AN UNCOUNTABLE GROUP OF FIBRE HOMOTOPY EQUIVALENCES

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1. Introduction

Let G be a compact connected Lie group which is not a torus and let T be a maximal torus of G. We have the following fibre bundle with structure group G

$$(1.1) \qquad G/T \to BT \to BG,$$

where BG and BT are classifying spaces of G and T respectively.

Then the set of all fibre homotopy classes of fibre homotopy equivalences of the total space BT of (1.1) forms a group under the multiplication defined by the composition of maps. This group is called the group of fibre homotopy equivalences, and we denote it by $\mathscr{L}(BT)$ (cf. [4], [7]).
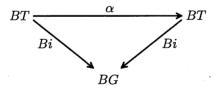
It is shown in [5, Theorem, p.422] that for the fibre space (E, p, B, F) with fibre F, $\mathscr{L}(E)$ is a finitely presented group, when E, B and F have the homotopy type of finite CW-complexes.

In this note we show that for the fibration (1.1), $\mathscr{L}(BT)$ is a split extension of an uncountable rational vector space by the Weyl group $W(G)$.

In a similar way, we can define the group of self homotopy equivalences of a topological space X, which we denote by $\mathscr{E}(X)$.

By [3, Theorem 3.6], the Weyl group of a given connected compact Lie group G is given as follows.

Let $i: T \subset G$ be the inclusion of a maximal torus, then the Weyl group $W(G)$ is isomorphic to the group of homotopy classes of homotopy equivalences $\alpha: BT \to BT$ over Bi such that the following diagram

$$(1.2)$$

is homotopy commutative.

Then, one can easily see that the forgetful homomorphism N

$$(1.3) \qquad \mathscr{L}(BT) \xrightarrow{N} W(G) \subset \mathscr{E}(BT) = GL(n; Z)$$

is surjective to the Weyl group $W(G)$, by using the covering homotopy property of the fibration (1.1).

Thus we have an example where the group of fibre homotopy equivalences is an infinite group whose image in the group of homotopy equivalences is finite, since $W(G)$ is finite for any compact connected Lie group G.

Let

$$(1.4) \qquad W(G) \times T \to T$$

be the natural action of the Weyl group on the torus.

$W(G)$ acts on BT and hence on

$$(1.5) \qquad \pi_1(\mathrm{map}(BT, BG), Bi).$$

We have the following

THEOREM (A). *For the fibre bundle (1.1), the group of fibre homotopy equivalences $\mathscr{L}(BT)$ is a semi-direct product of an uncountable rational vector space (1.5) and the Weyl group $W(G)$ with the operator as mentioned above.*

2. Proof of Theorem (A)

Let (E, p, B, F) be a fibration with fibre F. Then

$$(2.1) \qquad p^E \colon \mathrm{map}(E, E) \to \mathrm{map}(E, B), \ p^E(f) = pf,$$

is also a fibration with fibre $(p^E)^{-1}(p)$, which is the space of all fibre preserving maps of E to itself.

We apply this to the case that $(E, p, B, F) = (BT, Bi, BG, G/T)$.

Since $\pi_1(\mathrm{map}(BT, BG), Bi)$ acts on $\mathscr{L}(BT)$, the long exact sequence of the homotopy group restricts to the following exact sequence (cf. [7, Remark 7.9], [4, p.448])

$$(2.2) \qquad \pi_1(\mathrm{map}(BT, BT), 1) \to \pi_1(\mathrm{map}(BT, BG), Bi) \overset{\partial}{\longrightarrow} \mathscr{L}(BT) \overset{N}{\longrightarrow} \mathscr{E}(BT).$$

The image of $\mathscr{L}(BT)$ by N is $W(G)$ as stated in §1. And $\pi_1(\mathrm{map}(BT, BT), 1) = \pi_1(\mathrm{map}(K(\pi, 2), K(\pi, 2), 1)) = 0$ by R. Thom [6, Theorem 2].

We consider the homotopy group $\pi_1(\mathrm{map}(BT, BG), Bi)$, where i is an inclusion of a maximal torus T to G.

Let $i \colon T \to G$ be an inclusion of a maximal torus to the compact connected Lie group. Denote by $C(i)$ the centralizer of the image i. The obvious homomorphism

$$(2.3) \qquad C(i) \times T \to G$$

passes to a map of a classifying space, which has an adjoint

$$(2.4) \qquad ad(i) \colon BC(i) \to \mathrm{map}(BT, BG)_\rho (\rho = Bi)$$

Let us consider the following fibration (see [2, p.164])

$$(2.5) \qquad X_i \to BC(i) \xrightarrow{ad(i)} \mathrm{map}(BT, BG)_\rho \ (\rho = Bi),$$

where X_i is a homotopy fibre of $ad(i)$.

We have $BC(i) = BZ(T) = BT$, since T is a maximal torus ($Z(T)$ denotes the centralizer of T). Therefore by the homotopy exact sequence of (2.5), we have

$$(2.6) \qquad \pi_1(\mathrm{map}(BT, BG), Bi) \cong \pi_0(X_i).$$

By [2, Lemma 2.3, Remark 2.7 and Proposition 6.1],

$$(2.7) \qquad \pi_0(X_i) = \prod_{n \geq 1} H^n(BT; \pi_{n+1}(G) \otimes Z^\wedge/Z),$$

where $Z^\wedge$ is the completion of the integer Z, since $\pi_1(G/T)$ is trivial.

$\pi_{n+1}(G)(n \geq 1)$ contains a torsion free subgroup for some even n and $Z^\wedge/Z$ is a rational vector space of uncountable dimension. Hence $\pi_0(X_i)$ is an uncountable rational vector space.

Now we have the following exact sequence

$$(2.8) \qquad 1 \to \pi_1(\mathrm{map}(BT, BG), Bi) \to \mathscr{L}(BT) \to W(G) \to 1.$$

$W(G)$ acts on T as in (1.4), hence on BT and hence on $\pi_1(\mathrm{map}(BT, BG), Bi)$. Let

$$(2.9) \qquad \varphi: W(G) \to Aut\ \pi_1(\mathrm{map}(BT, BG), Bi)$$

be the operator induced by this action.

Let *Opext* denote the set of all congruence classes of extensions of the Abelian group (2.6) by $W(G)$ with operator φ as in (2.9). Since

$$(2.10) \qquad H^2_\varphi(W(G), \pi_1(\mathrm{map}(BT, BG), Bi)) = 0$$

by (2.6), (2.7) and [1, Corollary 5.2], *Opext* consists of a single element by [1, Theorem 4.1].

Therefore $\mathscr{L}(BT)$ is a semi-direct product of the vector space (2.6) and the Weyl group $W(G)$ with operator (2.9).

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