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ON EQUIVARIANT COHOMOLOGY

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1. Introduction

In a previous paper [5] machinery was developed for establishing isomorphism theorems (and n-equivalences) in ordinary cohomology. This machinery is based on a result called the uniqueness theorem in [5,6] and the comparison theorem in [7].

In the present paper a similar machinery is developed for equivariant cohomology on G spaces. The corresponding comparison theorem can be applied in situations similar to those where the original theorem for ordinary cohomology applies.

Our presentation is an elementary exposition of the subject. We present only one application of the comparison theorem. This is a simple proof of a Vietoris–Begle-like theorem for the Borel equivariant cohomology theory (a result proved in [4] by use of spectral sequences). Hopefully there will be other interesting applications of the comparison theorem.

The rest of the paper is divided into three sections. In Section 2 we summarize some properties of G spaces. In Section 3 we define equivariant cohomology theories. There is a bijection between equivariant cohomology theories on a G space X and ordinary cohomology theories on the quotient space X/G. Thus, the comparison theorem for equivariant cohomology theories is deduced from the comparison theorem for ordinary cohomology.

In Section 4 we define the Borel equivariant cohomology theory on the category of paracompact G spaces by a limiting procedure and deduce the Vietoris–Begle-like theorem for this cohomology from the comparison theorem.

In the sequel G will denote a compact topological group.

2. G spaces

In this section we recall some basic properties of G spaces, the main one being that a G space X is paracompact if and only if the quotient space X/G is paracompact.

A *G space* is a topological space X together with a continuous map

$$\mu : G \times X \to X$$

such that:

1) $\mu(e, x) = x$ for all $x \in X$ where e is the identity element of G,

2) $\mu(g_1 g_2, x) = \mu(g_1, \mu(g_2, x))$ for $g_1, g_2 \in G$, $x \in X$.

If we denote $\mu(g, x)$ by gx, then 1) becomes $ex = x$ for all $x \in X$ and 2) becomes the "associativity" $(g_1 g_2)x = g_1(g_2 x)$ for $g_1, g_2 \in G$, $x \in X$. From 1) and 2) it follows that $x \mapsto gx$ is a homeomorphism of X onto itself for every $g \in G$ (with inverse $x \mapsto g^{-1}x$).

If X is a G space, a *G set* $A \subset X$ is a subset such that $\mu(G \times A) \subset A$ (i.e. $ga \in A$ for $g \in G$, $a \in A$). Then A together with $\mu \mid G \times A : G \times A \to A$ is a

G space called a G *subspace* of X. Since $x \mapsto gx$ is a homeomorphism for each $g \in G$ it follows easily that if A is a G set so are $\bar{A}$ and $\operatorname{int} A$.

Let $p : X \to X/G$ be the canonical projection of X onto the space of orbits of G in X topologized by the quotient topology. It is clear that $A \subset X$ is a G set if and only if $A = p^{-1}(A')$ for some $A' \subset X/G$, in which case $A = p^{-1}(pA)$. Thus, p induces a bijection between G sets in X and subsets of X/G (with inverse p^{-1} from subsets of X/G to G sets in X).

Since G is compact, for every open neighborhood U of a G set $A \subset X$ there is an open G set V in X with $A \subset V \subset U$. It follows that the projection $p : X \to X/G$ is an open and closed continuous map. If X is Hausdorff, it follows that p is a perfect map (see the definition on p. 236 of [2]). Therefore, by 3.7.20 of [2], X/G is also a Hausdorff space, and, by 5.1.33 and 5.1.35 of [2], X is paracompact if and only if X/G is paracompact.

3. Equivariant cohomology

For G spaces equivariant cohomology is a natural generalization of ordinary cohomology. This section contains the relevant definitions for equivariant cohomology and the comparison theorem for homomorphisms between two equivariant cohomology theories on the same G space.

If X is a G space, $cl_G(X)$ will denote the category of closed G sets in X and inclusion maps between them. In case G is the trivial group, $cl_G(X) = cl(X)$ is the category of all closed sets in X.

A G *equivariant cohomology* theory H, δ on a G space X consists of:

i) a contravariant functor H from $cl_G(X)$ to the category of graded abelian groups $(H(A) = \{H^q(A)\}_{q \in \mathbf{Z}})$ such that $H(\emptyset) = 0$, and

ii) a natural transformation δ which assigns to every $A, B \in cl_G(X)$ a homomorphism of degree 1

$$\delta : H(A \cap B) \to H(A \cup B)$$

such that both of the following are satisfied:

Continuity. For every $A \in cl_G(X)$ the homomorphism

$$\rho : \varinjlim \{H^q(B) \mid B \text{ a neighborhood of } A \text{ in } cl_G(X)\} \to H^q(A)$$

defined by $\rho\{u\} = u \mid A$ for $u \in H^q(B)$ is an isomorphism.

