Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

ON THE CHERN CHARACTER HOMOMORPHISMS OF $SO(n)$ AND $Spin(n)$

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Introduction

We calculated the Chern character $ch : K^*(G) \to H^{**}(G;\mathbb{Q})$ for $G = Spin(n)$ with $n \leq 9$ in [6] and for $G = Spin(2n+1)$ in [7]. We have done it for $G = SO(2n+1)$ in [8]. The purpose of this paper is to do it for $G = SO(2n)$ and $Spin(2n)$.

1. Representation rings

In this section we collect some results on the complex representation rings of classical Lie groups that concern us. For details we refer to [1], [4] and [9].

Let G be a compact connected Lie group. Its representation ring $R(G)$ is the Grothendieck construction of the semiring of isomorphism classes $[V]$ of G-modules V over $\mathbb{C}$. It has an augmentation

$$\varepsilon : R(G) \to \mathbb{Z}$$

which assigns to each class $[V]$ the dimension of V. Let T be a maximal torus of G. The Weyl group $W(G) = N(T)/T$ of G acts on T and therefore on $R(T)$. The inclusion $i : T \to G$ induces a monomorphism $i^* : R(G) \to R(T)$ whose image is contained in the subring $R(T)^{W(G)}$ of elements invariant under the action of $W(G)$. Thus we can regard $R(G)$ as a subring of $R(T)$ through i^*. $R(G)$ forms a λ-ring (see [4; 12(1.1)]) with operations

$$\lambda^k : R(G) \to R(G) \quad \text{for } k \geq 0$$

induced by the exterior power functors $V \to \Lambda^k(V)$, which have the following properties: $\lambda^0(x) = 1$ for all $x \in R(G)$; if $\varepsilon(x) = n$, then

$$\varepsilon(\lambda^k(x)) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{and} \quad \lambda^k(x) = 0 \quad \text{for } k > n.$$

Let T_n be the maximal torus of diagonal matrices

$$\operatorname{diag}[\exp(\mathrm{i}\theta_1), \cdots, \exp(\mathrm{i}\theta_n)] \quad (\theta_i \in \mathbb{R})$$

in the unitary group $U(n) \subset M(n,\mathbb{C})$. If $\alpha_1, \cdots, \alpha_n$ denote the standard 1-dimensional representations of T_n, we have

$$(1.1) \qquad R(T_n) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_n, \alpha_n^{-1}]/(\alpha_1\alpha_1^{-1} - 1, \cdots, \alpha_n\alpha_n^{-1} - 1).$$

Put $\lambda_1 = [\mathbb{C}^n] \in R(U(n))$ and let $\lambda_k = \lambda^k(\lambda_1)$. Then

$$(1.2) \qquad R(U(n)) = \mathbb{Z}[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n, \lambda_n^{-1}]/(\lambda_n\lambda_n^{-1} - 1),$$

where $\varepsilon(\lambda_k) = \binom{n}{k}$ and in $R(T_n)$ the relation

$$\prod_{i=1}^{n}(1+\alpha_i t) = \sum_{k=0}^{n} \lambda_k t^k \tag{1.3}$$

or equivalently

$$\prod_{i=1}^{n}(t+\alpha_i) = \sum_{k=0}^{n} \lambda_{n-k} t^k \tag{1.4}$$

holds, where t is the indeterminate (see [4; 13(3.1)]).

For the rotation groups $SO(n) \subset M(n, \mathbb{R})$, there are inclusions

$$i_n : SO(n) \to SO(n+1).$$

Let

$$i'_n : U(n) \to SO(2n)$$

be the realification; that is, $i'_n = i'_1 \oplus \cdots \oplus i'_1$, where $i'_1 : U(1) \to SO(2)$ is the map defined by

$$\mathbb{C} \ni a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M(n, \mathbb{R}).$$

Then $T_n = i'_n(T_n)$ is a maximal torus of $SO(2n)$.

