Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

MODULAR REPRESENTATIONS AND THE COHOMOLOGY OF FINITE CHEVALLEY GROUPS

By Charles B. Thomas

In memory of José Adem

1. Motivation and Preliminaries

The structure of the representation rings of the classical compact Lie groups in characteristic zero is well-known, see for example the book of T. Bröcker and T. tom Dieck [2]. In particular we have

$$RSU(n) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_{n-1}]\,,$$

and

$$RSp(n) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_n]$$

where λ_i corresponds to the ith exterior power of the natural representation of the group on $\mathbb{C}^n$ (respectively on $\mathbb{C}^{2n} = \mathbb{H}^n$). For the smallest exceptional group we have

$$RG_2 = \mathbb{Z}[\sigma, adj]\,,$$

where σ is the real irreducible representation of degree 7 given by the inclusion of G_2 in Spin (7), the degree of the adjoint representation equals 14 and

$$\lambda^2\sigma = (adj) + \sigma\,.$$

With minor modifications these assertions remain true for the finite groups $SL(n, \mathbb{F}_q)$, $Sp(2m, \mathbb{F}_q)$ and $G_2(\mathbb{F}_q)$, where $q = p^t$; indeed there is a general result of this kind, which is valid for representations in the natural characteristic p for any simply connected, simple algebraic group. The argument is given in the book of P. Kleidman and M. Liebeck [8, §5.4], and in the typical case of $G = SL(n, \mathbb{F}_q)$ can be summarised as follows.

G contains a maximal torus $T_n(\mathbb{F}_q)$ consisting of diagonal matrices with elements $\xi_1, \ldots, \xi_n \in \mathbb{F}_q^\times$ such that $\xi_1\xi_2 \ldots \xi_n = 1$. As an abstract group $T_n(\mathbb{F}_q)$ is a direct product of $(n-1)$ copies of a cyclic group of order $(q-1)$, hence has order coprime with p. Working, if necessary, over an extension field $\mathbb{F}_{q'}$, the representation theory of $T_n(\mathbb{F}_q)$ is formally similar to the theory over the complex numbers. Furthermore, again copying the argument from characteristic 0 the restriction map from $R\mathbb{F}_{q'}G$ to $R\mathbb{F}_{q'}T_n$ is injective. Since by elementary linear algebra any matrix in G is conjugate to one in T_n, if two representation modules have the same Brauer character over T_n, they also have the same character over G. The equivalence relation on the modules generating $R\mathbb{F}_{q'}(\cdot)$ is expressed in terms of short exact sequences rather than direct sums, so the Brauer character of a representation determines its equivalence class. Clearly the image is contained in $(R\mathbb{F}_{q'}T_n)^W$, where W denotes the Weyl group, and inspection of the representations λ_i $(i = 1, \ldots, n)$ shows that this image is as large as possible. Perhaps the hardest part of the argument is to descend from $\mathbb{F}_{q'}$ to $\mathbb{F}_q$, i.e. to show that $\mathbb{F}_q$ is a splitting field for G, for this see [8, Prop. 5.4.4], and compare [10, Thm. 2.7] on Galois invariance. In this case one actually has

THEOREM (1). *If $G = SL(n, \mathbb{F}_q)$ the modular representation ring $R\mathbb{F}_q(G)$ equals $\mathbb{Z}[\lambda_1, \lambda_2, \ldots]$ subject to the relations* (i) $\lambda_d = 0$ *if* $d \geq n$, (ii) $\lambda_n = 1$ *and* (iii) $\psi^q \lambda_i = \lambda_i$.

The third relation enters since under Brauer lifting modular representations are invariant under the action of the Adams operation ψ^q. With the relations particularly in mind, Theorem 1 should be compared with the results in section three of the paper [3] by D. Carlisle and N. Kuhn for the general linear group $GL(n, \mathbb{F}_q)$.

