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THE TOP OF A SYSTEM OF EQUATIONS

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Dedicated to the memory of Professor José Adem

Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, and let $\mathbf{f} = f_1, \dots, f_m$ be a set of polynomials in R. This paper seeks means to determine the isolated zeros of the system of equations $\mathbf{f} = 0$. The motivation stems from the fact that often those zeros are the 'most singular' ones.

There are already techniques to accomplish this goal. For instance, in [2] methods are developed to compute radicals of ideals and which as a byproduct would isolate that set. It works by successively computing the higher dimensional components of the variety $V(\mathbf{f})$ first. We look here for more direct means. An example of the results obtained is the following explicit formula (mod some restrictions on the characteristic of k):

THEOREM (3.1). *Let $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ be a set of n polynomials, and let Δ be its Jacobian determinant. Then*

$$(f_1, \dots, f_n) : \Delta$$

is the radical of the minimal primary components of dimension 0.

The notation means that the elements of R which multiply Δ into $(\mathbf{f})$ is an ideal $(g_1, \dots, g_n)$ of a finite set of points. In addition, its excess multiplicities have been stripped away. This last feature is important when one wants to solve the system of equation by analytic methods, such as the path continuation approach.

To extend this to any ideal, we sketch a method to make slightly more effective the classical proof that ideals of $k[x_1, \dots, x_n]$ are set-theoretically generated by n polynomials ([1], [8]). This means that for any ideal I there are elements $\mathbf{f} = \{f_1, \dots, f_n\}$ with $\sqrt{I} = \sqrt{(\mathbf{f})}$. Since $(\mathbf{f})$ and I have the same minimal prime ideals, the formula applied to $\mathbf{f}$ yields the isolated zeros of I.

1. Regular sequences and radicals

Before we outline the technique that will be used, we recall the definition of the Jacobian ideals attached to a given ideal I and of the socle of an algebra.

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Let $R = k[x_1, \ldots, x_n]$, k is a perfect field, and let $I = (f_1, \ldots, f_m)$ be an ideal of R. We shall recall some basic properties of the module of Kähler differentials of the algebra $A = R/I$. Our basic reference will be [5].

The module of k–differentials of the algebra A will be denoted by $\Omega_{A/k}$. Although this module is independent on how it is presented as a quotient of a polynomial ring, it can be conveniently described by the exact sequence of modules of differentials

$$I/I^2 \xrightarrow{d} \Omega_{R/k} \otimes_R A \longrightarrow \Omega_{A/k} \to 0,$$

where d is the universal derivation: $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$.

The Jacobian ideals of I are the determinantal ideals of the matrix

$$\varphi = \left(\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)} \right) \bmod I.$$

We denote them by

$$J_a(I) := I_{n-a}(\varphi).$$

If $r = \operatorname{codim} I$, the *Jacobian ideal* proper is the ideal of $r \times r$ minors of φ; it shall be denoted by J. Because the $J_a(I)$ are the Fitting ideals of the module $\Omega_{A/k}$, they behave well with regard to many processes: localization, completion.

Let $(A, \mathbf{m})$ be a local algebra of dimension zero. The *socle* of A is the annihilator of $\mathbf{m}$. For an example, let k be a field and let $0 \neq f \in k[x]$ be a polynomial. Suppose that the characteristic of k does not divide the degrees of the irreducible factors of f. The algebra A decomposes into a finite product of local algebras

$$A = A_1 \times \cdots \times A_r,$$

and the image of $f'(x)$, the derivative of $f(x)$, in each A_i generates its socle. This can be formulated as an explicit radical formula:

$$\sqrt{(f)} = (f) : f'$$

For more general algebras, it is not well known how to predict, from the generators and relations of the algebra, which elements will generate the socle. An important exception is the following result of Scheja and Storch [7]:

THEOREM (1.1). *Let k be a field and let $A = k[[x_1, \ldots, x_n]]/I$ be a finite dimensional k–algebra. Assume $\dim_k A$ is not divisible by the characteristic of k. Denote by J the Jacobian ideal of A. If A is a complete intersection then J generates the socle of A. Conversely, if k has characteristic zero and A is not a complete intersection then $J = 0$.*

By conveniently extending the notion of socles to generic socle generators, the following Nullstellensatz is given in [2] (see also [10], [11]):

THEOREM (1.2). *Let I be an ideal whose primary components all have codimension m. Let $\{f_1, \ldots, f_m\}$ be a regular sequence in I, and let J_0 be the Jacobian ideal of $I_0 = (f_1, \ldots, f_m)$. Then*

$$\sqrt{I} = (I_0 \colon J_0) \colon ((I_0 \colon J_0) \colon I).$$

Used repeatedly this result can pick out the isolated zeros of a set of equations by deleting the higher dimensional components, one at a time. But used with this purpose it is obviously very onerous since it will compute many unnecessary components.

