psi = (\Theta \otimes \Psi_{F_q})\gamma : H \to C^*.$$

By induction we obtain a $(q-1)$-dimensional induced representation of B, $\mathrm{Ind}_H^B(\Theta \otimes \Psi)$. We may identify the underlying vector space of this as a mapping space in the following manner. There is an isomorphism

$$\lambda : W \xrightarrow{\cong} \mathrm{Ind}_H^B(\Theta \otimes \Psi)$$

where

$$W = \{g : B \to C | g(X \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}) = \overline{\Theta}(\alpha)\overline{\Psi}(\beta/\alpha)g(X)\}$$

where $\overline{f}$ denotes the complex conjugate function. Explicitly λ is given by

$$\lambda(g) = \sum_{X \in B/H} X \otimes g(X) \in C[B] \otimes_{C[H]} C_{\Theta\otimes\Psi}$$

where $C_{\Theta\otimes\Psi}$ denotes the complex numbers with H-action via $\Theta \otimes \Psi$. λ is well-defined since

$$\begin{aligned} &X \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \otimes g(X \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}) \\ &= X \otimes \Theta(\alpha)\Psi(\beta/\alpha) g X \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}) \\ &= X \otimes g(X). \end{aligned}$$

Define an action of B on W by the formula ($g \in W, X \in B$)

$$(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} g)(X) = g(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}^{-1} X) = g(\begin{pmatrix} \alpha^{-1} & -\beta(\alpha\delta)^{-1} \\ 0 & \delta^{-1} \end{pmatrix} X).$$

PROPOSITION (2.5). *With this B-action on W*

$$\lambda : W \xrightarrow{\cong} \mathrm{Ind}_H^B(\Theta \otimes \Psi)$$

is an isomorphism of B-representations.

(2.6). Since the matrices of the type

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

form a set of coset representatives for B/H we may define an isomorphism of vector spaces

$$A : \mathcal{H} \to W$$

by the formula, where $b \in F_{q^2}^*$ and $N_{F_{q^2}/F_q}(b) = \alpha^{-1}$,

$$A(h)\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) = \Theta(b)h(b). \tag{2.7}$$

Notice that (2.7) is well-defined because, if $N_{F_{q^2}/F_q}(t) = 1$,

$$\Theta(tb)h(tb) = \Theta(t)\Theta(b)\Theta(t)^{-1}h(b).$$

PROPOSITION (2.8). *Define a B-action on $\mathcal{H}$ by the formula*

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}(h) = \{x \mapsto \Theta(\delta)\Psi(\beta N(x)\delta^{-1})h(\lambda x)\Theta(\lambda)\}$$

where $N = N_{F_{q^2}/F_q}$ is the norm and $N(\lambda) = \alpha\delta^{-1}$. Then, with this B-action, A yields an isomorphism of B-representations

$$A : \mathcal{H} \xrightarrow{\cong} \mathrm{Ind}_H^B(\Theta \otimes \Psi).$$

Remark (2.9). From (2.8) it is not difficult to show that the B-action on $\mathcal{H}$ is given by

$$\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} h\right)(x) = \Psi(uN(x))h(x),$$

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h\right)(x) = h(\lambda x)\Theta(\lambda),$$

where $N(\lambda) = \alpha$ and

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} h\right)(x) = h(xF(\mu)^{-1}\Theta(\mu)$$

where $N(\mu) = \delta$ since, if $N(\lambda) = \delta^{-1}$,

$$\begin{aligned} \Theta(\delta)h(\lambda x)\Theta(\lambda) &= \Theta(\delta^{-1})\Theta(F(\delta)^{-1})h(\lambda x)\Theta(\lambda) \\ &= h(xF(\mu)^{-1})\Theta(\mu). \end{aligned}$$

These formulae coincide with those of ([6] p.122).

Definition (2.10). The Fourier Transform on $\mathcal{H}$.

In the notation of (2.1) and (2.3) suppose that $f \in \mathcal{H}$. We define the *Fourier transform*, $\widehat{f} \in \mathcal{H}$, of f by means of the formula

$$\widehat{f}(z) = -q^{-1}\Sigma_{y \in F^*_{q^2}} f(y)\Psi_{F_q}(yF(z) + zF(y)) \tag{2.12}$$

where Ψ_{F_q} is the additive character of (2.4).

LEMMA (2.12) *The map which sends $f \in \mathcal{H}$ to its Fourier transform is a C-linear endomorphism of order four.*

Definition (2.13). Let $\Theta : F^*_{q^2} \to C^*$ be a non-trivial character, as in (2.3). The following three formulae characterise the Weil representation associated to Θ

$$r(\Theta) : GL_2F_q \to \mathrm{Aut}_C(\mathcal{H}) \cong GL_{q-1}C$$

(i)

$$(r(\Theta)(w)f)(x) = \widehat{f}(x) \quad (f \in \mathcal{H}, x \in F^*_{q^2})$$

where w is as in (2.1), (ii)

$$(r(\Theta)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f)(x) = \Psi_{F_q}(uN_{F^2_q/F_q}(x))f(x)$$

and

(*iii*)

$$(r(\Theta)\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} f)(x) = \Theta(\beta)f(\beta x)$$

where $\alpha \in F_q, \beta \in F_{q^2}$ and $N_{F_{q^2}/F_q}(\beta) = \alpha$.

