Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

ON THE WHITEHEAD SQUARE, CAYLEY-DICKSON ALGEBRAS, AND RATIONAL FUNCTIONS

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José Adem was interested in certain algebras known as the Cayley-Dickson algebras [1,4,5,6,7]. Namely, given an algebra R with involution, there is a "doubling process" due to L.E. Dickson [4] which gives $R \oplus R$ the structure of an algebra with involution. If R is specialized to the real numbers with trivial involution, then Dickson's doubling process provides an explicit multiplication on $\mathbb{R}^{2^n}$ which is, of course, not norm preserving if $n \geq 4$. One feature of these algebras is that they are closely tied to a classical open problem in homotopy theory, the so-called strong form of the Kervaire invariant conjecture [2,9]. It is the purpose of this note to elaborate on this connection as described in [3]. There are explicit models for studying this last conjecture obtained by considering relations between the Cayley-Dickson algebras and spaces of rational functions. In particular, there are specific and elementary maps which are potentially useful in the study of this problem

The motivation for writing this note is the interest expressed by several people even though this program is unfinished. A deeper analysis is joint work in progress with Wu-Teh Hsiang. We emphasize that the results in the current paper are both elementary and immediate; these results merely provide a framework for further study.

1. Cayley-Dickson algebras

We first recall a construction due to L.E. Dickson [4] and used by J. Adem to construct non-singular bilinear maps [1]. Let R be an algebra with involution χ. Define $CD(0, R) = R$, and

$$CD(1, R) = R \oplus R$$

as a left R-module and give $CD(1, R)$ the structure of a (possibly non-associative) ring with conjugation χ by the formulas

$$(a, b) \cdot (c, d) = (a \cdot c - \chi(d) \cdot b, da + b \cdot \chi(c)) \quad \text{and}$$
$$\chi(a, b) = (\chi a, -b)$$

for (a, b) and (c, d) in $CD(1, R)$. Inductively, define $CD(n, R)$ to be $CD(1, R')$ where $R' = CD(n-1, R)$ if $n > 1$.

*Partially supported by the NSF.

EXAMPLE (1.1) If $R = \mathbb{R}$, the real numbers with $\chi a = a$, then $CD(1, \mathbb{R})$ is isomorphic to the ring of complex numbers, $CD(2, \mathbb{R})$ is isomorphic to the ring of quaternions, and $CD(3, \mathbb{R})$ is isomorphic to the ring of Cayley numbers.

In this note, we restrict attention to $CD(n, \mathbb{R})$. In Adem's paper [1], he studies a basis $\{e_0, \dots e_q\}$, $q = 2^n - 1$, for $CD(n, \mathbb{R})$. This basis satisfies the multiplicative properties

$$\begin{aligned} &(1) \quad e_0 = 1 \\ &(2) \quad e_i^2 = -1 \quad \text{if} \quad i > 0 \\ &(3) \quad \text{if } i, j > 0, \text{ then } e_i e_j = -e_j e_i = \sigma(i,j) e_k \end{aligned}$$

for $\sigma(i,j) = \pm 1$ and some $k = f(i,j)$. Thus if $a = \sum_i a_i e_i$ and $b = \sum_i b_i e_i$, then

$$a \cdot b = (a_0 b_0 - \sum_{i>0} a_i b_i) e_0 + \sum_{k \geq 1} \{a_0 b_k + a_k b_0 + \sum_{0<i<j, e_i e_j = \sigma(ij) e_k} \sigma(i,j)(a_i b_j - a_j b_i)\} e_k.$$

We collect some consequences below.

The following observation is useful as in [3].

LEMMA (1.2) *If a is in $CD(n, \mathbb{R})$, then $a^2 = 0$ if and only if $a = 0$.*

Proof. By the formulas above,

$$a^2 = \left(a_0^2 - \sum_{i>0} a_i^2\right) e_0 + \sum_{k \geq 1} (2 a_0 a_k) e_k.$$

Thus, if $a^2 = 0$, $a_0^2 - \sum_{i>0} a_i^2 = 0$ and $2 a_0 a_k = 0$ for $k \geq 1$. Thus $a = 0$.