Whereas the modular representations of a simple algebraic group G in characteristic p reflect the influence of the Cartan-Weyl theory for finite dimensional representations of a compact group, the ordinary representation theory in characteristic 0 has some of the flavour of infinite dimensional representations of semi-simple Lie groups, such as $SL(n, \mathbb{R})$. The aim of this paper is to look at this dichotomy from the point of view of group cohomology, and to show that, for both untwisted and twisted Chevalley groups, the passage from $R(G)$ to $R\mathbb{F}_q(G)$ corresponds to neglecting p-torsion in integral cohomology, i.e. to studying $H^*(G, \mathbb{Z}[\frac{1}{p}])$ rather than $H^*(G, \mathbb{Z})$. More precisely: Let G be a Chevalley group defined over the field $\mathbb{F}_q$. Then away from the prime p the Chern subring of $H^*(G, \mathbb{Z})$ is generated by the Chern classes of the Brauer lifts of the irreducible $\mathbb{F}_{q'}$-representations ($q' \geq q$). This is proved in section two, and then illustrated by examination of the cases $SL(n, \mathbb{F}_q)$ ($n \leq 4$), $Sp(4, \mathbb{F}_q)$, $G_2(q)$ and ${}^2G_2(\mathbb{F}_q)$ ($q = 3^{2m+1}$). (In a subsequent paper we propose to return to the exceptional groups of types F_4 and E_n, $n = 6, 7, 8$.) The proof of the main theorem depends on the completion theorem of D. Rector, and in a final section we prove analogues of this for orthogonal and symplectic representations. With the orthogonal groups arising in this last section particularly in mind let us fix our notation as follows:

$SL(n, \mathbb{F}_q)$ is the group of invertible ($n \times n$) matrices over the field $\mathbb{F}_q$ having determinant equal to 1, $SO(2m, \mathbb{F}_q)$ of those matrices with determinant 1 which preserve the quadratic form $x_1 x_{m+1} + \cdots + x_m x_{2m}$ on $\mathbb{F}_q^{\oplus 2m}$, and $Sp(2m, \mathbb{F}_q)$ of those matrices which preserve the skew-symmetric bilinear form described by the matrix $\begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$. In some places it will also be necessary to assume that q is odd; in this case the elements of $SO(2m, \mathbb{F}_q)$ preserve the bilinear symmetric form $\begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$. Finally $G_2(q)$ denotes the finite group of exceptional Lie type, associated with the Dynkin diagram

$$\bigcirc\!\!\equiv\!\!\equiv\!\!\bigcirc\ .$$

This paper has its roots in a talk given at the Adams memorial symposium in July 1990. The half-hour talk was devoted to rather special cohomological calculations (see section three below), as an illustration of what one can do with characteristic classes. In conversation with Nick Kuhn it then became clear that the argument applied to $SL(n, \mathbb{F}_q)$; Jan Saxl then referred me to [8], out of which grew the generalisation to more general algebraic groups. The final version was written while I was a guest at the ETH in Zürich in June 1992.

2. The Chern Subring of $SL(n, \mathbb{F}_q)$

We start by recalling the main result of [10]:

THEOREM (2). *If G is a finite group and $R\mathbb{F}_q(G)^\wedge$ the completion of the modular representation ring with respect to powers of the ideal of elements of virtual dimension zero, then there is a continuous isomorphism*

$$R\mathbb{F}_q(G)^\wedge \xrightarrow{\alpha^\wedge} K\mathbb{F}_q(BG) ,$$

which is natural for group homomorphisms and field extensions.

The importance of this theorem is that the right hand side can be studied by means of the Atiyah-Hirzebruch spectral sequence for the cohomology theory $K\mathbb{F}_q^*(\cdot)$, which has coefficients in $K_*(\mathbb{F}_q)$, a sequence of groups which in positive dimensions are alternately zero and cyclic of order coprime with p. This has the effect that at the E_2-level we can neglect p-torsion in cohomology. Furthermore at least for both the untwisted and twisted Chevalley groups the whole construction can be lifted to characteristic zero, provided the field $\mathbb{F}_q$ is a sufficiently large extension of $\mathbb{F}_p$. The point here is that if $\ell \neq p$ divides the order of G, then ℓ divides some $q^j - 1$ for some power j. Hence, if necessary extending $\mathbb{F}_q$ to $\mathbb{F}_{q'}$, we can guarantee that $H^{2k-1}(G, K_{2k-1}(\mathbb{F}_{q'}))$ carries $\oplus_{\ell \neq p} H^{2k}(G, \mathbb{Z})_{(\ell)}$.