Now $T'_n = i_{2n}(T_n)$ is a maximal torus of $SO(2n+1)$, and $R(T'_n)$ is the same as in (1.1). Put $\mu'_1 = [\mathbb{R}^{2n+1} \otimes \mathbb{C}] \in R(SO(2n+1))$ and let $\mu'_k = \lambda^k(\mu'_1)$. Then

$$R(SO(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \cdots, \mu'_n], \tag{1.5}$$

where $\varepsilon(\mu'_k) = \binom{2n+1}{k}$ and in $R(T'_n)$ the relation

$$(1+t)\prod_{i=1}^{n}(1+\alpha_i t)(1+\alpha_i^{-1} t) = \sum_{k=0}^{2n+1} \mu'_k t^k \tag{1.6}$$

holds (see [4; 13(10.3)]). We have $\mu'_{2n+1-k} = \mu'_k$ for $k = 0, , \cdots, 2n+1$, because

$$\begin{aligned}\sum_{k=0}^{2n+1} \mu'_{2n+1-k} t^k &= (t+1)\prod_{i=1}^{n}(t+\alpha_i)(t+\alpha_i^{-1}) \\ &= (t+1)\prod_{i=1}^{n}(\alpha_i^{-1}t+1)\alpha_i\alpha_i^{-1}(\alpha_i t+1) \\ &= (t+1)\prod_{i=1}^{n}(\alpha_i^{-1}t+1)(\alpha_i t+1) \\ &= \sum_{k=0}^{2n+1} \mu'_k t^k.\end{aligned}$$

For $n \geq 3$, the universal covering group of $SO(n)$ is the spinor group $Spin(n)$. Let

$$p_n : Spin(n) \to SO(n)$$

be the (twofold) covering map. Then there is a commutative diagram of principal fibre bundles:

$$\begin{array}{ccccc} Spin(n) & \xrightarrow{\tilde{i}_n} & Spin(n+1) & \xrightarrow{\tilde{q}_n} & S^n \\ \downarrow{p_n} & & \downarrow{p_{n+1}} & & \downarrow{=} \\ SO(n) & \xrightarrow{i_n} & SO(n+1) & \xrightarrow{q_n} & S^n. \end{array} \tag{1.7}$$

Now $\widetilde{T}'_n = {p_{2n+1}}^{-1}(T'_n)$ is a maximal torus of $Spin(2n+1)$. There are 1-dimensional representations $\alpha_1, \cdots, \alpha_n$ of $\widetilde{T}'_n$ such that

$$\begin{aligned} R(\widetilde{T}'_n) = &\mathbb{Z}[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_n, \alpha_n^{-1}, (\alpha_1 \cdots \alpha_n)^{1/2}]/ \\ &(\alpha_1\alpha_1^{-1} - 1, \cdots, \alpha_n\alpha_n^{-1} - 1, ((\alpha_1 \cdots \alpha_n)^{1/2})^2 - \alpha_1 \cdots \alpha_n) \end{aligned} \tag{1.8}$$

and $p_{2n+1} : \widetilde{T}'_n \to T'_n$ induces the inclusion $R(T'_n) \to R(\widetilde{T}'_n)$ under the descriptions of (1.1) and (1.8) (see [4; 13(8.3)]). Put $\mu'_k = {p_{2n+1}}^*(\mu'_k) \in R(Spin(2n+1))$ and let Δ_{2n+1} be the spin representation of dimension 2^n. Then

$$R(Spin(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \cdots, \mu'_{n-1}, \Delta_{2n+1}], \tag{1.9}$$

where the relation

$$(\Delta_{2n+1})^2 = \sum_{k=0}^{n} \mu'_k \tag{1.10}$$

holds (see [4; 13(10.3)]) and in $R(\widetilde{T}'_n)$ the relation

$$\Delta_{2n+1} = \prod_{i=1}^{n} (\alpha_i^{1/2} + \alpha_i^{-1/2}) = \sum_{\varepsilon_i = \pm 1} \alpha_1^{\varepsilon_1/2} \cdots \alpha_n^{\varepsilon_n/2} \tag{1.11}$$