MV Exactness. For every $A, B \in cl_G(X)$ the following sequence is exact

$$\cdots \xrightarrow{\delta} H^q(A \cup B) \xrightarrow{\alpha} H^q(A) \oplus H^q(B) \xrightarrow{\beta} H^q(A \cap B) \xrightarrow{\delta} H^{q+1}(A \cup B) \xrightarrow{\alpha} \cdots$$

where $\alpha(u) = (u \mid A, u \mid B)$ for $u \in H^q(A \cup B)$ and $\beta(u, v) = u \mid A \cap B - v \mid A \cap B$ for $u \in H^q(A)$, $v \in H^q(B)$.

The equivariant theory is *nonnegative* if $H^q(A) = 0$ for all $q < 0$ and all $A \in cl_G(X)$. It is said to be *additive* if for every discrete family $\{A_j\}_{j \in J}$ in $cl_G(X)$ the homomorphism

$$\sigma : H^q(\cup_{j \in J} A_j) \to \Pi_{j \in J} H^q(A_j)$$

defined by $\sigma(u) = \{u \mid A_j\}_{j\in J}$ for $u \in H^q(\cup_{j\in J} A_j)$ is an isomorphism. It is *weakly additive* if for every discrete family $\{A_j\}_{j\in J}$ in $cl_G(X)$ and for every $u \in H^q(\cup_{j\in J} A_j)$ there is some finite subset $F \subset J$ such that $u \mid \cup_{j\notin F} A_j = 0$. This implies there is an isomorphism

$$\oplus_{j\in J} H^q(A_j) \approx H^q(\cup_{j\in J} A_j)$$

so the definition above is equivalent to the one in [7].

A cohomology theory H, δ on X is defined to be a G equivariant cohomology theory for G equal to the trivial group. It follows that there is a bijection between G equivariant cohomology theories on X and cohomology theories on X/G induced by the obvious isomorphism between the categories $cl_G(X)$ and $cl(X/G)$. This bijection preserves nonnegativity and additivity or weak additivity (because discrete families in $cl_G(X)$ correspond to discrete families in $cl(X/G)$).

Example (3.1). If G' is a closed subgroup of G and X is a G space then X is also a G' space and $cl_G(X) \subset cl_{G'}(X)$. Thus, every G' equivariant cohomology theory on X determines by restriction a G equivariant cohomology theory on X.

Example (3.2). Let X be a G space where G is a compact Lie group and let k be a covariant coefficient system for G over a ring R [3]. For $A \in cl_G(X)$ define $H^q(A) = H^G_{-q}(X, X - A; k)$ (equivariant singular homology with coefficients k [3]). With a suitable definition of δ this is a weakly additive G equivariant cohomology theory on X (not usually nonnegative).

Example (3.3). Let E_G be any compact Hausdorff G space. If X is a paracompact G space, define H on X by $H^q(A) = \check{H}^q((A \times E_G)/G; R)$ (Cěch cohomology with coefficients R of the quotient space of $A \times E_G$ by the diagonal action of G) for $A \in cl_G(X)$. With a suitable definition of δ this is a nonnegative additive G equivariant cohomology theory on X. ($X \times E_G$ is paracompact by 5.1.36 of [2] so $(X \times E_G)/G$ is paracompact. If A is a closed G set in X the compactness of E_G implies that every closed neighborhood of $(A \times E_G)/G$ in $(X \times E_G)/G$ contains a closed neighborhood of the form $(N \times E_G)/G$ where N is a closed neighborhood of A in X which is a G set.)

A homomorphism of degree 0, $\varphi : \Gamma \to \Gamma'$ between graded abelian groups is called an *n-equivalence* if $\varphi : \Gamma^i \to \Gamma'^i$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.

If H, δ and H', δ' are two G equivariant cohomology theories on the same G space X, a *homomorphism* $\varphi : H, \delta \to H', \delta'$ consists of a natural transformation from H to H' (both being functors on $cl_G(X)$) which commutes up to sign with δ, δ'. Because of the bijection between G equivariant cohomology theories on X and cohomology theories on X/G the comparison theorem for cohomology theories [7] translates to the following comparison theorem for G equivariant cohomology theories.

THEOREM (3.4) *Let $\varphi : H, \delta \to H', \delta'$ be a homomorphism between G equivariant cohomology theories on the same G space X both of which are additive or both are weakly additive. Suppose there is n such that $\varphi_A : H(A) \to H'(A)$ is an n-equivalence for every orbit A in X. If both H, δ and H', δ' are nonnegative or if X/G is locally finite dimensional then $\varphi_A : H(A) \to H'(A)$ is an n-equivalence for all $A \in cl_G(X)$.* □

4. The Borel cohomology theory

In this section we use the comparison theorem of the last section and the construction in Example (3.3) to define the Borel chomology theory, a particular Čech-like G equivariant cohomology theory on every paracompact G space where G is a compact Lie group. The main result of the section is a mapping theorem for Borel cohomology theory which generalizes a result of Kosniowski.