THEOREM (3). *Let G be a finite untwisted or twisted Chevalley group defined over $\mathbb{F}_q$. Then the subring $Ch(G)_{\ell \neq p}$ of $H^{even}(G, \mathbb{Z})_{\ell \neq p}$ is generated by the classes $\{c_i(\rho) : i = 1, 2, \dots, \deg(\rho)\}$, where ρ runs through the irreducible representations of G over $\mathbb{F}_{q'}$, some sufficiently large extension of $\mathbb{F}_q$.*

Proof. As in our earlier papers $Ch(G)$ denotes the subring in even-dimensional cohomology generated by the Chern classes of *all* representations in characteristic 0. The point of Theorem 3 is that away from $\ell = p$ we can restrict attention to the smaller class of p-modular representations, defining Chern classes as in our earlier paper [13].

The proof is by comparison of Rector's spectral sequence with the analogue of Atiyah's spectral sequence in algebraic K-theory. This takes the form

$$E_2^{r,s} = H^r(G, K_{-s}(\bar{\mathbb{Q}})) \Longrightarrow K\bar{\mathbb{Q}}^{r+s}(BG) ,$$

where $K\bar{\mathbb{Q}}^0(BG)$ may be identified with the I-adic completion of the characteristic 0 representation ring $R(G)$, I = kernel of the augmentation map. Now since G is finite, we can neglect the uniquely divisible summand in $K_{-s}(\bar{\mathbb{Q}})$, and take the coefficients in $E_2^{r,s}$ as

$$K_{-s}(\bar{\mathbb{Q}}) = \begin{cases} \mathbb{Q}/\mathbb{Z}\,, & s = odd\,, \\ 0\,, & s = even\,. \end{cases}$$

For this see [11]. Using the work of Quillen and Suslin we can therefore construct a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \longmapsto & K_{2i-1}(\bar{\mathbb{Q}}) \\ \uparrow & & \uparrow{\scriptstyle Br} \\ \mathbb{Z}/q^i - 1 & \cong & K_{2i-1}(\mathbb{F}_q)\,, \end{array}$$

in which the vertical arrow on the left is the inclusion of the unique cyclic group of order $(q^i - 1)$ and that on the right is induced by Brauer lifting. (As explained in [9] a fixed choice of embedding $\bar{\mathbb{F}}_q^* \to \bar{\mathbb{Q}}^* \subseteq \mathbb{C}^*$ gives rises to a map of classifying spaces $BGL(\mathbb{F}_q)^+ \to BGL(\bar{\mathbb{Q}})^+$; now take homotopy groups.) In its turn $Br : K_{-s}(\mathbb{F}_q) \to K_{-s}(\bar{\mathbb{Q}})$ induces a map of spectral sequences, such that in characteristic zero $\{E_\infty^{r,-r} : r \geq 0\}$ is isomorphic to the topologically graded object determined by the image of $R\mathbb{F}_q(G)$ under Brauer lifting. Now extend $\mathbb{F}_q$ to $\mathbb{F}_{q'}$ far enough to ensure that $E_2^{r,-r}(\mathbb{F}_{q'})$ maps onto $E_2^{r,-r}(\bar{\mathbb{Q}})_{\ell \neq p}$ (see the remark preceeding the statement of the theorem). At the $E_\infty^{r,-r}$-level we can now replace the Brauer image by $R(G)^{\psi^{q'}}$, when $\psi^{q'}$ is the appropriate Adams operation. Note that in algebraic K-theory the term $E_2^{r,-r}$ is non-zero for $r = 2i - 1(odd)$; translation into the language of topological K-theory is made by means of the Bockstein isomorphism

$$\beta : H^{2i-1}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^{2i}(G, \mathbb{Z})\,,$$

compare the discussion in section 2 of [13]. Algebraically the topological filtration is *odd*, explaining the shift in dimension brought about by operation with β.

Remarks: (1) The splitting field $\mathbb{F}_{q'}$ equals $\mathbb{F}_q$ except when $G = {}^2B_2$, ${}^2G_2$ or ${}^2F_4$, see the remarks following 5.4.1 in [8]. For the case $SL(n, \mathbb{F}_q)$ this is clear from the discussion below.