THEOREM (2.14). *The formulae of (2.13) characterise a unique, well-defined $(q-1)$-dimensional, irreducible representation, $r(\Theta)$, of GL_2F_q.*

(2.15). We shall now construct the remaining irreducible representations of $GL_2 F_q$. Suppose that we are given characters of the form

$$\chi, \chi_1, \chi_2 : F^*_q \to C^*$$

then we clearly have a one-dimensional representation, $L(\chi)$, given by

$$L(\chi) = \chi \cdot \det : GL_2F_q \xrightarrow{\det} F^*_q \xrightarrow{\chi} C^*. \tag{2.16}$$

If χ_1 and χ_2 are *distinct* define

$$\mathrm{Inf}^B_T(\chi_1 \otimes \chi_2) : B \to C^*$$

by inflating $\chi_1 \otimes \chi_2$ from the diagonal torus, T, to the Borel subgroup, B. That is

$$\mathrm{Inf}_T^B(\chi_1 \otimes \chi_2)(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}) = \chi_1(\alpha)\chi_2(\delta).$$

Define a $(q+1)$-dimensional representation, $R(\chi_1, \chi_2)$, by

$$R(\chi_1, \chi_2) = \mathrm{Ind}_B^{GL_2F_q}(\mathrm{Inf}_T^B(\chi_1 \otimes \chi_2)). \tag{2.17}$$

When $\chi = \chi_1 = \chi_2$ we have

$$\mathrm{Inf}_T^B(\chi \otimes \chi) = \mathrm{Res}_B^{GL_2F_q}(L(\chi)) : B \rightarrow C^*$$

so that there is a canonical surjection of the form

$$\mathrm{Ind}_B^{GL_2F_q}(\mathrm{Inf}_T^B(\chi \otimes \chi)) \rightarrow \mathrm{Ind}_{GL_2F_q}^{GL_2F_q}(L(\chi)) = L(\chi).$$

Therefore we may define a q-dimensional representation, $S(\chi)$, by means of the following short exact sequence of representations (which is *split*, by semisimplicity)

$$0 \rightarrow S(\chi) \rightarrow \mathrm{Ind}_B^{GL_2F_q}(\mathrm{Inf}_T^B(\chi \otimes \chi)) \rightarrow L(\chi) \rightarrow 0. \tag{2.18}$$

THEOREM (2.19). *A complete list of all the irreducible representations of GL_2F_q is given by*

(i) $L(\chi)$ of (2.16) for $\chi : F_q^* \rightarrow C^*$,

(ii) $S(\chi)$ of (2.18) for $\chi : F_q^* \rightarrow C^*$,

(iii) $R(\chi_1, \chi_2) = R(\chi_2, \chi_1)$ of (2.17) for any pair of distinct characters $\chi_1, \chi_2 : F_q^* \rightarrow C^*$ and

(iv) $r(\Theta) = r(F^*(\Theta))$ of (2.13) for any character $\Theta : F_{q^2}^* \rightarrow C^*$ which is distinct from its Frobenius conjugate, $F^*(\Theta)$.

(2.20). For future use let us record the conjugacy class information concerning GL_2F_q.

The conjugacy class of a matrix, $X \in GL_2F_q$ is determined by its minimal polynomial. The minimal polinomial of X must have degree one or two. The representatives of each conjugacy class, together with their minimal polynomial and the number of elements within each class, are tabulated below.

(2.21) Conjugacy Classes in GL_2F_q

Type	Minimal Polynomial	Conjugacy Class Representative	Number in Class
I	$(t-\alpha)(t-\beta)$ $\alpha \neq \beta \in F^*q$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$q(q+1)$
II	$(t-\alpha)^2$ $\alpha \in F_q^*$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	q^2-1
III	$(t-\alpha)$ $\alpha \in F_q^*$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	1
IV	$t^2-(x+F(x))t+xF(x)$ $F(x) \neq x \in F_{q^2}^*$	$\begin{pmatrix} 0 & -xF(x) \\ 1 & x+F(x) \end{pmatrix}$	q^2-q

THEOREM (2.22). *With the notation of (2.21) and (2.19) the character values of the irreducible representations of GL_2F_q are given by the following table:*

Type	$L(x)$	$R(\chi_1,\chi_2)$	$S(\chi)$	$r(\Theta)$
I	$\chi(\alpha\beta)$	$\chi_1(\alpha)\chi_2(\beta)+\chi_2(\alpha)\chi_1(\beta)$	$\chi(\alpha\beta)$	0
II	$\chi(\alpha)^2$	$\chi_1(\alpha)\chi_2(\alpha)$	0	$-\Theta(\alpha)$
III	$\chi(\alpha)^2$	$(q+1)\chi_1(\alpha)\chi_2(\alpha)$	$q\chi(\alpha)^2$	$(q-1)\Theta(\alpha)$
IV	$\chi(N(x))$	0	$-\chi(N(x))$	$-\{\Theta(x)+\Theta(F(x))\}$

where $N = N_{F_{q^2}/F_q}$ denotes the norm.

3. The Shintani correspondence

In this section we shall describe the Shintani correspondence for GL_2F_q (see (3.11)) in terms of the maximal self-normalising elements of $\mathcal{M}_{GL_2F_q}$ which appear in the Explicit Brauer Induction formula for the irreducible representation under discussion. Then main result is to be found in (3.25).