Partition of the unity

This method to compute radicals requires an efficient mechanism to select regular sequences. Regular sequences can be found in generic combinations of the generators of an ideal. More precisely, if $I = (\mathbf{f}) = (f_1, \ldots, f_m)$ is an ideal of codimension g and $\varphi = (c_{ij})$ is a sufficiently generic $m \times g$ matrix with entries in k, then the entries of $\mathbf{f} \cdot \varphi$ generate an ideal of codimension g. A drawback in this approach lies in the loss of whatever sparseness is present in the data, which is a resource which must be preserved. A much more effective approach in case of sparse data is given in [3].

Another source of regular sequences in ideals are some of the characteristic sets of Wu and Ritt (see [4]). But this aspect of the subject deserves additional examination.

We discuss next a mechanism that calls on the algorithm above itself to generate its own regular sequences. Actually it only produces an ideal which is locally a complete intersection; but this is obviously all that is required to use Theorem (1.2). It is more of an approach than an algorithm proper. It permits however a great deal of *manual* control. Its shape and usage are based on the following elementary observations.

PROPOSITION (1.3). *Let R be a reduced Noetherian ring with n minimal prime ideals. Let I be an ideal of codimension at least one. Then*

(a) *For $a \in I$ there exists $b \in I \cap (0 \colon a)$ such that $a + b$ is a regular element.*
(b) *If $a \neq 0$ for each $0 \neq b \in I \cap (0 \colon a)$, $(0 \colon a) \neq (0 \colon (a + b))$.*
(c) *Iteration leads to a regular element contained in I in at most n steps.*

Proof. (a): Any prime ideal $\wp \supseteq (a, I \cap (0 \colon a))$ either contains I, which has codimension at least one, or will contain $(a, (0 \colon a))$. Since R is reduced the latter has also codimension at least 1. This means that there exists a regular element of the form $ra + b$, with $b \in I \cap (0 \colon a)$. Again using that R is reduced, $a + b$ is regular as well.

(b): This is immediate since any annihilator of $a + b$ must annihilate both a and b.

(c): Let $a_1, a_2, \ldots, a_r$ be obtained by this process: in the notation above, we repeatedly set $a = a_1 + \cdots + a_{j-1}$, $b = a_j$. The descending chain of ideals

$$0\colon a_1 \supset 0\colon (a_1 + a_2) \supset \cdots \supset 0\colon (a_1 + a_2 + \cdots + a_r)$$

gives rise to a similar sequence in the total ring of fractions S of R. Since S has a composition series of length n, the last ideal vanishes if $r \geq n$. ∎

REMARK (1.4). In practice this runs as follows. Suppose $f_1, \ldots, f_s \in I$ have been chosen so that they generate an ideal of codimension $s <$ height I. Let $L = \sqrt{(f_1, \ldots, f_s)}$. We follow the scheme above, but with colon ideals computed relative to L. We note by f_{s+1} the element of I obtained from the lift.

The measure of manual control over the sparseness comes in because in selecting the a_j's we may also use reduction modulo the previously chosen a_i for $i < j$.

2. The top radical of an ideal

Let I be an ideal of codimension m, minimally generated by $m + r$ elements; r is the *deviation* of I. According to Krull's Theorem, any minimal prime of I has codimension at most $m + r$. We are going to key on those primes. (A cloud of ambiguity lies around the word *minimal*: this being a local question, we take it to be the supremum of the least number of generators of I in all the localizations of R.)

Given a primary decomposition of I, we collect into I_i those primary components of a given codimension $m + i$.

Definition (2.1). In the representation

$$I = I_0 \cap I_1 \cap \cdots \cap I_r \cap \cdots,$$

I_i is called the ith equi–dimensional component of I.

This decomposition can be refined by breaking up each I_i into two pieces: I_i', corresponding to minimal primes of I, and I_i'' derived from the embedded primes of I. If one of these is not present, such as I_0'' or I_i', $i > r$, by abuse of notation we set it equal to R.

The radical of I is then given by the expression

$$\sqrt{I} = \sqrt{I_0'} \cap \sqrt{I_1'} \cap \cdots \cap \sqrt{I_r'}.$$

It suggests that one focus on obtaining formulas for the $\sqrt{I_i'}$'s.