(2) It would be interesting to construct a direct proof of Theorem 3, without passing through algebraic K-theory. Given the identification of $R\mathbb{F}_q(G)$ with $R(G)^{\psi^q}$, this should be possible by comparing cohomology operations in ordinary cohomology and K-theory.

Let us interpret theorem 3 in the special case of $G = SL(n, \mathbb{F}_q)$ of order $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-2})q^{n-1}$. As a λ-ring $R\mathbb{F}_q(G)$ is generated by the class of the identity representation over $\mathbb{F}_q$ (Theorem 1), and if ρ denotes its

Brauer lift to characteristic zero, then $Ch(SL(n,\mathbb{F}_q))_{\ell\neq p}$ is generated by the Chern classes $\{c_i(\rho) : i = 1,2,\ldots,n\}$. Without explicit mention of algebraic K-theory we can explain this as follows: G contains a cyclic subgroup $S = \langle s\rangle$, generated by the so-called "Singer cycle" of order $q^n - 1/q - 1$, which has the following properties:

(i) S is self-centralising, $Z_G(S) = S$,

(ii) the group of automorphisms of S induced by inner automorphisms of G, $N_G(S)/S$, is cyclic of order n, and

(iii) if $x \in S$ and for each k with $k < n$ and $k|n$ the order of x does not divide $q^k - 1$, then $N_G(\langle x\rangle) = N_G(S)$. Property (iii) implies that if G_ℓ is an ℓ-Sylow subgroup contained in S, then $H^*(G,\mathbb{Z})_{(\ell)} \subseteq H^*(G_\ell,\mathbb{Z})$ has period $2n$ and is generated by $c_n(\rho)_{(\ell)}$, where ρ is the Brauer lift of the identity representation of G over $\mathbb{F}_q$. For the group theory see [7]. The class $c_n(\rho)_{(q^n-1/q-1)}$ actually restricts to a generator of $H^{2n}(S,\mathbb{Z})$, but if ℓ divides q^k-1 for some $k < n$, S no longer contains an ℓ-Sylow subgroup, and we have only obtained part of the ℓ-cohomology of G. Hence for an arbitrary prime $\ell \neq p$ it is possible to regard the ℓ-torsion in $Ch(G)_{(\ell)}$ as being carried by the Singer cycles for the ascending chain of subgroups $\{SL(k,\mathbb{F}_q) : k \leq n\}$. Thus we obtain particularly good information about $H^*(G,\mathbb{Z})_{(\ell)}$ when an ℓ-Sylow subgroup G_ℓ is cyclic, i.e. the ℓ-torsion in cohomology is periodic. Such primes ℓ can be picked out as follows (see [7, page 190]):

Let ℓ be an odd prime number such that,

1) the order t of $q \pmod{\ell}$ is greater than $\frac{n}{2}$, and

2) ℓ divides $q^t - 1$ to the first power only.

Then if $m \leq n$ is the smallest positive integer such that $2 > \frac{m}{t} \geq 1$, $H^*(SL(n,\mathbb{Z}_q),\mathbb{Z})_{(\ell)}$ is periodic and is generated by $c_m(\rho)_{(\ell)}$.

For more detail of what is going on consider the special cases of $n = 2, 3, 4$.

Example (1). $SL(2,\mathbb{F}_q)$.

Away from the prime p this group has cohomological period 4, and $H^*(SL(2,\mathbb{F}_q),\mathbb{Z})_{\ell\neq p}$ is generated by $c_2(\rho)$, where ρ denotes the standard action on $\mathbb{F}_q^{\oplus 2}$. Lifted to characteristic 0, $\rho = \rho_1 - \rho_2$, where ρ_1 (of degree $q+1$) and ρ_2 (of degree $q-1$) are irreducible representations constructed by a transfer argument.

Example (2). $SL(3,\mathbb{F}_q)$, $q \equiv -1 \pmod{12}$.