(3.1). Maximal Pairs in $\mathcal{M}_{GL_2F_q}$

As in (1.16)–(1.17) let $\mathcal{M}_{GL_2F_q}$ denote the poset of pairs (J,φ) with $J \leq GL_2F_q$ and $\varphi : J \to C^*$. From (1.20), if $v \in R(GL_2F_q)$ and (J,φ) is *maximal* in $\mathcal{M}_{GL_2F_q}$, then the coefficient of $(J,\varphi)^{GL_2F_q}$ in $a_{GL_2F_q}(v)$ is given by

$$(3.2)\qquad \begin{cases} \{\text{multiplicity of } (J,\varphi)^{GL_2F_q} \text{ in } a_{GL_2F_q}(v)\} \\ = [N_{GL_2F_q}(J,\varphi) : J]^{-1} < \varphi, \mathrm{Res}_J^{GL_2F_q}(v) > . \end{cases}$$

Here

$$N_{GL_2F_q}(J,\varphi) = \{X \in GL_2F_q | (XJX^{-1}, (X^{-1})^*(\varphi)) = (J,\varphi)\}.$$

In other words, it is easy to calculate the multiplicity of maximal pairs, $(J,\varphi)^{GL_2F_q}$, in $a_{GL_2F_q}(v)$. For this reason we will now introduce four types of maximal pairs in $\mathcal{M}_{GL_2F_q}$. Type A: $(GL_2F_q, \chi \cdot \det)$ for $\chi : F_q^* \to C^*$. Type B: $(H, \lambda \otimes \mu)$ where $\mu : F_q \to C^*$ is *non-trivial* and $\lambda : F_q^* \to C^*$ is any homomorphism. Here

$$\lambda : H = \{\begin{pmatrix} z & y \\ 0 & z \end{pmatrix} \in GL_2F_q\} \stackrel{\cong}{\to} F_q^* \times F_q$$

is as in (2.3). Type C: $(B, \mathrm{Inf}_T^B(\lambda_1 \otimes \lambda_2))$, in the notation of (2.17), where λ_1, λ_2: $F_q^* \to C$ are *distinct*. Type D: $(F_{q^2}^*, \rho)$ where ρ and $F^*(\rho)$ are *distinct*. Here we consider $F_{q^2}^*$ to be the cyclic subgroup generated by the matrix

$$\begin{pmatrix} 0 & -xF(x) \\ 1 & x+F(x) \end{pmatrix} \tag{3.3}$$

of (2.21) where $x \in F_{q^2}^*$ is a generator. Up to conjugation this subgroup is independent of the choice of x. On this subgroup the Frobenius map, $F \in G(F_{q^2}/F_q)$, corresponds to conjugation by the matrix

$$f = \begin{pmatrix} 1 & x+F(x) \\ 0 & -1 \end{pmatrix}. \tag{3.4}$$

This is seen as follows. With respect to the F_q-basis, $\{1, x\}$ of F_{q^2} multiplication by x is represented by the matrix of (3.3). However

$$\begin{aligned} &\begin{pmatrix} 1 & x+F(x) \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -xF(x) \\ 1 & x+F(x) \end{pmatrix}\begin{pmatrix} 1 & x+F(x) \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} x+F(x) & -xF(x)+(x+F(x))^2 \\ -1 & -x-F(x) \end{pmatrix}\begin{pmatrix} 1 & x+F(x) \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} x+F(x) & xF(x) \\ -1 & 0 \end{pmatrix} \end{aligned}$$

which represents multiplication by $F(x)$ with respect to this basis.

PROPOSITION (3.5). *Each of the pairs $(J, \varphi) \in \mathcal{M}_{GL_2F_q}$, listed in types A/D of (3.1), is maximal. In addition, for each of types A/D,*

$$N_{GL_2F_q}(J, \varphi) = J.$$

Proof. The result is obvious for type A, $(GL_2F_q, \chi \cdot \det)$.

From the classification of maximal subgroups of SL_2F_q, which is given in ([4] p.286, §§262), it is straightforward to see that any proper subgroup of GL_2F_q which contains H must lie in $B = N_{GL_2F_q}H$. However,

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\begin{pmatrix} z & y \\ 0 & z \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \alpha z & \alpha y + \beta z \\ 0 & \delta z \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\beta\delta^{-1} \\ 0 & \delta^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} z & \alpha z \delta^{-1} \\ 0 & z \end{pmatrix}$$

Hence the action of this matrix on $H \cong F_q^* \times F_q$ is to multiply the second coordinate by $\alpha\delta^{-1}$. However μ and $\mu(\alpha\delta^{-1}-)$ are distinct unless $\alpha = \delta$, which shows that $N_{GL_2F_q}(H, \lambda \otimes \mu) \leq H$ and proves the result for type B.

The result for type C is immediate from the facts that $N_{GL_2F_q}B = B$ and that the conjugate, $xvx^{-1}(x, v \in B)$, has the same diagonal as v.

From the classification of maximal subgroups of SL_2F_q ([4] p.286, §§262) one readily finds that $N_{GL_2F_q}F_{q^2}^* = \langle F_{q^2}^*, f\rangle$ where f is as in (3.4). However, conjugation by f induces the Frobenius map of $F_{q^2}^*$ so that $f \notin N_{GL_2F_q}(F_{q^2}^*, \rho)$ which easily yields the result for type D.

