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CLOSEDNESS OF COMPRESSIONS OF FAMILIES OF HILBERT SPACE OPERATORS

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Dedicated to the memory of Professor José Adem

1. Notations and the statement of the problem

$B(H)$ denotes the algebra of all linear, bounded operators on a complex Hilbert space H. I stands for the identity operator in H. Three topologies in $B(H)$ will be considered: the norm topology, the strong (operator) topology, i.e., the locally convex topology defined by the family of seminorms $q_x(T) = \|Tx\|$, $T \in B(H)$, $x \in H$, and the weak (operator) topology defined by the family of seminorms $q_{xy}(T) = |(Tx, y)|$, $T \in B(H)$, x, $y \in H$. The vector space operations in $B(H)$ are continuous in all three topologies. The multiplication of operators is continuous in the norm topology, but only separately continuous in the strong and weak topology. The basic facts about the three topologies can be found in any text on von Neumann algebras, e.g., [3], [5] as well as in [1], [4].

Let $\mathcal{S}$ be a subset of $B(H)$. The set of all operators in $B(H)$ that commute with every element of $\mathcal{S}$ is called the *commutant* of $\mathcal{S}$, it is denoted by $\mathcal{S}'$, and it is known to be a weakly closed subalgebra of $B(H)$. The closure of $\mathcal{S}$ will be denoted by $\operatorname{cl}\mathcal{S}$ and the topology will be specified. The weak and strong closures of a convex subset of $B(H)$ coincide. An operator $E \in B(H)$ is called a *projection* if $E = E^2 = E^*$. The *compression* of a set $\mathcal{S} \subset B(H)$ to the subspace EH is the set $\mathcal{S}_E = \{ES|_{EH} : S \in \mathcal{S}\}$, which is a subset of $B(EH)$. A subspace EH of H, or the projection E, is *invariant* for $\mathcal{S}$ if $SEH \subset EH$, or, algebraically, $ESE = SE$, for each $S \in \mathcal{S}$. A subspace EH of H *reduces* $\mathcal{S}$ if EH is invariant for both $\mathcal{S}$ and $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$, or, algebraically, if $E \in \mathcal{S}'$.

When one studies compressions, the following topological problem arises naturally:

Problem. Let $\mathcal{S}$ be a subset of $B(H)$ and let $E \in B(H)$ be a projection. If $\mathcal{S}$is closed (in norm, strongly, weakly, respectively), what are the conditions that guarantee that $\mathcal{S}_E$ is closed (in norm, strongly, weakly, respectively)?

Even though the problem looks rather basic, I have not seen it stated anywhere before.

There is only one fairly general case in which the problem is solved. This case is, however, important enough to make the above general problem interesting and worth investigating. Namely, almost since the beginning of the von Neumann algebra theory it has been known that if $\mathcal{S}$ is a von Neumann (i.e., a symmetric, weakly closed) algebra and E is a projection in $\mathcal{S}$ or in $\mathcal{S}'$, then $\mathcal{S}_E$ is a von Neumann algebra, thus $\mathcal{S}_E$ is weakly closed. (cf. e.g. [3], Part I, Ch.2, Proposition 1, [6], 5.5.6). The simplicity of the formulation of the general problem may deceive one to think that the solution should not be all that hard. Perhaps it is so, but a careful study of the von Neumann algebra case gives indications to the contrary. Namely, one realizes that the proofs of the above mentioned result on von Neumann algebras with $E \in \mathcal{S}'$, i.e., if EH reduces $\mathcal{S}$, always rely on one of the two elementary but fundamental, and by no means trivial results about von Neumann algebras: the Double Commutant theorem (in [3]), or the Kaplansky density theorem (in [6]). Thus the proofs depend in an essential way upon the assumption that $\mathcal{S}$ is a von Neumann algebra.

In this paper some answers to the problem will be presented without the assumption that $\mathcal{S}$ is a von Neumann algebra. The von Neumann algebra case will also be approached in a manner different from the ones known so far.