This example is more interesting. The group has order $(q^3-1)(q^2-1)q^3 = (q^2+q+1)(q+1)(q-1)^2q^3$, where the first factor gives the order of the Singer cycle and the third that of the maximal torus $T^3\mathbb{F}_q$. Suppose first that $\ell \neq 2, 3, p$ is a prime not dividing $q-1$, so that if ℓ divides the order of G, ℓ divides either q^2+q+1 or $q+1$. There is a corresponding Sylow subgroup G_ℓ contained either in the subgroup generated by the Singer cycle for $SL(2,\mathbb{F}_q)(q+1\ \textit{case})$ or for $SL(3,\mathbb{F}_q)(q^2+q+1\ \textit{case})$. The ℓ-torsion in cohomology is periodic with period equal to 2 or 3, and a generator is provided by the appropriate Chern class of ρ. In order to obtain an easily expressible answer for the primes 2 and 3, we restrict attention to primes q such that $q \equiv -1 \pmod 4$ and $\equiv -1$ $(mod\ 3)$. The first restriction implies that a 2-Sylow subgroup is semidihedral, a class of groups whose cohomology is well-understood. In particular, since G is perfect, $H^{even}(G,\mathbb{Z})_{(2)}$ is generated by $c_2(\rho)_{(2)}$ of dimension 4. Finally if $q+1$ is divisible by 3, a 3-Sylow subgroup must be cyclic, and has been covered in the previous discussion.

It remains to consider primes $\ell \geq 5$ dividing $q-1$. There is a representative ℓ-Sylow subgroup contained in the rank 2 maximal torus, and the cohomology has been described by several authors, see for example [9]. The polynomial part is generated by Chern classes —associated with the two Singer cycles. Putting everything together we have quite explicitly shown that, with $q \equiv -1$ $(mod\ 12)$,

$$H^{even}(SL(3,\mathbb{F}_q),\mathbb{Z})_{p\neq\ell} = \langle c_2(\rho), c_3(\rho)\rangle \ .$$

Working directly in characteristic 0 rather than in the natural characteristic would make this result much less transparent. Thus as a virtual representation $\rho = \rho_1 - \rho_2 + \rho_3$, where

$$\deg(\rho_1) = q^2+q+1,\ \deg(\rho_2) = (q-1)(q^2+q+1) \text{ and } \deg\rho_3 = (q-1)^2(q+1)\ .$$

All three representations are obtained by transfer from proper subgroups, ρ_3 restricts to a fixed point free representation of the 3-Singer cyclic subgroup and $\rho_1 - \rho_2$ to a similar representation of the Singer cycle of a suitably embedded copy of $SL(2,\mathbb{F}_q)$.

For $n \geq 4$ the situation becomes progressively more complicated. Thus if $n = 4$ the primes ℓ for which the cohomology is periodic are those dividing q^2+1 (period 8) and q^2+q+1 (period 6), and in these cases $c_j(\rho)(j = 3,4)$ provides a generator.

3. Further Examples

First consider the symplectic group $Sp(2m,\mathbb{F}_q)$, for which the representation ring in the natural characteristic is generated by the exterior powers λ_j of the identity representation acting the vector space $\mathbb{F}_q^{\oplus 2m}$. Stably one knows that $H^*(BSp(\infty),\mathbb{Z})$ contains the subalgebra $\mathbb{Z}[p_1,p_2,\ldots]/\{(q^{2k}-1)p_k = 0\}$, where $p_k \in H^{4k}$ is the kth symplectic Pontrjagin class (see [4]). Non-stably the situation is illustrated by

Example (3). $Sp(4,\mathbb{F}_q), p \geq 5$.

The group has order $(q^2+1)(q+1)^2(q-1)^2q^4$, and we suppose that ℓ is a prime distinct from 2 or p. The relations $q^2+1 = (q\pm 1)^2 \mp 2q$ show that ℓ can divide at most one of the factors (q^2+1), $(q+1)$ and $(q-1)$. In particular 3 divides either $(q+1)$ or $(q-1)$ and a 3-Sylow subgroup is of "toral type" with rank equal to 2. In general we have

(i) ℓ divides $q-1$, and $H^*(G,\mathbb{Z})_{(\ell)}$ is obtained by restriction to a maximal torus,
(ii) ℓ divides $q+1$, and $H^*(G,\mathbb{Z})_{(\ell)}$ is obtained by restriction to the subgroup $SL(2,\mathbb{F}_q)\times SL(2,\mathbb{F}_q)$. Finally
(iii) ℓ divides q^2+1, and an ℓ-Sylow subgroup is generated by some power of the Singer cycle.