COROLLARY (3.6). *Suppose that $g \in GL_2F_q$ and that $(J, \varphi) \in \mathcal{M}_{GL_2F_1}$ is one of the maximal pairs of type A/D, as in (3.5). If $(gJg^{-1}, (g^{-1})^*(\varphi)) = (J, \varphi_1)$ then $\varphi = \varphi_1$ in the case of types A and C. For type B, $\varphi = \lambda \otimes \mu$ and $\varphi_1 = \lambda \otimes \mu(u \cdot -)$ for some $u \in F_q^*$ while, for type D, $\varphi_1 = \varphi$ or $\varphi_1 = F^*(\varphi)$ where F is the Frobenius of $G(F_{q^2}/F_q)$.*

Proof. We must have $g \in N_{GL_2F_q}J = GL_2F_q, B, B, \langle F_{q^2}^*, f\rangle$ for types A/D, respectively. From this observation the result follows easily from the computations which were used in the proof of (3.5).

Definition (3.7). Suppose that $\nu : GL_2F_q \to GL(V)$ is an *irreducible* representation. We will write

$$(3.8)\qquad \begin{aligned} a_{GL_2F_q}(\nu) \; = \; & \Sigma_r a_r(H, \lambda_r \otimes \mu_r)^{GL_2F_q} \\ & + \Sigma_s b_s(B, \mathrm{Inf}_T^B(\lambda_{1,s} \otimes \lambda_{2,s}))^{GL_2F_q} \\ & + \Sigma_t c_t(F_{q^2}^*, \rho_t)^{GL_2F_q} \\ & + \Sigma_u d_u(GL_2F_q, \chi_u \cdot \det)^{GL_2F_q} \\ & + \cdots \end{aligned}$$

to signify that the multiplicities in $a_{GL_2F_q}(\nu)$ of the maximal pairs of types A/D in (3.1) are as shown in (3.8) (the ellipsis denoting the sum of all the terms of other types). In (3.8) the sum over r, s, t, u are taken over all the terms of types A, B, C, D respectively

THEOREM (3.9). *With the notation of (2.19) and (3.8)*

(i)

$$a_{GL_2F_q}(L(\chi)) = (GL_2F_q, \chi \cdot \det)^{GL_2F_q}.$$

(ii)

$$\begin{aligned} a_{GL_2F_q}(R(\chi_1,\chi_2)) &= \Sigma_{1\neq\mu}(H,\chi_1\chi_2\otimes\mu)^{GL_2F_q} \\ &+(B,\mathrm{Inf}_T^B(\chi_1\otimes\chi_2))^{GL_2F_q} \\ &+(B,\mathrm{Inf}_T^B(\chi_2\otimes\chi_1))^{GL_2F_q} \\ &+\Sigma_{\mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\rho)=\chi_q\chi_2}(F_{q^2}^*,\rho)^{GL_2F_q} \\ &+\cdots. \end{aligned}$$

(iii)

$$\begin{aligned} a_{GL_2F_q}(S(\chi)) &= \Sigma_{1\neq\mu}(H,\chi^2\otimes\mu)^{GL_2F_q} \\ &+(B,\mathrm{Inf}_T^B(\chi\otimes\chi))^{GL_2F_q} \\ &+\Sigma_{\mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\rho)=\chi^2}(F_{q^2}^*,\rho)^{GL_2F_q} \\ &+\cdots. \end{aligned}$$

(iv)

$$\begin{aligned} a_{GL_2F_q}(r(\Theta)) &= \Sigma_{v\in F_q^*}(H,\mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\Theta)\otimes\Psi_{F_q}(v\cdot -))^{GL_2F_q} \\ &+\Sigma_{\rho\notin\{\Theta,F^*(\Theta)\},\mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\rho)=\mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\Theta)}(F_{q^2}^*,\rho)^{GL_2F_q} \\ &+\cdots. \end{aligned}$$

Proof. Part (i) follows from (1.8)(ii).

Parts (ii) and (iii) are similar and therefore we will only prove part (ii). By (3.2) and (3.5), the multiplicity of a term of type B from (3.1) in $a_{GL_2F_q}(R(\chi_1,\chi_2))$ is equal to

$$\begin{aligned} &\langle\lambda\otimes\mu,\mathrm{Res}_H^{GL_2F_q}(\mathrm{Ind}_B^{GL_2F_q}(\mathrm{Inf}_T^B(\chi_1\otimes\chi_2)))\rangle \\ &=\Sigma_{z\in H\backslash GL_2F_q/B}\langle\lambda\otimes\mu,\mathrm{Ind}_{H\cap zBz^{-1}}^H(\mathrm{Res}_{H\cap zBz^{-1}}^{Bz^{-1}}((z^{-1})^* \\ &\quad(\mathrm{Inf}_T^B(\chi_1\otimes\chi_2)))))\rangle \\ &=\Sigma_{z=1,w}\langle\lambda\otimes\mu,\mathrm{Ind}_{H\cap zBz^{-1}}^H(\mathrm{Res}_{H\cap zBz^{-1}}^{zBz^{-1}}((z^{-1})^*(\mathrm{Inf}_T^B(\chi_1\otimes\chi_2)))))\rangle \end{aligned}$$

by the Bruhat decomposition,

$$=\langle\lambda\otimes\mu,(\chi_1\chi_2\otimes 1)+\mathrm{Ind}_{F_q^*}^H(\chi_2\otimes\chi_1)\rangle$$

$$= \langle \lambda \otimes \mu, \chi_1\chi_2 \otimes \mathrm{Ind}_{F_q^*}^{H}(1)\rangle$$
$$= \begin{cases} 1 & \text{if } \lambda = \chi_1\chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

This accounts for the first part of the formula in part (ii).