A bounded subset of $B(H)$ is understood to be norm-bounded. The following notations will also be used. If $T \in B(H)$, then $\sigma(T)$ stands for the spectrum of T, $\mathcal{A}_u(T)$, $\mathcal{A}_s(T)$ denote the closure of the algebra of all polynomials in T in the norm and strong (or weak) topology, respectively. $\mathcal{R}_u(T)$ stands for the norm-closure of the set $\{u(S) : u \text{ is a rational function with poles off } \sigma(T)\}$. $\mathbf{C}$ denotes the complex plane. If K is a compact subset of $\mathbf{C}$ and $f\colon K \longrightarrow \mathbf{C}$ is a continuous function, then $\|f\|_K = \sup\{|f(z)| : z \in K\}$. Moreover, $\hat{K} = \{z \in \mathbf{C} : |p(z)| \leq \|p\|_K, \text{ for all polynomials } p\}$ is the *polynomially convex hull* of K. It is plain that $\|p\|_K = \|p\|_{\hat{K}}$, for each polynomial p.

2. Preliminary results

Let $E \in B(H)$ be a projection. In what follows it will be understood that the topologies in $B(H)$ and in $B(EH)$ are of the same kind, i.e., if $B(EH)$ is considered to be a subalgebra of $B(H)$, or, more precisely, $B(EH)$ is embedded isometrically into $B(H)$ by the mapping $T \longrightarrow T \oplus 0$ with respect to the decomposition $H = EH \oplus (I - E)H$, then the topology in $B(EH)$ is the restriction of the topology in $B(H)$ to $B(EH)$. The compression mapping $\Phi : B(H) \longrightarrow B(EH)$ defined by $\Phi(T) = ET|_{EH}$ is linear and continuous in any of the three topologies, because multiplication of operators is separately continuous.

In practice, proving that a compression of a set of operators is closed, one often uses the following

PROPOSITION (2.1). *Let $\mathcal{S} \subset B(H)$, let $E \in B(H)$ be a projection. Then $(\mathrm{cl}\,\mathcal{S})_E$ is closed if and only if $(\mathrm{cl}\,\mathcal{S})_E = \mathrm{cl}(\mathcal{S}_E)$, i.e., if the operations of compression and closure can be interchanged. The closure is taken in any of the three topologies.*

Proof. $(\mathrm{cl}\,\mathcal{S})_E \subset \mathrm{cl}(\mathcal{S}_E)$, because the mapping Φ is continuous. If $(\mathrm{cl}\,\mathcal{S})_E$ is closed, then $\mathrm{cl}(\mathcal{S}_E) \subset \mathrm{cl}((\mathrm{cl}\,\mathcal{S})_E) = (\mathrm{cl}\,\mathcal{S})_E$. The opposite implication is clear. Q.E.D.

This proposition is merely a restatement of an elementary property of continuous functions known from the general topology.

The following simple example shows that, in general, the compression of a closed set does not need to be closed.

Example (2.2). Take $\alpha_n = 2^{-n}$, $\beta_n = 2^n$. In $H = \mathbf{C} \oplus \mathbf{C}$ define $S_n = [\alpha_n] \oplus [\beta_n]$, for all integers $n \geq 0$. Let $\mathcal{S} = \{S_n : n \geq 0\}$ and let E be the projection of H onto the first $\mathbf{C}$. Then the set $\mathcal{S}$ is closed in $B(H)$, but $\mathcal{S}_E = \{[\alpha_n] : n \geq 0\}$ is not closed in $B(EH)$. One could also take $\alpha_n = n^{-1}$, $\beta_n = n$, $n > 0$.

In this example the topology in $B(H)$ is any of the three topologies, because for a finite-dimensional space H all three topologies in $B(H)$ are identical and $B(H)$ is homeomorphic to $\mathbf{C}^{n^2}$ with the Euclidean norm topology, if $\dim H = n$. Notice that in Example (2.2) all operators are self-adjoint, $\mathcal{S}$ is a semigroup with unit generated by a single operator S_1, and EH reduces $\mathcal{S}$. However, $\mathcal{S}$ is not bounded, and, indeed, it cannot be bounded because of the following elementary

PROPOSITION (2.3). *If H is finite-dimensional, $E \in B(H)$ is a projection, and $\mathcal{S} \subset B(H)$ is bounded and closed, then $\mathcal{S}_E$ is compact, and thus closed.*

Proof. $\mathcal{S}$ is compact, Φ is continuous. Thus $\Phi(\mathcal{S}) = \mathcal{S}_E$ is compact. Q.E.D.