For a compact Lie group G there is a sequence $E_1 \subset E_2 \subset \cdots$ of path connected compact G spaces such that E_n is an n-universal fibration over $B_n = E_n/G$ [8]. Then $\pi_i(E_n) = 0$ for $1 \leq i < n$. From the exactness of the homotopy sequence of a fibration and the "5-lemma" it follows that $\pi_i(B_n) \to \pi_i(B_{n+1})$ is an isomorphism for $i \leq n$.

Let X be a paracompact G space and consider $(X \times E_n)/G$. This has a natural projection to $E_n/G = B_n$. This projection

$$p_n : (X \times E_n)/G \to B_n$$

is a fibration with fiber X (see 1.3 on p. 50 of [1]). Again using the exactness of the homotopy sequence of a fibration and the "5-lemma" it follows that the natural map

$$(X \times E_n)/G \subset (X \times E_{n+1})/G$$

induces a homomorphism $\pi_i((X \times E_n)/G) \to \pi_i((X \times E_{n+1})/G)$ which is an isomorphism for $1 \leq i < n$ and an epimorphism for $i = n$. Therefore, $((X \times E_{n+1})/G, (X \times E_n)/G)$ is n-connected so that, in singular cohomology with integer coefficients,

$$H^i((X \times E_{n+1})/G) \to H^i((X \times E_n)/G)$$

is an isomorphism for $1 \leq i < n$ and a monomorphism for $i = n$. In particular, if A is an orbit of G in X, then

$$H^*((A \times E_{n+1})/G) \to H^*((A \times E_n)/G)$$

is an n-equivalence. But if G_A is the isotropy subgroup of A, $(A \times E_{n+1})/G \approx E_{n+1}/G_A$ and $(A \times E_n)/G \approx E_n/G_A$. Since both E_n, E_{n+1} and their quotient spaces E_n/G_A, E_{n+1}/G_A are manifolds [8] it follows that their singular and Čech cohomology groups are isomorphic so that

$$(*) \qquad \check{H}^*((A \times E_{n+1})/G; R) \to \check{H}^*((A \times E_n)/G; R)$$

is an n-equivalence (in Cěch cohomology with coefficients R).

Let H_n, δ_n be the G equivariant cohomology theory defined on the paracompact G space X using E_n as in Example (3.3). Then $H_n^i(A) = \check{H}^i((A \times E_n)/G; R)$ for $A \in cl_G(X)$. There is a natural map

$$H_{n+1}(A) \to H_n(A)$$

induced by the inclusion map $(A \times E_n)/G \subset (A \times E_{n+1})/G$ so there is a homomorphism of G equivariant cohomology theories on X

$$\varphi_n : H_{n+1}, \delta_{n+1} \to H_n, \delta_n.$$

By $(*)$ above this is an n-equivalence for all orbits A of G in X. It follows from Theorem (3.4) that φ_n is an n-equivalence for all $A \in cl_G(X)$ (both cohomology theories are nonnegative and additive). Therefore, we can define a G equivariant cohomology theory H_∞, δ_∞ on X by $H_\infty(A) = \varprojlim_n \{H_n(A)\}$ and $\delta_\infty = \varprojlim_n \{\delta_n\}$. We call H_∞, δ_∞ the *Borel G equivariant cohomology theory* on X with coefficients R. It is nonnegative and additive and is a contravariant functor on the category of paracompact G spaces and G maps between them.

Example (4.1). Let $f : X \to Y$ be a closed continuous map between paracompact G spaces. For $B \in cl_G(Y)$ define $H_f(B) = H_\infty(f^{-1}(B))$. With a suitable definition of δ_f we obtain a nonnegative additive G equivariant cohomology theory H_f, δ_f on Y.

The following was obtained by Kosniowski (comparison theorem in [4]) using spectral sequences.

THEOREM (4.2) *Let $f : X \to Y$ be a closed continuous G map between paracompact G spaces. Suppose there is n such that $f^* : H_\infty(B) \to H_\infty(f^{-1}(B))$ is an n-equivalence for every orbit B of G in Y. Then $f^* : H_\infty(B) \to H_\infty(f^{-1}(B))$ is an n-equivalence for all $B \in cl_G(Y)$.*

Proof. Clearly f^* is a homomorphism from H_∞, δ_∞ on Y to the G equivariant cohomology theory H_f, δ_f on Y constructed in Example (4.1). Then the result is a consequence of the comparison theorem Theorem (3.4). □

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