By (3.2) and (3.5), the multiplicity of a term of type C from (3.1) in $a_{GL_2F_q}(R(\chi_1, \chi_2))$ is equal to

$$\begin{aligned}
&\langle \mathrm{Inf}_T^B(\lambda_1 \otimes \lambda_2), \mathrm{Res}_B^{GL_2F_q}(\mathrm{Ind}_B^{GL_2F_q}(\mathrm{Inf}_T^B(\chi_1 \otimes \chi_2)))\rangle \\
&= \langle \mathrm{Inf}_T^B(\lambda_1 \otimes \lambda_2), \mathrm{Inf}_T^B(\chi_1 \otimes \chi_2) + \mathrm{Ind}_T^B(\chi_2 \otimes \chi_1)\rangle \\
&= \langle \lambda_1, \chi_1\rangle\langle \lambda_2, \chi_2\rangle + \langle \lambda_2, \chi_1\rangle\langle \lambda_1, \chi_2\rangle \\
&= \begin{cases} 1 & \text{if } \{\lambda_1, \lambda_2\} = \{\chi_1, \chi_2\} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This accounts for the remaining part of the formula in part (ii).

Clearly there are no terms of type A in $a_{GL_2F_q}(R(\chi_1, \chi_2))$ and the multiplicity of a term of type D is equal to

$$\begin{aligned}
&\langle \rho, \mathrm{Res}_{F_{q^2}^*}^{GL_2F_q}(R(\chi_1, \chi_2))\rangle \\
&= (q^2-q)^{-1}\Sigma_{x \in F_{q^2}^* - F_q^*}\overline{\rho}(x) \cdot 0 + (q^2-1)^{-1}\Sigma_{x \in F_q^*}(q+1)\overline{\rho}(x)\chi_1(x)\chi_2(x) \\
&= \langle \mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\rho), \chi_1\chi_2\rangle
\end{aligned}$$

by the character values of (2.22). This completes the proof of part (ii).

For part (iv) we observe that, since $r(\Theta)$ is irreducible, there can be no terms of type A. Also there can be no terms of type C, since

$$\begin{aligned}
&\langle \mathrm{Inf}_T^B(\lambda_1 \otimes \lambda_2), \mathrm{Ind}_H^B(\Theta \otimes \Psi_{F_q})\rangle \\
&= \langle \lambda_1\lambda_2 \otimes 1, \Theta \otimes \Psi_{F_q}\rangle \\
&= 0.
\end{aligned}$$

By (3.2) and (3.5), the multiplicity of a term of type B from (3.1) in $a_{GL_2F_q}(r(\Theta))$ is equal to

$$\begin{aligned}
&\langle \lambda \otimes \mu, \mathrm{Res}_H^{GL_2F_q}(r(\Theta))\rangle \\
&= \langle \lambda \otimes \mu, \mathrm{Res}_H^B(\mathrm{Ind}_H^B(\Theta \otimes \Psi))\rangle \\
&= \Sigma_{z \in B/H}\langle \lambda \otimes \mu, (z^{-1})^*(\Theta \otimes \Psi)\rangle \\
&= \Sigma_{v \in F_q^*}\langle \lambda \otimes \mu, \mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\Theta)\Psi_{F_q}(v \cdot -)\rangle
\end{aligned}$$

which accounts for the first part of the formula in part (iv).

Finally, by (3.2) and (3.5), the multiplicity of a term of type D from (3.1) in $a_{GL_2F_q}(r(\Theta))$ is equal to

$$\begin{aligned}
&\langle \rho, \mathrm{Res}^{GL_2F_q}_{F^*_{q^2}}(r(\Theta))\rangle \\
&= (q^2-1)^{-1}\Sigma_{x\in F^*_q}\overline{\rho}(x)(q-1)\Theta(x) - (q^2-1)^{-1}\Sigma_{x\in F^*_{q^2}-F^*_q}\overline{\rho}(x)\Theta(x)+\Theta(F(x)) \\
&\quad \text{by (2.22)} \\
&= (q+1)(q^2-1)^{-1}\Sigma_{x\in F^*_q}\overline{\rho}(x)\Theta(x) - \langle\rho,\Theta\rangle - \langle\rho, F^*(\Theta)\rangle \\
&= \langle \mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\rho), \mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\Theta)\rangle - \langle\rho,\Theta\rangle - \langle\rho, F^*(\Theta)\rangle
\end{aligned}$$

which accounts for the remaining terms in the formula for part (iv) and completes the proof.

COROLLARY (3.10) *The irreducible representations, v, of GL_2F_q are uniquely characterised by the terms of types A/D, in the terminology of (3.1), which occur in the Explicit Brauer Induction formula*

$$a_{GL_2F_q}(\nu) \in R_+(GL_2F_q).$$

These (maximal) terms are given by the formulae of (3.9).