Example (2.2) can be generalized to arbitrary dimensions as follows:

Example (2.4). Let H_1, H_2 be Hilbert spaces, let $A_n \in B(H_1)$, $B_n \in B(H_2)$ be sequences of operators. Let $H = H_1 \oplus H_2$, $S_n = A_n \oplus B_n$, $n \geq 0$. Assume: $\|A_n\| \longrightarrow 0$, $A_n \neq 0$, for each n, the set $\{B_n : n \geq 0\}$ has no weak cluster point, and all B_n's are different. By Lemma (2.5) applied to the compression mapping $B(H) \longrightarrow B(H_2)$, the set $\mathcal{S} = \{S_n : n \geq 0\}$ has no weak cluster point. Hence $\mathcal{S}$ is weakly (therefore strongly and norm-) closed. On the other hand, if E is a projection of H onto H_1, then 0 is in the norm-closure of $\mathcal{S}_E$, but not in $\mathcal{S}_E$. Thus $\mathcal{S}_E$ is not closed in any topology. One can get $\mathcal{S}$ to be a multiplicative semigroup generated by a single operator choosing $A \in B(H_1)$, $\|A\| < 1$, A not nilpotent, $B \in B(H_2)$ such that the set $\{B^n : n \geq 0\}$ has no weak cluster point, all B^n's different, and $S = A \oplus B$, $S_n = S^n$, $n \geq 0$.

LEMMA (2.5). *Let X, Y be topological spaces, let $A \subset X$, let $f\colon X \longrightarrow Y$ be a continuous function whose restriction to A is one-to-one. If X is a T_1-space and $f(A)$ has no cluster point, then A has no cluster point.*

Proof. It is clear that no point in A is a cluster point of A. Let $\operatorname{cl} A$ denote the closure of A. Now, $f(A) = f(\operatorname{cl} A)$, because f is continuous, $f(A)$ is closed, and $f(A) \subset f(\operatorname{cl} A) \subset \operatorname{cl} f(A) = f(A)$. Suppose x is a cluster point of A, not in A. Then $x \in \operatorname{cl} A$. Thus $f(x) \in f(\operatorname{cl} A) = f(A)$, hence $f(x) = f(a)$ for some $a \in A$.

Since $f(A)$ has no cluster point, there exists a neighborhood V of $f(x) = f(a)$ such that $V \cap f(A) = \{f(a)\}$. Hence

$$f^{-1}(V) \cap A = f^{-1}(V) \cap f^{-1}(f(A)) \cap A = f^{-1}(f(a)) \cap A = \{a\}.$$

The last equality holds, because f is one-to-one on A. Since $x \neq a$ and X is a T_1 space, there exists a neighborhood U of x such that $a \notin U$. Thus $U \cap f^{-1}(V)$ is a neighborhood of x, and $U \cap f^{-1}(V) \cap A = U \cap \{a\} = \emptyset$. This contradicts the assumption that x is a cluster point of A. Q.E.D.

The next result has been known under the assumption that $\mathcal{S}$ is a von Neumann algebra. Much less, however, suffices.

PROPOSITION (2.6). *Consider any of the three topologies in $B(H)$. Let $\mathcal{S}$ be a multiplicative semigroup and let E be a projection in $\mathcal{S}$. If $\mathcal{S}$ is closed, so is $\mathcal{S}_E$.*

Proof. Take $T \in \mathrm{cl}(\mathcal{S}_E)$. There exists a net $S_n \in \mathcal{S}$ such that $ES_n|_{EH} \longrightarrow T$. The net ES_nE converges. Call its limit S. Now $S \in \mathcal{S}$, because $\mathcal{S}$ is a closed, multiplicative semigroup, and $E \in \mathcal{S}$. Finally, $ES|_{EH} = T$. Q.E.D.

$B(H)$ with the norm topology is a Banach (thus metric) space, hence it would suffice to deal with sequences only. Unfortunately, neither strong, nor weak topology in $B(H)$ satisfies the first axiom of countability (cf.[3], p.49, Exercise 4), therefore for these topologies one must use nets.