Definition (2.2). Given an ideal I of deviation r, $\sqrt{I'_r}$ is the *top radical* of I (Notation: topradical(I).)

Conjecture (2.3). Let I be an ideal of codimension m and deviation r, and denote by L the ideal generated by the minors of size $m + r$ of the Jacobian matrix of I. Then

$$\text{topradical}(I) = I\colon L.$$

The following provides a measure of support (assume from now on that k is a field of characteristic zero):

THEOREM (2.4). *This conjecture holds if every embedded prime of I has codimension at most $m + r$. In particular it holds if $v(I) \geq$ proj dim R/I.*

Proof. We have

$$I\colon L = (I_0\colon L) \cap (I'_1\colon L) \cap (I''_1\colon L) \cap \cdots \cap (I'_r\colon L) \cap (I''_r\colon L),$$

since by assumption $I''_j = R$ for $j > r$. We claim that all the quotients on the right side, with the possible exception of $I'_r\colon L$, are equal to R. This will suffice to establish the claim since at each minimal prime $\wp$ of I'_r, we have $I_\wp = (I'_r)_\wp$ and $L_\wp$ is nothing but the Jacobian ideal of $(I'_r)_\wp$; we may then apply Theorem (1.2).

We begin by showing that $I''_r\colon L = R$. If this is not so, any associated prime of the left–hand side has codimension $m + r$. Let $\wp$ be then one of its primes and localize R (but keep the notation); denote also $A = R/I$. (We warn the reader about a possible confusion: Sometimes we shall say that an ideal is zero when it would be more appropriate to say it is zero mod I.)

As before we may assume that $A = k[[x_1, \ldots, x_n]]/I$, and $\wp$ is the maximal ideal of A. We consider the Jacobian ideals of the Artin algebras $B_s = A/\wp^s$. Because A has dimension > 0, L will map into the Jacobian ideal of B_s, for each s. If B_s is not a complete intersection, by Theorem (1.1), its Jacobian ideal vanishes and thus $L \subset \wp^s$. On the other hand, if B_s is a complete intersection its socle must be $\wp^{s-1}B_s$ and $L \subset \wp^{s-1}$. This means that

$$L \subset \bigcap_{s \geq 0} \wp^s,$$

which by Krull's intersection theorem implies $L = 0$.

The case of a component such as $I'_j\colon L$, for $j < r$, or $I''_j\colon L$, $1 \leq j < r$, is easier to deal with. In fact, if $\wp$ is a minimal prime of say I'_j and $(I'_j)_\wp$ is not a complete intersection, then already the $m + j$-sized minors of the Jacobian matrix of $(I'_j)_\wp$ vanish by Theorem (1.1). On the other hand, if $(I'_j)_\wp$ is a complete intersection, the rank of its Jacobian matrix is $m + j < m + r$, so L vanishes at $(I'_j)_\wp$ anyway.

The last assertion follows from the Auslander-Buchsbaum formula [6] and standard facts on depth. ∎

The natural place to first examine Conjecture (2.3) is among almost complete intersections. These must be considered with care in view of [9, Proposition 4] asserting that many such ideals will satisfy the conditions of the previous theorem.

3. The top of a system of equations

One application of the previous result is the following explicit formula that retrieves exactly the isolated zeros of a set of polynomials.

THEOREM (3.1). *Let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ be a set of n polynomials, and let Δ be its Jacobian determinant. Then*

$$(f_1, \ldots, f_n) \colon \Delta$$

is the radical of the minimal primary components of dimension 0.

To illustrate, suppose $I = \big(f(x,y)g(x,y), f(x,y)h(x,y)\big)$, where $f(x,y)$ is the gcd of I. Denote by Δ the Jacobian determinant of these two generators of I, and denote by Δ_0 the Jacobian of g, h. A simple calculation shows that

$$I \colon \Delta = \big((g,h) \colon \Delta_0\big) \colon f,$$

and therefore

$$I \colon \Delta = \sqrt{(g,h)} \colon f,$$

in agreement with the assertion of the theorem.

Let us recall a result of Eisenbud–Evans [1] and Storch [8]:

THEOREM (3.2). *Let S be a Noetherian ring of dimension $d-1$. Then the radical of any ideal I of $R = S[x]$ is the radical of a d-generated ideal.*

Their proof is fairly constructive already. It is based on the notion of pseudo-division of polynomials and an appropriate induction argument on the dimension of S. We will just rewrite it in sketching out some moves to make it more amenable for a Gröbner basis computation.