Proof. This follows easily by inspection of the formulae of (3.9). For example, the Weil representations are the only ones for which a term of type D appears. The type B terms in $a_{GL_2F_q}(r(\Theta))$ determine the sum over which the type D terms are taken and the characters Θ and $F^*(\Theta)$ are characterised by being the only two characters on $F^*_{q^2}$ with the prescribed restriction to F^*_q which *do not* appear in the sum. Of course, $r(\Theta) = r(F^*(\Theta))$.

(3.11). The Shintani Correspondence for GL_2F_q.

Let $\Sigma \in G(F_{q^n}/F_q)$ denote the Frobenius transformation. In [8] Shintani discovered a remarkable one-one correspondence of the following form

$$(3.12)\qquad \left\{\begin{array}{c} \{\text{irreducible representation, } \nu, \text{ of} \\ GL_mF_{q^n} \text{ fixed under } \Sigma\} \\ \updownarrow \mathrm{Sh} \\ \{\text{irreducible representations, } \mathrm{Sh}(\nu), \text{ of } GL_mF_q\}. \end{array}\right.$$

In (3.12) the Frobenius, Σ, acts via its action upon the entries of a matrix. This correspondence, which was also treated by Shintani for GL_2 of a local field, is also sometimes called *Shintani descent* or *lifting* (see [6], for example).

The correspondence of (3.12) may be characterised by means of the *Shintani norm*. For $X \in GL_m F_{q^n}$ define

$$N(X) = \Sigma^{n-1}(X)\Sigma^{n-2}(X)\dots\Sigma(X)X. \tag{3.13}$$

Although $N(X)$ lies in $GL_m F_{q^n}$, its conjugacy class contains a unique $GL_m F_q$-conjugacy class, which depends only on the conjugacy class of X. This gives a meaning to the equation

$$\mathrm{Trace}\,(\mathrm{Sh}(\nu)(N(X))) = \mathrm{Trace}\,(\nu(X)). \tag{3.14}$$

The correspondence of (3.12) is characterised by the fact (3.14) holds for all $X \in GL_m F_{q^n}$.

When $m = 1$ this correspondence is consequence of Hilbert's Theorem 90, which states that $H^1(G(L/K); L^*) = 0$. When L/K is an extension of finite fields we obtain an exact sequence of the form

$$F_{q^n}^* \xrightarrow{\Sigma/1} F_{q^n}^* \xrightarrow{N} F_q^* \longrightarrow \{1\}. \tag{3.15}$$

If $\nu : F_{q^n}^* \to C^*$ satisfies $\nu = \Sigma^*(\nu)$ then, by (3.15), there exists a unique $\mathrm{Sh}(\nu) : F_q^* \to C^*$ such that

$$\mathrm{Sh}(\nu)N(x)) = \nu(x) \ (x \in F_{q^n}^*). \tag{3.16}$$

(3.17). We shall now use Explicit Brauer Induction and (3.9)–(3.10) to describe a correspondence of type of (3.12) in the modest circumstances of $GL_2 F_{q^n}$. As it happens, our correspondence will coincide with that of (3.11) although no mention of the Shintani norm appears in our contruction. My correspondence will be effected by applying Hilbert's Theorem 90, in the sense of (3.15)–(3.16), to the maximal one-dimensional characters which appear in the fomrula for $a_{GL_2 F_{q^n}}(\nu)$.

We begin by observing that, if ν is irreducible and $\Sigma^*(\nu) = \nu$, then

$$a_{GL_2 F_{q^n}}(\nu) = a_{GL_2 F_{q^n}}(\Sigma^*(\nu)) = \Sigma^*(a_{GL_2 F_{q^n}}(\nu))$$

where $\Sigma^* : R_+(GL_2 F_{q^n}) \to R_+(GL_2 F_{q^n})$ is given by the formula

$$\Sigma^*(J, \varphi)^{GL_2 F_{q^n}} = (\Sigma(J), \varphi(\Sigma^{-1} \cdot -))^{GL_2 F_{q^n}}.$$

Since $\Sigma^*(\nu)$ is also irreducible the maximal terms of types A/D in $a_{GL_2 F_{q^n}}(\Sigma^*(\nu))$ will be obtained by applying Σ^* to the maximal terms of type A/D in $a_{GL_2 F_{q^n}}(\nu)$.

Now let us describe our contruction of the correspondence, which will be denoted by Υ.

If $\Sigma^*(L(\chi)) = L(\chi)$ then $\Sigma^*(\chi) = \chi$ and there exists a unique $\overline{\chi} : F_q^* \to C^*$ such that $\chi(z) = \overline{\chi}(N(z))$ where N is the norm. In this case we set

$$\Upsilon(L(\chi)) = L(\overline{\chi}). \tag{3.18}$$

Next suppose that $\Sigma^*(R(\chi_1, \chi_2)) = R(\chi_1, \chi_2)$ then, in (3.9)(ii),

$$\sum_{1\neq\mu}(H, \chi_1\chi_2 \otimes \mu)^{GL_2F_{q^n}} = \sum_{1\neq\mu}(H, \Sigma^*(\chi_1\chi_2) \otimes \mu)^{GL_2F_{q^n}}$$

and

$$\begin{aligned}&(B, \mathrm{Inf}_T^B(\chi_1 \otimes \chi_2))^{GL_2F_{q^n}} + (B, \mathrm{Inf}_T^B(\chi_2 \otimes \chi_1))^{GL_2F_{q^n}}\\ &= (B, \mathrm{Inf}_T^B(\Sigma^*(\chi_1) \otimes \Sigma^*(\chi_2)))^{GL_2F_{q^n}} + (B, \mathrm{Inf}_T^B(\Sigma^*(\chi_2) \otimes \Sigma^*(\chi_1)))^{GL_2F_{q^n}}.\end{aligned}$$

By (3.6), these equations imply that either

(a) $\chi = \Sigma^*(\chi_1)$ and $\chi_2 = \Sigma^*(\chi_2)$

or

(b) $\chi_1 = \Sigma^*(\chi_2)$, $\chi_2 = \Sigma^*(\chi_1)$ and $\chi_1\chi_2 = \Sigma^*(\chi_1\chi_2)$.