holds (see [4; 13(9.4)]). Therefore, the behavior of ${p_{2n+1}}^* : R(SO(2n+1)) \to R(Spin(2n+1))$ is given by

$$\begin{aligned} {p_{2n+1}}^*(\mu'_k) &= \mu'_k \quad for\ k = 1, \ldots, n-1, \\ {p_{2n+1}}^*(\mu'_n) &= (\Delta_{2n+1})^2 - \sum_{k=0}^{n-1} \mu'_k. \end{aligned} \tag{1.12}$$

If T_n is the above torus of $SO(2n)$, then $R(T_n)$ is the same as in (1.1). Put $\mu_1 = [\mathbb{R}^{2n} \otimes \mathbb{C}] \in R(SO(2n))$ and let $\mu_k = \lambda^k(\mu_1)$. We have $\varepsilon(\mu_k) = \binom{2n}{k}$ and

$\mu_{2n-k} = \mu_k$ for $k = 0, 1, \ldots, 2n$. Besides, μ_n can be halved; that is, there are two representations μ_n^+, μ_n^- of $SO(2n)$ such that $\varepsilon(\mu_n^+) = \varepsilon(\mu_n^-) = \frac{1}{2}\binom{2n}{n}$ and $\mu_n = \mu_n^+ + \mu_n^-$. Then

$$R(SO(2n)) = \mathbb{Z}[\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n^+, \mu_n^-]/(\gamma_n), \tag{1.13}$$

where

$$\gamma_n = (\mu_n^+ + \sum_{i\geq 1} \mu_{n-2i})(\mu_n^- + \sum_{j\geq 1} \mu_{n-2j}) - (\mu_{n-1} + \sum_{k\geq 1} \mu_{n-1-2k})^2$$

(see [4; 13(10.3)]). Here the summations in the right side of the last equality end at $\cdots + \mu_4 + \mu_2 + 1$ or $\cdots + \mu_3 + \mu_1$. Furthermore, in $R(T_n)$ the relation

$$\prod_{i=1}^{n} (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n} \mu_k t^k \tag{1.14}$$

holds.

Now $\widetilde{T}_n = p_{2n}{}^{-1}(T_n)$ is a maximal torus of $Spin(2n)$. Similarly, $R(\widetilde{T}_n)$ is the same as in (1.8) and $p_{2n} : \widetilde{T}_n \to T_n$ induces the inclusion $R(T_n) \to R(\widetilde{T}_n)$ (see [4; 13(8.3)]). Put $\mu_k = p_{2n}{}^*(\mu_k), \mu_n^+ = p_{2n}{}^*(\mu_n^+), \mu_n^- = p_{2n}{}^*(\mu_n^-) \in R(Spin(2n))$. Let $\Delta_{2n}^+, \Delta_{2n}^-$ be the half spin representations, each of dimension 2^{n-1}. Then

$$R(Spin(2n)) = \mathbb{Z}[\mu_1, \mu_2, \cdots, \mu_{n-2}, \Delta_{2n}^+, \Delta_{2n}^-], \tag{1.15}$$

where the relations

$$\begin{aligned} \Delta_{2n}^+ \Delta_{2n}^- &= \mu_{n-1} + \sum_{k\geq 1} \mu_{n-1-2k}, \\ (\Delta_{2n}^+)^2 &= \mu_n^+ + \sum_{k\geq 1} \mu_{n-2k}, \\ (\Delta_{2n}^-)^2 &= \mu_n^- + \sum_{k\geq 1} \mu_{n-2k} \end{aligned} \tag{1.16}$$