Guillermo Moreno has pointed out that a more conceptual proof of Lemma (1.2) follows by decomposing a into its real and imaginary parts together with McCrimmon's formula given in the proof of Lemma (1.3) below.

We are indebted to Ted Erickson for pointing out the next lemma which is based on work of K. McCrimmon [6].

LEMMA (1.3) (T. Erickson) *If $a \cdot b = 0$ in $CD(n, \mathbb{R})$, then $a_0 = b_0 = 0$.*

Proof. Following McCrimmon [6], define $N(a) = a \cdot \chi(a)$ and $T(a) = a + \chi(a) = a_0$ to obtain the formulas $a^2 - T(a)a + N(a) \cdot 1 = 0$ and $N(a^2) = (N(a))^2$. Further define $N(a, b) = T(a \cdot \chi b)$ to obtain the equations

$$\begin{aligned} N(a \cdot b, c) &= N(a, c \cdot \chi b), \\ N(ab, a) &= N(a)T(b) = (ba, a), \quad \text{and} \\ T((a \cdot b) \cdot c) &= T(a \cdot (b \cdot c)). \end{aligned}$$

Thus if $a \cdot b = 0$ and $a, b \neq 0$, then

$$0 = N(0, a) = N(a \cdot b, a) = N(a)T(b). \quad \text{Hence} \quad b_0 = T(b) = 0.$$

A similar calculation gives $a_0 = T(a) = 0$ and the lemma follows.

The next lemma follows immediately from the multiplicative properties of the e_i and Lemma (1.3). Recall that $a = \sum_i a_i e_i$ and $b = \sum_i b_i e_i$ with a and b in $CD(n, \mathbb{R})$. Write $c = \sum_i c_i e_i = a \cdot b$.

LEMMA (1.4) *The following formulas hold in $CD(n, \mathbb{R})$:*

(1) $c_0 = a_0 b_0 - \sum_{i>0} a_i b_i$, *the Lorentz form,*

(2) *if* $k \geq 1$,

$$c_k = a_o b_k + a_k b_o + \sum_{0<i<j, e_i e_j = \sigma(i,j) e_k} \sigma(i,j)(a_i b_j - a_j b_i)$$

(3) $a \cdot b = 0$ *if and only if*

(i) $a_o = b_o = 0$,

(ii) $\sum_{i>0} a_i b_i = 0$, *and*

(iii) $\sum_{0<i<j, e_i e_j = \sigma(i,j) e_k} \sigma(i,j)(a_i b_j - a_j b_i) = 0.$

REMARK. Although (1.4) is an immediate consequence, the orthogonality relations in (1.4)(3) provide an elementary description of zero divisors in terms of simultaneously vanishing quadratic forms. A deeper analysis of the properties of the Cayley-Dickson algebras are given by P. Eakin and A. Sathaye in [5]. It follows quickly from their work that the space of ordered pairs (a, b) with (1) a, b in $CD(4, \mathbb{R})$, (2) $\|a\| = \|b\| = 1$, and (3) $a \cdot b = 0$ is homeomorphic to the Lie group G_2. Paul Yiu has discovered an independent proof of this fact [11]. We finish this remark with an example of (1.4)(3)(iii). Let $k = 1$, then one has

$$\sum_{j \geq 1} (-1)^{1+\alpha(j)}(a_{2j+1} b_{2j} - a_{2j} b_{2j+1}) = 0$$

where $\alpha(j)$ denotes the number of ones in the dyadic expansion of j.