In case (a) there exist unique homomorphims, $\overline{\chi}_i : F_q^* \to C^*(i = 1, 2)$ such that $\chi_i(z) = \overline{\chi}_i(N(z))$ for each $i = 1, 2$. In this case we set

$$\Upsilon(R(\chi_1, \chi_2)) = R(\overline{\chi}_1, \overline{\chi}_2). \tag{3.19}$$

In case (b) we have a surjective homomorphism

$$\lambda : G(F_{q^n}/F_q) \cong Z/n \to \{\pm 1\}$$

given by $\lambda(g) = (-1)^{i-1}$ if $g(\chi_1) = \chi_i$. Hence $n = 2d$, $\mathrm{Ker}(\lambda) = G(F_{q^n}/F_{q^2})$ and each χ_i is fixed by $\mathrm{Ker}(\lambda)$. Hence there exists a unique $\overline{\chi}_1 : F_{q^2}^* \to C^*$ such that $\chi_1(z) = \overline{\chi}_1(N(z))$ where $N : F_{q^n}^* \to F_{q^2}^*$ is the norm. Also, if F generates $G(F_{q^2}/F_q)$ and $\overline{\chi}_2 = F^*(\overline{\chi}_1)$ then $\chi_2(z) = \overline{\chi}_2(N(z))$.

Notice also that, in case (b), $\Sigma^*(\chi_1\chi_2) = \chi_1\chi_2$ so that there exists a unique $\overline{\chi}_{1,2} : F_q^* \to C^*$ such that $\chi_1(z)\chi_2(z) = \overline{\chi}_{1,2}(N(z))$. In fact we have

$$\mathrm{Res}_{F_{q^2}^*}^{F_q^*}(\overline{\chi}_1) = \overline{\chi}_{1,2} = \mathrm{Res}_{F_{q^2}^*}^{F_q^*}(\overline{\chi}_2).$$

For, if $w \in F_q^*$, $v \in F_{q^2}^*$ and $r \in F_{q^n}^*$ satisfy $N(v) = w$, $N(r) = v$ then

$$\begin{aligned}
\overline{\chi}_1(w) &= \overline{\chi}_1(vF(v)) \\
&= \overline{\chi}_1(v)\overline{\chi}_1(F(v)) \\
&= \overline{\chi}_1(v)\overline{\chi}_2(v) \\
&= \overline{\chi}_1(N(r))\overline{\chi}_2(N(r)) \\
&= \chi_1(r)\chi_2(r) \\
&= \overline{\chi}_{1,2}(N(r)) \\
&= \overline{\chi}_{1,2}(w).
\end{aligned}$$

From these characters it is natural to form

$$\begin{aligned}
&\Sigma_{v \in F_q^*}(H, \overline{\chi}_{1,2} \otimes \Psi_{F_q}(v \cdot -))^{GL_2F_q} \\
&+\Sigma_{\rho \notin \{\overline{\chi}_1, \overline{\chi}_2\}\, \mathrm{Res}_{F_q^*}^{F_{q^2}^*}(\rho) = \overline{\chi}_{1,2}} (F_{q^2}^*, \rho)^{GL_2F_q} \\
&+ \cdots .
\end{aligned}$$

in $R_+(GL_2F_q)$. These are the maximal terms of types A/D in $a_{GL_2F_q}(r(\overline{\chi}_1))$ and therefore, in case (b), we set

$$\Upsilon(R(\chi_1, \chi_2)) = r(\overline{\chi}_1) = r(\overline{\chi}_2). \tag{3.20}$$

If $\Sigma^*(S(\chi)) = S(\chi)$ then, as in case (a) above, we see that $\Sigma^*(\chi) = \chi$ and that there exists a unique $\overline{\chi} : F_q^* \to C^*$ such that $\chi(z) = \overline{\chi}(N(z))$. In this case we set

$$\Upsilon(S(\chi)) = S(\overline{\chi}). \tag{3.21}$$

Finally suppose that $\Sigma^*(r(\Theta)) = r(\Theta)$. Hence, by (3.9)(iv),

$$\Sigma_{v \in F_{q^n}^*}(H, \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\Sigma^*(\Theta)) \otimes \Psi_{F_{q^n}}(v \cdot -))^{GL_2F_{q^n}}$$

$$= \Sigma_{v \in F_{q^n}^*}(H, \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\Theta) \otimes \Psi_{F_{q^n}}(v \cdot -))^{GL_2F_{q^n}}$$

and

$$\Sigma_{\rho \notin \{\Theta, F^*(\Theta)\}, \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\rho) = \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\Theta)} (F_{q^{2n}}^*, \Sigma^*(\rho))^{GL_2F_{q^n}}$$

$$= \Sigma_{\rho \notin \{\Theta, F^*(\Theta)\}, \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\rho) = \mathrm{Res}_{F_{q^n}^*}^{F_{q^{2n}}^*}(\Theta)} (F_{q^{2n}}^*, \rho)^{GL_2F_{q^n}}.$$