hold (see [4; 13(10.3)]) and in $R(\widetilde{T}_n)$ the relations

$$\Delta_{2n}^+ = \sum_{\Pi \varepsilon_i = 1} \alpha_1^{\varepsilon_1/2} \cdots \alpha_n^{\varepsilon_n/2}, \quad \Delta_{2n}^- = \sum_{\Pi \varepsilon_i = -1} \alpha_1^{\varepsilon_1/2} \cdots \alpha_n^{\varepsilon_n/2} \tag{1.17}$$

hold (see [4; 13(9.4)]). Therefore, the behavior of $p_{2n}{}^* : R(SO(2n)) \to R(Spin(2n))$ is given by

$$\begin{aligned} p_{2n}{}^*(\mu_k) &= \mu_k \quad for \ k = 1, \ldots, n-2, \\ p_{2n}{}^*(\mu_{n-1}) &= \Delta_{2n}^+ \Delta_{2n}^- - \sum_{k\geq 1} \mu_{n-1-2k}, \\ p_{2n}{}^*(\mu_n^+) &= (\Delta_{2n}^+)^2 - \sum_{k\geq 1} \mu_{n-2k}, \\ p_{2n}{}^*(\mu_n^-) &= (\Delta_{2n}^-)^2 - \sum_{k\geq 1} \mu_{n-2k}. \end{aligned} \tag{1.18}$$

PROPOSITION (1.19). (1) $i_{2n}^* : R(SO(2n+1)) \to R(SO(2n))$ *satisfies*

$$i_{2n}^*(\mu_1') = \mu_1 + 1.$$

(2) $i_{2n-1}^* : R(SO(2n)) \to R(SO(2n-1))$ *satisfies*

$$i_{2n-1}^*(\mu_1) = \mu_1' + 1.$$

Proof. Since the homomorphism $R(T_n') \to R(T_n)$ induced by the inclusion $i_{2n} : T_n \to T_n'$ is the identity, (1) follows from (1.6) and (1.14). Since the homomorphism $R(T_n) \to R(T_{n-1}')$ induced by the inclusion $i_{2n-1} : T_{n-1}' \to T_n$ sends α_i to α_i for $i = 1, \dots, n-1$ and α_n to 1, (2) follows.

PROPOSITION (1.20). (1) $\tilde{i}_{2n}^* : R(Spin(2n+1)) \to R(Spin(2n))$ *satisfies*

$$\tilde{i}_{2n}^*(\mu_1') = \mu_1 + 1 \quad and \quad \tilde{i}_{2n}^*(\Delta_{2n+1}) = \Delta_{2n}^+ + \Delta_{2n}^-.$$

(2) $\tilde{i}_{2n-1}^* : R(Spin(2n)) \to R(Spin(2n-1))$ *satisfies*

$$\tilde{i}_{2n-1}^*(\mu_1) = \mu_1' + 1 \quad and \quad \tilde{i}_{2n-1}^*(\Delta_{2n}^+) = \tilde{i}_{2n-1}^*(\Delta_{2n}^-) = \Delta_{2n-1}.$$

Proof. The first relations of (1) and (2) follow from Proposition (1.19). The remaining relations follow from (1.11) and (1.17).

PROPOSITION (1.21). $i_n'^* : R(SO(2n)) \to R(U(n))$ *satisfies*

$$i_n'^*(\mu_k) = \lambda_n^{-1} \sum_{i=0}^{k} \lambda_i \lambda_{n-k+i} \quad for \ k = 1, \dots, n.$$

Proof. In $R(T_n)$ we have

$$\prod_{i=1}^{n}(1+\alpha_i^{-1}t) = \prod_{i=1}^{n}\alpha_i^{-1}(\alpha_i + t) = \prod_{i=1}^{n}\alpha_i^{-1}\prod_{i=1}^{n}(t+\alpha_i)$$
$$= \lambda_n^{-1}\sum_{j=0}^{n}\lambda_{n-j}t^j \quad \text{by (1.4)}.$$