Proposition (2.6) has several applications for non-symmetric operator algebras, some of which are listed below:

COROLLARY (2.7). *a. Let H_i be a Hilbert space, $A_i \in B(H_i)$, $i = 1, 2$. Let E be the projection of $H = H_1 \oplus H_2$ onto H_1, $S = A_1 \oplus A_2$, $\mathcal{S} = \mathcal{A}_u(S)$. Suppose $\hat{\sigma}(A_1) \cap \hat{\sigma}(A_2) = \emptyset$.*

b. With A_i, S, E as above, let $\mathcal{S} = \mathfrak{R}_u(S)$. Suppose $\sigma(A_1) \cap \sigma(A_2) = \emptyset$.

c. Let $\mathcal{S} \subset B(H)$ be a commutative, reflexive algebra, let E be a maximal antisymmetric projection for $\mathcal{S}$.

Then $\mathcal{S}_E$, $\mathcal{S}_{I-E}$ are norm-closed in a, b, and weakly closed in c.

Proof. a, b. follow from Theorem 2.1 and 2.2, respectively, in [2], where more situations can be found, to which Proposition (2.6) applies. c. follows from Corollary 2 in [9], or Theorem 4 in [10], where one also finds all necessary definitions. The proof is completed by applying the following

REMARK (2.8). *Let $\mathcal{S} \subset B(H)$ be an algebra with I and let $E \in B(H)$ be a projection. Then $\mathcal{S} = \mathcal{S}_E \oplus \mathcal{S}_{I-E}$ if and only if $E \in \mathcal{S} \cap \mathcal{S}'$.*

To prove this assume first that $\mathcal{S} = \mathcal{S}_E \oplus \mathcal{S}_{I-E}$. Then $ES(I - E) = (I - E)SE = 0$, $S \in \mathcal{S}$, thus $E \in \mathcal{S}'$. Moreover, $E = I|_{EH} \oplus 0|_{(I-E)H} \in \mathcal{S}$. Conversely, if $E \in \mathcal{S} \cap \mathcal{S}'$, then $\mathcal{S} \subset \mathcal{S}_E \oplus \mathcal{S}_{I-E}$. Now take $S, T \in \mathcal{S}$. Then $S|_{EH} \oplus T|_{(I-E)H} = SE + T(I - E) \in \mathcal{S}$. Q.E.D.

Finally, notice that if $\mathcal{S} \subset B(H)$ is a linear subspace and $E \in B(H)$ is a finite-dimensional projection, then $\mathcal{S}_E$ is a linear subspace of a finite-dimensional space $B(EH)$, thus it is closed in all three (identical) topologies.

3. The norm topology

In this section $B(H)$ has the norm topology, thus it is a Banach space. If $\mathcal{S}$ is a closed, linear subspace of $B(H)$, then the question asked in the problem is the question, when the compression mapping $\Phi: \mathcal{S} \longrightarrow B(EH)$ has closed range. This type of questions is usually not easy to answer. Some answers are known.

Recall that if X, Y are Banach spaces, then a bounded, linear mapping $\Psi: X \longrightarrow Y$ is called bounded below (by $c > 0$) if $\|\Psi x\| \geq c\|x\|$ for all $x \in X$. The following known result, whose proof uses Banach's closed graph theorem, will be needed:

THEOREM (3.1). *Let X, Y be Banach spaces and let $\Psi: X \longrightarrow Y$ be a bounded, linear mapping. Ψ is bounded below if and only if Ψ has closed range and* $\ker \Psi = 0$.

Some solutions of the problem follow from this result.

THEOREM (3.2). *Let $E \in B(H)$ be a projection invariant for $T \in B(H)$. If $\sigma(T) \subset \hat{\sigma}(T|_{EH})$ and $\|p(T)\| \leq c\|p\|_{\sigma(T)}$ for some $c > 0$ and all polynomials p, then $\mathcal{A}_u(T)_E$ is norm-closed.*

Proof. Let p be a polynomial. Then

$$\|p(T)|_{EH}\| = \|p(T|_{EH})\| \geq \|p\|_{\sigma(T|_{EH})} = \|p\|_{\hat{\sigma}(T|_{EH})} \geq \|p\|_{\sigma(T)} \geq c^{-1}\|p(T)\|.$$

The first inequality follows from the spectral theorem for polynomials. If $S \in \mathcal{A}_u(T)$, then there is a sequence p_n of polynomials such that $p_n(T) \longrightarrow S$. Passing to the limit in the above inequality written for p_n's one gets $\|S|_{EH}\| \geq c^{-1}\|S\|$. This proves that the compression mapping $\Phi: \mathcal{A}_u(T) \longrightarrow B(EH)$ is bounded below. By Theorem (3.1) applied to Φ, $\mathcal{A}_u(T)_E$ is norm-closed. Q.E.D.