2. Unimodular rows and rational functions

Fix an integer q and let $f_i(z)$ be in the polynomial ring $\mathbb{C}[z]$ with $f_i(z) = \sum_{j=0}^{q} a_{j,i} z^j$. A row $\Phi = (f_0, \ldots, f_n)$ is unimodular provided $f_0(z), \ldots, f_n(z)$

do not simultaneously vanish for any value of z. Any unimodular row Φ determines a continuous function

$$\Phi \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C}P^n$$

given by $\Phi(z) = L$ where L is the line through the origin in $\mathbb{C}^{n+1}$ containing $(f_0(z), \dots, f_n(z))$. In [8], G. Segal defined the space $\mathrm{Rat}_q(\mathbb{C}P^n)$ which is the space of unimodular rows $(f_0, \dots f_n)$ with $a_{q,i} = 1$. There is a related continuous function

$$S \colon \mathrm{Rat}_q(\mathbb{C}P^n) \to \Omega^2 \mathbb{C}P^n$$

defined by $S(\Phi)(z) = \Phi(z)$. The space $\Omega^2 \mathbb{C}P^n$ is not connected; the map S takes values in $\Omega^2_{(q)} \mathbb{C}P^n$ corresponding to maps of degree q. Segal proved that $S \colon \mathrm{Rat}_q(\mathbb{C}P^n) \to \Omega^2_{(q)} \mathbb{C}P^n$ is a homotopy equivalence through a range depending on n and increasing with q.

Next, observe that if λ_i is any element in the principal ideal generated by $f_0(z)$, then $(f_0, f_1 + \lambda_1, f_2 + \lambda_2, \dots, f_n + \lambda_n)$ is again a unimodular row. Thus one obtains operations on the space of unimodular rows. If $n \geq 1$, let $r \colon S^{2n+1} \to S^{2n+1}$ be any choice of orientation preserving homotopy equivalence such that $r(x_0, x_1, 0, \dots, 0) = (1, 0, \dots, 0)$ for all x_i such that $x_0^2 + x_1^2 = 1$. An example of such an r which we shall use later is as follows. Let $k \geq 3$ and let $S^{k-1} = \{(x_0, \dots, x_{k-1}) \in \mathbb{R}^k \mid x_0^2 + \dots + x_{k-1}^2 = 1\}$. Define r by the formula

$$r(x_0, \dots, x_{k-1}) = \begin{cases} (1, 0, \dots, 0) & \text{if } 1 \geq x_3 \geq 0 \\ (2x_2 + 1, \lambda x_0, \lambda x_1, \dots, \lambda x_{k-2}) & \text{if } 0 \geq x_2 \geq -1 \\ \quad \text{with } \lambda = \sqrt{\frac{4x_2}{x_2 - 1}}. & \end{cases}$$

Let t denote the operation on unimodular rows given by

$$t(f_0, \dots, f_n) = (f_0, f_1 - f_0, f_2 - f_0, \dots, f_n - f_0).$$

Define

$$\theta \colon \mathrm{Rat}_q(\mathbb{C}P^n) \to \Omega^2 S^{2n+1}$$

as follows. Let z be a point in $S^2 = \mathbb{C} \cup \{\infty\}$. Then

$$\theta(f_0, \dots, f_n)(z) = \begin{cases} r\left[\frac{t(f_0(z), \dots f_n(z))}{\|t(f_0(z), \dots, f_n(z))\|}\right] & \text{if } z \neq \infty \\ (1, 0, \dots, 0) & \text{if } z = \infty. \end{cases}$$

We remark that composition with r is essential to obtain a continuous map as $\lim\limits_{z \to \infty} \dfrac{t(f_0(z), \dots, f_n(z))}{\|t(f_0(z), \dots, f_n(z))\|}$ is not well-defined and depends on the choice of path.

Our main interest here lies with $\mathrm{Rat}_2(\mathbb{C}P^n)$, but much of the remainder applies to $\mathrm{Rat}_q(\mathbb{C}P^n)$ for $q \geq 2$. We use $\mathrm{Rat}_2(\mathbb{C}P^n)$ in conjunction with the natural multiplication in $CD(m, \mathbb{R})$. Thus it is necessary to give real analogues of $\mathrm{Rat}_2(\mathbb{C}P^n)$.