If as before ρ describes the Brauer lift of the identity representation we have proved that $H^{even}\left(Sp(4,\mathbb{F}_q),\mathbb{Z}\right)\otimes\mathbb{Z}[1/2p]$ is generated by the symplectic Pontrjagin classes $p_k(\rho)$, $k=1,2$.

Example (4). $G_2(\mathbb{F}_q)$, $q\geq 3$.

In his thesis [6] D. Green determines the structure of the cohomology ring of $G_2(q)$ away from the primes 2,3 and p. Thus

$$H^*(G_2(q),\mathbb{Z})\otimes\mathbb{Z}[\frac{1}{6p}]\cong\mathbb{Z}[\frac{1}{6p}][\alpha,\beta]\otimes E(\chi)\,,$$

where the generators have degrees (respectively orders) equal to 4, 12, 15 (respectively q^2-1, (q^6-1), q^2-1). Furthermore the even dimensional part is generated by Chern classes. The relation between the representations in characteristic 0 and the natural representation σ mentioned in the introduction is clearly a complicated one. In particular, if $q=5$, the cohomology is periodic for $\ell=7,31$, and in both cases the period is 12. (Note that α makes no appearence, since $q^2-1=24$.) The required 6th Chern class can be read off from the Big Red Book; in order to handle $\ell=31$ we need the character χ_{21} of degree $12096=2^6\cdot 3^3\cdot 7$.

There is a similar result for the twisted group ${}^2G_2(\mathbb{F}_q)$, $q=3^{2m+1}$. Here $H^{even}({}^2G_2(\mathbb{F}_q),\mathbb{Z})\otimes\mathbb{Z}[\frac{1}{6}]\cong\mathbb{Z}[\frac{1}{6}][\alpha,\beta]$, where α and β have the same degrees as before and orders $(q-1)$, (q^3+1). Again the precise relation between representations in characteristics 0 and 3 would be interesting.

4. Real and Symplectic Versions of Rector's Theorem

Theorem 2, quoted in section two, is the analogue in characteristic p to Atiyah's theorem on the completion of the representation ring of a finite group G over the complex numbers. This holds both for real and symplectic representations, although in the first case one must prove the original theorem in the framework of "representations with involution", and in the second appeal to Bott periodicity in the form $KO^{i+4}(point)=KSp^i(point)$, see [12]. In characteristic p, as lucidly explained by Quillen in the appendix to [9], one can also define Grothendieck groups of orthogonal and symplectic representations, starting from homomorphisms $\rho:G\rightarrow Aut(\mathbb{F}_q^{\oplus n})$ which respect a preassigned symmetric or antisymmetric bilinear form. From the point of view of K-theory one can define cohomology theories $KO\mathbb{F}_q^*(\cdot)$ and $KSp\mathbb{F}_q^*(\cdot)$ with coefficients given by the homotopy groups of the appropriate fibre $F\psi^q$. (At this point one makes essential use of the initial assumption that q is *odd*, since one needs to use the fact that an odd tensor power of a symplectic bundle is symplectic.) The coefficients are listed in the following table, in which (r) denotes $\mathbb{Z}/r$, see [5]:

$i \pmod 8$	$\pi_i(BO(\mathbb{F}_q)^+)$
0	(2)
1	$(2) \oplus (2)$
2	(2)
3	$(q^{(i+1)/2} - 1)$
4	0
5	0
6	0
7	$(q^{(i+1)/2} - 1)$

The groups $\pi_i(BSp(\mathbb{F}_q)^+)$ can be calculated from $\pi_{i+4}(BO(\mathbb{F}_q)^+)$; in particular the copies of $\mathbb{Z}/2$ arise in dimensions 4,5,6 —compare the calculations of Fiedorowicz and Priddy already alluded to at the start of §3.

THEOREM (4). *There are flat bundle homomorphisms α, whose completions define isomorphisms*

$$\hat{\alpha} : RO\mathbb{F}_q(G)^\wedge \longrightarrow KO\mathbb{F}_q(BG) \quad and$$
$$\hat{\alpha} : RSp\mathbb{F}_q(G)^\wedge \longrightarrow KSp\mathbb{F}_q(BG)\ ,$$

having the naturality properties of Theorem 1.