A similar result, whose proof is like the one above, holds for $\mathcal{R}_u(T)$.

THEOREM (3.3). *Let $E \in B(H)$ be a projection invariant for $T \in B(H)$. If $\sigma(T) \subset \sigma(T|_{EH})$, and $\|u(T)\| \leq c\|u\|_{\sigma(T)}$ for some $c > 0$ and all rational functions u with poles off $\sigma(T)$, then $\mathcal{R}_u(T)_E$ is norm-closed.*

Here are consequences of the last two theorems for subnormal operators, whose up-to-date theory is presented in [1], where all necessary definitions can be found.

COROLLARY (3.4) *Suppose $N \in B(H)$ is a normal operator, $E \in B(H)$ is a projection invariant for N, and N is the minimal normal extension of $N|_{EH}$. Then $\mathscr{A}_u(N)_E$, $\mathscr{R}_u(N)_E$ are norm-closed.*

Proof. By the spectral theorem for normal operators, $\|u(N)\| = \|u\|_{\sigma(N)}$ for each continuous function u on $\sigma(N)$. Moreover, $\sigma(N) \subset \sigma(N|_{EH})$, by the minimality condition - cf. [1], Ch. II, Theorem 2.11. Now Theorems (3.2) and (3.3) complete the proof. Q.E.D.

COROLLARY (3.5). *Suppose $T \in B(H)$ is a subnormal operator and $E \in B(H)$ is a projection invariant for T.*
a. If $\sigma(T) \subset \hat{\sigma}(T|_{EH})$, then $\mathscr{A}(T)_E$ is norm-closed.
b. If $\sigma(T) \subset \sigma(T|_{EH})$, then $\mathscr{R}_u(T)_E$ is norm-closed.

Proof. Use Theorems (3.2), (3.3), and Proposition 9.2 of [1], Ch. II. Q.E.D

Now, one more solution to the problem.

THEOREM (3.6). *Let $\mathscr{S}$ be a linear subspace of $B(H)$, let $V \in \mathscr{S}'$ be bounded below, and let E be the projection of H onto VH. If $\mathscr{S}$ is norm-closed, so is $\mathscr{S}_E$.*

Proof. By Theorem (3.1), VH is closed. Also, VH is invariant for $\mathscr{S}$, because $V \in \mathscr{S}'$. Moreover, $EV = V$. Let $c > 0$ be a constant such that $\|Vx\| \geq c\|x\|$, $x \in H$. Take $S \in \mathscr{S}$, $x \in H$ such that $\|x\| \leq 1$. Then

$$c\|Sx\| \leq \|VSx\| = \|SVx\| = \|S|_{EH}Vx\| \leq \|S|_{EH}\| \, \|V\|.$$

Thus $c\|V\|^{-1}\|S\| \leq \|S|_{EH}\|$, i.e., the compression mapping $\Phi\colon \mathscr{S} \longrightarrow B(EH)$ is bounded below. By Theorem (3.1) again, the proof is finished. Q.E.D.

The subspace EH above, invariant for $\mathscr{S}$, does not need to reduce $\mathscr{S}$, in general, as the following corollary illustrates.

COROLLARY (3.7). *Let H^2 be the Hardy space on the unit circle, let $T_z \in B(H^2)$, $T_z f = zf$, $f \in H^2$ be the unilateral shift. If E is the projection onto an invariant subspace for T_z, then $\mathscr{A}_u(T_z)_E$ is norm-closed.*

Proof. Assume $E \neq 0$. By Beurling's theorem, there is a function $q \in H^\infty$, $|q| = 1$ almost everywhere relative to the Lebesgue measure on the circle such that $EH^2 = T_q H^2$, where $T_q f = qf$, $f \in H^2$. T_q is an isometry (hence bounded below) which commutes with T_z. Theorem (3.6) finishes the proof. Moreover, E does not commute with T_z, unless $q = 1$, because T_z has no non-trivial reducing subspace. Q.E.D.