So

$$\sum_{k=0}^{2n}\mu_k t^k = \prod_{i=1}^{n}(1+\alpha_i t)(1+\alpha_i^{-1}t) \quad \text{by (1.14)}$$
$$= (\sum_{i=0}^{n}\lambda_i t^i)(\lambda_n^{-1}\sum_{j=0}^{n}\lambda_{n-j}t^j) \quad \text{by (1.3)}$$
$$= \sum_{k=0}^{2n}\lambda_n^{-1}(\sum_{i+j=k}\lambda_i\lambda_{n-j})t^k.$$

LEMMA (1.22). *In $R(\widetilde{T}_n)$ the relation*

$$(\Delta_{2n}^+)^2 - (\Delta_{2n}^-)^2 = \lambda_n^{-1}(\sum_{i=0}^{n} \lambda_i)(\sum_{j=0}^{n}(-1)^{n-j}\lambda_j)$$

holds.

Proof. We have

$$\begin{aligned}(\Delta_{2n}^+)^2 - (\Delta_{2n}^-)^2 &= (\Delta_{2n}^+ + \Delta_{2n}^-)(\Delta_{2n}^+ - \Delta_{2n}^-) \\ &= \prod_{i=1}^{n}(\alpha_i^{1/2} + \alpha_i^{-1/2}) \prod_{i=1}^{n}(\alpha_i^{1/2} - \alpha_i^{-1/2}) \quad \text{by } (1.17) \\ &= \prod_{i=1}^{n}(\alpha_i - \alpha_i^{-1}) = \prod_{i=1}^{n} \alpha_i^{-1}(\alpha_i^2 - 1) \\ &= \prod_{i=1}^{n} \alpha_i^{-1} \prod_{i=1}^{n}(\alpha_i + 1) \prod_{i=1}^{n}(\alpha_i - 1) \\ &= \lambda_n^{-1}(\sum_{i=0}^{n} \lambda_i)(\sum_{j=0}^{n}(-1)^j \lambda_{n-j}) \quad \text{by } (1.4) \text{ with } t = \pm 1.\end{aligned}$$

2. Cohomology rings

In this section we fix some notation concerning the integral cohomology of our groups G. For details, see [5].

First, if $G = U(n)$, there exist elements $x_{2i-1} \in H^{2i-1}(U(n);\mathbb{Z})$ for $i = 1,\dots,n$ such that

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1, x_3, \cdots, x_{2n-1})$$

and

$$PH^*(U(n);\mathbb{Z}) = \mathbb{Z}\{x_1, x_3, \cdots, x_{2n-1}\},$$

where P denotes the primitive module functor. Using such x_{2i-1}'s, we have

(2.1) $$H^*(U(n);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1, x_3, \cdots, x_{2n-1}).$$

$H^*(SO(n);\mathbb{Z})$ has only 2-torsion for every $n \geq 2$. If $G = SO(2n+1)$, by consequences of the Poincaré duality, there exist elements $x_{4i-1} \in H^{4i-1}(SO(2n+1);\mathbb{Z})$ for $i = 1,\dots,n$ such that

$$H^*(SO(2n+1);\mathbb{Z})/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \cdots, x_{4n-1})$$

and

$$PH^*(SO(2n+1);\mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \cdots, x_{4n-1}\}.$$

Using such x_{4i-1}'s, we have

$$H^*(SO(2n+1);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \cdots, x_{4n-1}). \tag{2.2}$$

If $G = SO(2n)$, there exist elements $x_{4i-1} \in H^{4i-1}(SO(2n);\mathbb{Z})$ for $i = 1, \ldots, n-1$ and $x'_{2n-1} \in H^{2n-1}(SO(2n);\mathbb{Z})$ such that

$$H^*(SO(2n);\mathbb{Z})/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1})$$

and

$$PH^*(SO(2n);\mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1}\}.$$

Using such x_{4i-1}'s and x'_{2n-1}, we have

$$H^*(SO(2n);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1}). \tag{2.3}$$