Here, let $\mathbb{R}[x, y]$ denote the polynomial ring over $\mathbb{R}$ with indeterminates x and y. Given $\varepsilon > 0$, define

$$K(n, \varepsilon) = \{(g_0(x, y), \ldots, g_{2n-1}(x, y)) \mid g_i(x, y) \text{ satisfy conditions (1)–(6) below.}\}$$

(1) $g_0(x, y) = x^2 - y^2 + c_0$

(2) $g_1(x, y) = 2xy + c_1$

(3) $g_{2j}(x, y) = a_{2j}x - b_{2j}y + c_{2j}, j \geq 1$

(4) $g_{2j+1}(x, y) = b_{2j}x + a_{2j}y + c_{2j+1}, j \geq 1$

(5) $|c_0|, |c_1| < \varepsilon$, and

(6) $g_0(x, y), \ldots, g_{2n-1}(x, y)$ do not simultaneously vanish.

The space $K(n, \varepsilon)$ is the natural real analog of $\mathrm{Rat}_2(\mathbb{C}P^n)$ with t operating on it. The parametrized version is useful in what follows. By high school algebra, we have

LEMMA (2.1) *If $\varepsilon > 0$, the space $K(n, \varepsilon)$ is homotopy equivalent to $\mathrm{Rat}_2(\mathbb{C}P^n)$ and thus if $n \geq 2$, $K(n, \varepsilon)$ is homotopy equivalent to the $(4n - 1)$-skeleton of $\Omega^2 S^{2n+1}$.*

Proof. Define $H: I \times \mathrm{Rat}_2(\mathbb{C}P^n) \to \mathrm{Rat}_2(\mathbb{C}P^n)$ by $H(t, f_0, \ldots, f_m))(z) = (f_0(z + t\lambda), \ldots, f_n(z + t\lambda))$ where $f_i(z) = z^2 + A_i z + B_i$ and $\lambda = -\frac{A_0}{2}$. Using the operation t, and the homotopy H, we see that the space of unimodular rows $(f_0, \ldots f_n)$ with $f_0(z) = z^2 + D_0$ and $f_i(z) = E_i z + F_i, i > 0$, say $\overline{\mathrm{Rat}_2}(\mathbb{C}P^n)$, is homeomorphic to a strong deformation retract of $\mathrm{Rat}_2(\mathbb{C}P^n)$. This suffices by [8].

3. On the divisibility of the Whitehead square

Let w_k denote the Whitehead product $[i_k, i_k]$ where i_k is a choice of fundamental cycle for the k-sphere S^k. The element w_k is the Whitehead square; it has been known for many years (≥ 35) that w_k is not divisible by 2 if k is not one less than a power of 2. The strong form of the Kervaire invariant conjecture is that w_k is divisible by 2 if $k = 2^n - 1$ [2, 9]. In this section, we use the structures in sections 1–2 to give a reformulation of this conjecture in terms of specific formulas. A naive attempt to solve the conjecture is shown to fail because of the geometry of the zero divisors described in Lemma (1.4)(3)).

The ingredients here are two self-maps of $\Omega^2 S^k$ where $k = 2^n - 1$. Thus S^k may be taken to be the unit sphere in $CD(n, \mathbb{R})$. The first self-map is induced by $\mathrm{Sq}: CD(n, \mathbb{R}) \to CD(n, \mathbb{R})$ given by $\mathrm{Sq}(x) = x \cdot x$.

LEMMA (3.1) *The map* Sq *restricts to a self-map of the unit sphere* S^k *which preserves* $e_0 = 1$. *Furthermore,* Sq *is of degree 2.*

Proof. If $x = \sum_{i\geq 0} x_i e_i$, then $\mathrm{Sq}(x) = \left(x_0^2 - \sum_{i>0} x_i^2\right) e_0 + \sum_{i>0} 2x_0 x_i e_i$ as in the proof of (2.1). If x is in the unit sphere, then $x_0^2 + \ldots + x_k^2 = 1$ and $\mathrm{Sq}(x) = (2x_0^2 - 1, 2x_0x_1, \ldots, 2x_0x_k)$. Since k is odd, the antipodal map A on S^k is homotopic to the identity and $A \cdot \mathrm{Sq}(x) = (1 - 2x_0^2, -2x_0x_1, \ldots, -2x_o x_{k+1})$ which is the map of degree 2 given in the proof of Lemma (5.4) [10, page 14]. Thus Lemma (3.1) follows.