4. The strong and weak topologies

In this section mostly bounded subsets of $B(H)$ will be considered. The case of the weak topology is rather simple because of an elementary but deep theorem, which is a version of the Banach-Bourbaki-Alaoglu theorem, and which states that each ball $B(H)_r = \{S \in B(H) : \|S\| \leq r\}$ is weakly compact (cf. e.g. [4], (3) after Problem 107, or [5], 5.1.3). Since the weak topology is Hausdorff, a closed subset of a compact set is compact, and a compact set is closed. Moreover, the compression mapping Φ is weakly continuous. This proves the following perfect generalization of the finite-dimensional case described in Proposition (2.3).

PROPOSITION (4.1). *If $\mathcal{S} \subset B(H)$ is bounded and weakly closed, and $E \in B(H)$ is a projection, then $\mathcal{S}_E$ is weakly compact, and thus weakly closed.*

For a von Neumann algebra $\mathcal{S}$, a particular case of this proposition applied to the unit ball in $\mathcal{S}$ and $E \in \mathcal{S}'$ combined with the Kaplansky density theorem essentially constitutes the proof that $\mathcal{S}_E$ is a von Neumann algebra - cf. [5], 5.5.6.

The strong topology poses more difficulties. The compactness argument, used for the weak topology, cannot be used, simply, because balls $B(H)_r$, even though obviously closed, are not strongly compact (cf. e.g. [4], solution to Problem 115). Fortunately, each bounded and strongly closed subset $\mathcal{S}$ of $B(H)$ is complete in the uniform structure associated with the strong topology. Namely, each ball $B(H)_r$ is complete ([5], 2.5.11). If $\mathcal{S}$ is bounded, then $\mathcal{S} \subset B(H)_r$ for some $r > 0$. Thus $\mathcal{S}$ is complete, if closed (cf. e.g. [6], Ch.6, Theorem 22). This allows one to prove the following analogue of Theorem (3.6):

THEOREM (4.2). *Let $\mathcal{S}$ be a bounded subset of $B(H)$, let $V \in \mathcal{S}'$ be bounded below, and let E be the projection of H onto VH. If $\mathcal{S}$ is strongly closed, so is $\mathcal{S}_E$.*

Proof. Take $T \in \mathrm{cl}(\mathcal{S}_E)$. Then there is a net $S_n \in \mathcal{S}$ such that $S_n|_{EH} \longrightarrow T$. Let $c > 0$ be such that $\|Vx\| \geq c\|x\|$, $x \in H$. Then

$$c\|S_nx - S_mx\| \leq \|V(S_nx - S_mx)\| = \|S_nVx - S_mVx\| = \|S_n|_{EH}Vx - S_m|_{EH}Vx\|.$$

Thus S_n is a Cauchy net. Since $\mathcal{S}$ is complete, S_n converges strongly. Let S be the limit. Then $S \in \mathcal{S}$, and $S|_{EH} = T$. Q.E.D.

One obtains immediately the following corollary, whose proof is similar to that of Corollary (3.7), but uses Theorem (4.2) instead of (3.6).

COROLLARY (4.3). *Let T_z be the unilateral shift. If E is an invariant projection for T_z and $\mathcal{S} \subset \mathcal{A}_s(T_z)$ is bounded and strongly closed, then $\mathcal{S}_E$ is strongly closed.*

Finally, how close to the amazingly complete result described in Proposition (4.1) can one get for the strong topology? Here is an answer:

THEOREM (4.4). *Let $\mathcal{S} \subset B(H)$ be bounded. Let $E \in \mathcal{S}'$ be a projection. Then for each $T \in \mathrm{cl}(\mathcal{S}_E)$ in the strong topology there is $S \in W^*(\mathcal{S}) =$ the von Neumann algebra generated by $\mathcal{S}$, such that $S|_{EH} = T$.*

The proof of this theorem requires some preparation. It is somewhat surprising that the canonical decomposition method developed in [8] is of help here. The reader is referred to [8] and [7] for the details and some applications of this method. Two projections in a von Neumann algebra $\mathcal{R}$ are called *equivalent*-notation: $E \sim F(\mathrm{mod}\, \mathcal{R})$-if there is a partial isometry $U \in \mathcal{R}$ such that $U^*U = E$, $UU^* = F$. The projection onto the closed linear span of EH and FH is denoted by $E \vee F$.