We assume that S is a reduced ring. Let $I = (f_1, \ldots, f_m)$. In their proof it is argued that there exists a regular element u of S and an element h_1 of I such that

$$u \cdot I \subseteq (h_1).$$

The issue is how u is to be found. We employ the argument of Proposition (1.3). Given two elements f, g of I, with $\deg f(x) \geq \deg g(x)$ (as polynomials in x), let α be the leading coefficient of $g(x)$. For some power of α we have

$$\alpha^s \cdot f = q \cdot g + r, \ \deg r < \deg g.$$

Replace then f by r and keep processing the list of generators of I until we have a nonzero element $a_1 \in S$ such that $a_1 \cdot I \subseteq (g_1)$, with $g_1 \in I$.

Repeat this step on the generators of $I \cap (0\colon a_1)$, if the latter is nonzero. This produces a sequence $a_1, \ldots, a_r \in S$, with corresponding elements $g_i \in I$ such that

$$\begin{cases} a_i \cdot a_j = 0 & \text{if } i \neq j \\ a_1 + a_2 + \cdots + a_r & \text{is regular on } S \\ a_i \cdot I \subseteq (g_i) \end{cases}$$

Set now

$$u = a_1 + a_2 + \cdots + a_r$$
$$h_1 = a_1 g_1 + a_2 g_2 + \cdots + a_r g_r,$$

and consider the image I^* of I in $(S/\sqrt{(u)})[x]$. By induction select $h_2, \ldots, h_d \in I$ whose images generate an ideal with the same radical as I^*.

It suffices to verify

$$\sqrt{I} = \sqrt{(h_1, h_2, \ldots, h_d)}.$$

Let $\wp$ be a prime ideal with $(h_1, \ldots, h_d) \subseteq \wp$. If $u \in \wp$, by hypothesis $I \subseteq \wp$. Assume otherwise; then $a_i \notin \wp$ for some i which implies that $a_j \in \wp$ for $j \neq i$. Since $h_1 \in \wp$, this means that $g_i \in \wp$ which from the equation $a_i \cdot I \subseteq (g_i)$ finally implies $I \subseteq \wp$, as desired.

REMARK (3.3). This arrangement leaves much to be desired: There is a great deal less of control than that afforded by Proposition (1.3).

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REFERENCES

[1] D. EISENBUD AND E. G. EVANS, JR., *Every algebraic set in n-space is the intersection of n hypersurfaces,* Invent. Math. **19** (1973), 107–112.

[2] ——— C. HUNEKE AND W. V. VASCONCELOS, *Direct methods for primary decomposition,* Invent. Math. 110 (1992), 207–235.

[3] ——— AND B. STURMFELS, *Finding sparse systems of parameters,* Preprint, 1992.

[4] G. GALLO AND B. MISHRA, *Efficient algorithms and bounds for Wu-Ritt characteristic sets,* in Effective Methods in Algebraic Geometry, Progress in Mathematics **94**, Birkhäuser, Boston-Basel-Berlin, 1991, 119–194.

[5] E. KUNZ, Kähler Differentials, Vieweg, Wiesbaden, 1986.

[6] H. MATSUMURA, Commutative Algebra, Benjamin/Cummings, Reading, Massachusetts, 1980.

[7] G. SCHEJA AND U. STORCH, *Über Spurfunktionen bei vollständigen Durchschnitten,* J. reine angew. Math. **278** (1975), 174–190.

[8] U. STORCH, *Bemerkung zu einem Satz von M. Kneser,* Arch. Math. **23** (1972), 403–404.

[9] B. ULRICH, *Remarks on residual intersections,* in Free Resolutions in Commutative Algebra and Algebraic Geometry, Proceedings, Sundance (1990) (D. Eisenbud and C. Huneke, Eds.), Research Notes in Mathematics **2**, Jones and Bartlett Publishers, Boston-London, 1992, 133–138.

[10] W. V. VASCONCELOS, *Constructions in Commutative Algebra,* in Computational Algebraic Geometry and Commutative Algebra, Proceedings, Cortona 1991: D. Eisenbud, et. al., Eds.), Cambridge Univ. Press, 1993, 151–197.

[11] ________ *Jacobian matrices and constructions in algebra,* Proceedings 9th AAEEC, Lecture Notes in Computer Science **539**, Springer-Verlag, Berlin-Heidelberg-New York, 1991, 48–64.