Proof. We have the identifications $RO\mathbb{F}_q(G) = Ker(\psi^q - 1)$, where $\psi^q : RO(G) \longrightarrow RO(G)$ is the Adams operation on the classical real representation ring, and $KO\mathbb{F}_q(BG) = Ker(\psi^q - 1)$, where ψ^q is the corresponding operation in real K-theory. For the first result see [9, 5.4.2], for the second [5, Corollary 1.5]. Furthermore by taking the field in which we work to be sufficiently large we can ensure that ψ^q is an idempotent operator, under which, for example, $RO(G)$ maps onto $RO\mathbb{F}_q(G)$. Consider the diagram

$$\begin{array}{ccccc} RO\mathbb{F}_q(G) & \longrightarrow & RO(G) & \xrightarrow{\psi^{q-1}} & RO(G) \\ \downarrow{\alpha} & & \downarrow{\alpha'} & & \downarrow{\alpha'} \\ KO\mathbb{F}_q(BG) & \longrightarrow & KO(BG) & \longrightarrow & KO(BG)\ , \end{array}$$

where α' is the flat bundle homomorphism for $O(n)$-bundles and α is its characteristic p analogue, defined as in [10, §3]. Complete the upper row with respect to the augmentation ideal in $RO(G)$, i.e. first give $RO\mathbb{F}_q(G)$ the structure of an $RO(G)$-module (using the surjective homomorphism ψ^q). Completion is left exact, so this gives the diagram

$$\begin{array}{ccccc} RO\mathbb{F}_q(G)^\wedge & \longrightarrow & RO(G)^\wedge & \xrightarrow{\psi^{q-1}} & RO(G)^\wedge \\ \big\downarrow \alpha^\wedge & & \big\downarrow \alpha' & & \big\downarrow \alpha'^\wedge \\ KO\mathbb{F}_q(BG) & \longrightarrow & KO(BG) & \xrightarrow{\psi^{q-1}} & KO(BG)\,. \end{array}$$

The assertion of the theorem will follow once we show that the topologies on $RO\mathbb{F}_q(G)$ as an $I\mathbb{C}_G$ and $I\mathbb{F}_{q_G}$-module coincide. This follows from the identification of ψ^q with the decomposition homomorphism d, again see [9, 5.4.2].

The symplectic argument is similar, using the bijection $\alpha'^\wedge$ in characteristic zero symplectic theory, constructed as already mentioned in [12]. In a way similar to that used in section two these isomorphisms, and their associated spectral sequences, can presumably be used to throw light on the orthogonal and symplectic representations of finite linear groups.

DPMMS, 16 Mill Lane
GB-Cambridge CB2 1SB
E-mail: c.b.thomas@pmms.cam.ac.uk.

References

[1] M. F. Atiyah, *Characters and the cohomology of finite groups*, Publ. Math. IHES **9** (1961), 23-64.

[2] Th. Bröcker, T.tom Dieck, Representations of compact Lie groups; Springer-Verlag (Heidelberg), 1985.

[3] D. Carlisle, N. Kuhn, *Subalgebras of the Steenrod algebra and the action of matrices on truncated polynomial algebras*, J. Algebra **121** (1989), 370-387.

[4] Z. Fiedorowicz, S. Priddy, *Homology of classical groups over a finite field*, Springer LN551 (1976), 269-277.

[5] E. Friedlander, *Computations of K-theories of finite fields*, Topology **15** (1976), 87-109.

[6] D. J. Green, Thesis (Cambridge University, 1991).

[7] B. Huppert, Endliche Gruppen I; Springer-Verlag (Heidelberg), 1966.

[8] P. Kleidman, M. Liebeck, *The subgroup structure of the finite classical groups*, LMS Lecture Notes **129**, Cambridge University Press (1990).

[9] D. Quillen, The Adams Conjecture; Topology **10** (1970), 67-80.

[10] D. Rector, *Modular characters and K-theory with coefficients in a finite field*, J. Pure and Appl. Algebra **4** (1974), 137-158.

[11] A. A. Suslin, *On the K-theory of local fields*, J. Pure and Appl. Algebra **34** (1984), 301-318.

[12] C. B. Thomas, *Maps between real classifying spaces*, Math. Ann. **253** (1980), 195-203.

[13] ——, *Characteristic classes and 2-modular representations of some sporadic simple groups*, Contemporary Math. **96** (1989), 303-318.