THEOREM (4.5). *Let $\mathcal{R} \subset B(H)$ be a von Neumann algebra and let $\mathcal{P}$ be a family of projections in $\mathcal{R}'$. If $E \vee F \in \mathcal{P}$ whenever $E, F \in \mathcal{P}$, then $E_0 = \sup \mathcal{P}$ exists, E_0 is a projection, and it is the strong limit of the increasing net $\mathcal{P}$. Moreover, if $E_0 \in \mathcal{P}$ and $\mathcal{P}$ satisfies the condition*

(*) for all E, F: $E \in \mathcal{P}$, $F \in \mathcal{R}'$, $F \sim E(\mathrm{mod}\, \mathcal{R}')$ implies $F \in \mathcal{P}$,

then $E_0 \in \mathcal{R}$.

The above theorem contains those results of [8] that are needed here.

THEOREM (4.6). *Let $T_n \in B(H)$ be a bounded net and let $\mathcal{M}$ be the set of all T_n's. Then there exists the largest projection $E_0 \in \mathcal{M}'$ with the property that T_nE_0 converges strongly. Moreover, $E_0 \in W^*(\mathcal{M})$.*

Proof. Let $H_1 = \{x \in H : T_nx \text{ converges}\}$. H_1 is a linear manifold. A standard argument, which uses boundedness of T_n and completeness of the Hilbert space H, proves that H_1 is closed. Let P_1 be the projection of H onto H_1. Define

$$\mathcal{P} = \{E \in W^*(\mathcal{M})' : E \text{ is a projection}, E \leq P_1\}.$$

To prove that $\mathcal{P}$ satisfies (*) with $\mathcal{R} = W^*(\mathcal{M})$, take $E \in \mathcal{P}$, $F \in \mathcal{R}'$, $F \sim E(\mathrm{mod}\, \mathcal{R}')$, and let $U \in \mathcal{R}'$ be the partial isometry such that $U^*U = E$, $UU^* = F$. Then $UE = U = FU$. Let T be the strong limit of T_nE. Take $x \in H$.

$$\begin{aligned} \|T_nFx - UTU^*x\| &= \|T_nUU^*Fx - UTU^*Fx\| = \|UT_nU^*Fx - UTU^*Fx\| \\ &= \|UT_nEU^*x - UTEU^*x\| = \|T_nEU^*x - TU^*x\|. \end{aligned}$$

The last equality is justified, because U maps unitarily EH onto FH. Thus T_nF converges strongly to UTU^* and hence $F \in \mathcal{P}$. It is easy to see that $\mathcal{P}$ satisfies all remaining assumptions of Theorem (4.5). Since $\mathcal{P}$ is strongly closed, $E_0 = \sup \mathcal{P} \in \mathcal{P}$. Theorem (4.5) completes the proof. Q.E.D.

Another proof of this theorem can be given using completeness of strongly closed, bounded subsets of $B(H)$ by noticing that a bounded net T_n converges strongly if and only if $\|T_n x - T_m x\| \longrightarrow 0$ for each $x \in H$ (cf. e.g. [6], Ch.6, Lemma 20). Then one needs to apply the Proposition of [7], whose proof uses in an essential manner the methods of [8].

Proof of Theorem (4.4). Take $T \in \mathrm{cl}(\mathcal{S}_E)$. Then there is a net $S_n \in \mathcal{S}$ such that $S_n|_{EH} \longrightarrow T$. By Theorem (4.5), there exists the largest projection $E_0 \in W^*(\mathcal{S})'$ such that $S_n E_0$ converges, and $S_n E_0 \in W^*(\mathcal{S})$. Thus the strong limit S of $S_n E_0$ belongs to $W^*(\mathcal{S})$. Now, $S|_{EH} = \lim S_n E_0|_{EH} = \lim S_n|_{EH} = T$, because $E \leq E_0$. Q.E.D.

If $\mathcal{R} \subset B(H)$ is a von Neumann algebra, then $\mathcal{R}_1 = \mathcal{R} \cap B(H)_1$ is strongly closed. In the particular case of $\mathcal{S} = \mathcal{R}_1$, Theorem (4.4) reads: if $E \in \mathcal{R}'$, then $(\mathcal{R}_1)_E$ is strongly closed (because $W^*(\mathcal{R}_1) = \mathcal{R}$). This particular case is known, but it has always been proved by "taking a trip" to the weak closure and using Proposition (4.1) for the unit ball (cf. e.g. of 5.5.6 in [5]). The proof above stays entirely within the strong topology.

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