The first equation implies that $\mathrm{Res}^{F^*_{q^{2n}}}_{F^*_{q^n}}(\Sigma^*(\Theta)) = \mathrm{Res}^{F^*_{q^{2n}}}_{F^*_{q^n}}(\Theta)$ and so there exists a unique $\widetilde{\Theta} : F^*_q \to C^*$ such that, for all $z \in F^*_{q^n}$, $\Theta(z) = \overline{\Theta}(N(z))$ where $N : F^*_{q^n} \to F^*_q$ is the norm. However, $\Sigma^*(\Theta)$ and Θ must be distinct on $F^*_{q^{2n}}$, since $F \in G(F_{q^{2n}}/F_{q^n})$ acts non-trivially on Θ, by assumption. The second equation shows that Σ^* permutes the set

$$\{\rho \notin \{\Theta, F^*(\Theta)\}, \mathrm{Res}^{F^*_{q^{2n}}}_{F^*_{q^n}}(\rho) = \mathrm{Res}^{F^*_{q^{2n}}}_{F^*_{q^n}}(\Theta)\}$$

so that we must have $\Sigma^*(\Theta) = F^*(\Theta)$. Since $G(F_{q^{2n}}/F_q) \cong Z/2n$ and $F = \Sigma^n$ we have

(3.22) $$(\Sigma^{n-1})^*(\Theta) = \Theta.$$

Since Θ is not Galois invariant $\langle\Sigma^{n-1}\rangle$ must be a proper subgroup of $\langle\Sigma\rangle$. However, $HCF(n-1, 2n) \in \{1, 2\}$ so that we must have $HCF(n-1, 1, 2n) = 2$ and therefore n *must be odd*. This means that

(3.23) $$Z/n \cong \langle\Sigma^{n-1}\rangle = G(F_{q^{2n}}/F_{q^2})$$

and, by (3.22), there exists a unique $\overline{\Theta} : F^*_{q^2} \to C^*$ such that $\Theta(w) = \overline{\Theta}(N(w))$ for all $w \in F^*_{q^{2n}}$. If $z \in F^*_q$ and $s \in F^*_{q^n}$ satisfy $N(s) = z$ then

$$\begin{aligned}
\mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\overline{\Theta})(z) &= \mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\overline{\Theta})(N_{F_{q^n}/F_q}(s)) \\
&= \overline{\Theta}(N_{F_{q^{2n}}/F_{q^2}}(s) \\
&= \mathrm{Res}^{F^*_{q^{2n}}}_{F^*_{q^n}}(\Theta)(s) \\
&= \widetilde{\Theta}(N_{F_{q^n}/F_q}(s)) \\
&= \widetilde{\Theta}(z)
\end{aligned}$$

so that $\mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\overline{\Theta}) = \widetilde{\Theta}$. From these characters it is natural to form

$$\begin{aligned}
&\Sigma_{v \in F^*_q}(H, \widetilde{\Theta} \otimes \Psi_{F_q}(v \cdot -))^{GL-2F_q} \\
&+\Sigma_{\rho \notin \{\overline{\Theta}, F^*(\overline{\Theta})\}, \mathrm{Res}^{F^*_{q^2}}_{F^*_q}(\rho) = \overline{\Theta}} (F^*_{q^2}, \rho)^{GL_2F_q} \\
&+\cdots.
\end{aligned}$$

These are the maximal terms of type A/D in $a_{GL_2F_q}(r(\overline{\Theta}))$ and therefore we set

(3.24) $$\Upsilon(r(\Theta)) = r(\overline{\Theta}).$$

Each of these recipes is reversible and one easily see that the process yields a one-one correspondence similar to that of (3.12). The discussion of (3.17) may be summarised as follows:

THEOREM (3.25). *The yoga of (3.17) yields a one-one correspondence of the form*

$$(3.26)\qquad \left\{\begin{array}{c} \{\textit{irreducible representations, } \nu\textit{, of} \\ GL_2F_{q^n} \textit{ fixed under } \Sigma\} \\ \updownarrow \Upsilon \\ \{\textit{irreducible representations, } \Upsilon(\nu)\textit{, of } GL_2F_q\}. \end{array}\right.$$

In fact, Υ coincides with the Shintani correspondence as described in ([8], p.410, §§4).

The fact that Υ satisfies the characterisation of Sh which is given by (3.16) is easily verified by means of the table of character value in (2.22).

Remark (3.27) It would be very interesting to develop for GL_mF_q a yoga similar to that which is given in (3.17) for GL_2F_q. In such an enterprise one would have to determine suitable generalisations of types A/D of (3.1). In this example the types were arrived at by considering first the maximal *abelian* pairs and then, should they prove not to be self-normalising, their normalisers. In the case of GL_2F_q what we have given is merely a calculation and in general one would wish for a more intrinsic proof; preferably one which, in the presence of a suitable Explcit Brauer Induction technique, would extend to the case of GL_mF where F is a local field.

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