Thus one obtains a self-map of $\Omega^2 S^k$ given by $\Omega^2(\mathrm{Sq})$. The second map is the H-space squaring map $2\colon \Omega^2 S^k \to \Omega^2 S^k$ for which we given an explicit formula in terms of the product structure in $CD(n, \mathbb{R})$.

Let $S^2 = \mathbb{C} \cup \{\infty\}$ and define $H\colon I \times S^2 \to S^2$ by the formula

$$H(t,z) = \begin{cases} z & \text{if } 0 \leq |z| \leq 1/2 \\ \frac{(1-2t+2t|z|)}{1-t} z & \text{if } 1/2 \leq |z| \leq 1 \text{ and } t < 1 \\ \frac{z}{1-t} & \text{if } 1 \leq |z| \text{ and } t < 1, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

Next, let $e = (1, 0)$ and define

$$H_1(t,z) = \begin{cases} H(t, z-e) + e & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty, \text{ and} \end{cases}$$

$$H_2(t,z) = \begin{cases} H(f, z+e) - e & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty. \end{cases}$$

Let θ be an element of $\Omega^2 S^k$. Thus θ is a function from S^2 to S^k with $\theta(\infty) = (1, 0, \ldots, 0)$. Consider

$$G(t,z)(\theta) = \{\theta \circ H_1(\epsilon, z)\} \cdot \{\theta \circ H_2(t,z)\}.$$

LEMMA (3.2)

(1) $G(0,z)(\theta) = \Omega^2(\mathrm{Sq})(\theta)$.

(2) $G(1,z)(\theta) = 2(\theta)$.

Proof. Since $H(0,z) = z$, $G(0,z)(\theta)$ is by definition $\Omega^2(\mathrm{Sq})(\theta)$. Also, $G(1,z)(\theta)$ is the definition of the H-space squaring map since $\theta \circ H_i(1,z) = (1, 0, \ldots, 0)$ if $\theta \circ H_j(1,z) \neq (1, 0, 0, \ldots, 0)$ if $i \neq j$.

The next lemma was pointed out in [3, Prop. 11.2].

LEMMA (3.3) *If $k = 2^n - 1$ and $n \geq 3$, then w_k is divisible by 2 provided the maps*

$$2 \quad \text{and } \Omega^2(\mathrm{Sq})\colon \Omega^2 S^k \to \Omega^2 S^k$$

are homotopic when restricted to the $(4k-3)$-skeleton.

Indeed, the two statements in 3.3 are equivalent [3,9]. By 2.1 and 3.3, the following is immediate.

PROPOSITION (3.4) *If $k = 2^n - 1$, then w_k is divisible by 2 if and only if the composites*

$$K\left(\frac{k-1}{2}, \varepsilon\right) \to \Omega^2 S^k \xrightarrow{\Omega^2(\mathrm{Sq})} \Omega^2 S^k \text{ and}$$
$$K\left(\frac{k-1}{2}, \varepsilon\right) \to \Omega^2 S^k \xrightarrow{2} \Omega^2 S^k$$

are homotopic.

The construction $G(t, z)$ is a naive attempt to construct such a homotopy which, as expected, fails.

PROPOSITION (3.5) *If $k = 2^n - 1$ and $n \geq 4$, then there are values for z and t such that $G(t, z) = 0$.*

Proof. It suffices to give elements θ in $K(\frac{k-1}{2}, \varepsilon)$ such that

$$\theta \cdot H_1(1/2, z) = e_1 + e_{12} \quad \text{and}$$
$$\theta \cdot H_2(1/2, z) = e_7 - e_{10}$$

as $(e_1 + e_{12})(e_7 - e_{10})$ is zero by (1.4). These are elementary and the proposition follows.

We close with some remarks. Using (1.4), one can measure how the product homotopy fails. The methods here then suggest specific ways to remedy this failure. The product homotopy can be changed "locally" to avoid the zero divisors described in (1.4). It remains open whether there is a global modification of $G(t, z)$ which satisfies (3.4).

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