$H^*(Spin(n);\mathbb{Z})$ has no odd torsion for every $n \geq 2$ and it has 2-torsion if and only if $n \geq 7$. If $G = Spin(2n+1)$, there exist elements $\widetilde{x}_{4i-1} \in H^{4i-1}(Spin(2n+1);\mathbb{Z})$ for $i = 1, \ldots, n$ such that

$$H^*(Spin(2n+1);\mathbb{Z})/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-1})$$

and

$$PH^*(Spin(2n+1);\mathbb{Q}) = \mathbb{Q}\{\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-1}\}.$$

Using such $\widetilde{x}_{4i-1}$'s, we have

$$H^*(Spin(2n+1);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-1}). \tag{2.4}$$

Finally, if $G = Spin(2n)$, there exist elements $\widetilde{x}_{4i-1} \in H^{4i-1}(Spin(2n);\mathbb{Z})$ for $i = 1, \ldots, n-1$ and $\widetilde{x}'_{2n-1} \in H^{2n-1}(Spin(2n);\mathbb{Z})$ such that

$$H^*(Spin(2n);\mathbb{Z})/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-5}, \widetilde{x}'_{2n-1})$$

and

$$PH^*(Spin(2n);\mathbb{Q}) = \mathbb{Q}\{\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-5}, \widetilde{x}'_{2n-1}\}.$$

Using such $\widetilde{x}_{4i-1}$'s and $\widetilde{x}'_{2n-1}$, we have

$$H^*(Spin(2n);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(\widetilde{x}_3, \widetilde{x}_7, \cdots, \widetilde{x}_{4n-5}, \widetilde{x}'_{2n-1}). \tag{2.5}$$

The following two propositions are quite easy.

PROPOSITION (2.6). $i_{2n-1}{}^* : H^*(SO(2n);\mathbb{Z}) \to H^*(SO(2n-1);\mathbb{Z})$ *satisfies*

$$i_{2n-1}{}^*(x_{4i-1}) = x_{4i-1} \quad \textit{for } i = 1, \ldots, n-1,$$
$$i_{2n-1}{}^*(x'_{2n-1}) = 0.$$

PROPOSITION (2.7). *$\tilde{i}_{2n-1}{}^* : H^*(Spin(2n);\mathbb{Z}) \to H^*(Spin(2n-1);\mathbb{Z})$ satisfies*

$$\tilde{i}_{2n-1}{}^*(\tilde{x}_{4i-1}) = \tilde{x}_{4i-1} \quad for\ i = 1, \ldots, n-1,$$
$$\tilde{i}_{2n-1}{}^*(\tilde{x}'_{2n-1}) = 0.$$

PROPOSITION (2.8). *$i'_n{}^* : H^*(SO(2n);\mathbb{Z}) \to H^*(U(n);\mathbb{Z})$ satisfies*

$$i'_n{}^*(x'_{2n-1}) = x_{2n-1}.$$

In particular, if $n = 2m$, $i'_{2m}{}^*(x'_{4m-1}) = x_{4m-1}$ and moreover

$$i'_{2m}{}^*(x_{4m-1}) = 0.$$

Proof. We recall the argument of [6; pp. 465-466]. There, in order to characterize a generator $x_j \in H^j(G;\mathbb{Z})$, its image $f_{j+1} \in H^{j+1}(BT;\mathbb{Z})$ under the transgression $\tau : H^*(G;\mathbb{Z}) \to H^{*+1}(BT;\mathbb{Z})$ in the Serre spectral sequence of the fibration $G \to G/T \to BT$ is investigated. Consider the commutative diagram of fibrations:

$$\begin{array}{ccccc} U(n) & \longrightarrow & U(n)/T_n & \longrightarrow & BT_n \\ \downarrow{\scriptstyle i'_{2n}} & & \downarrow & & \downarrow \\ SO(2n) & \longrightarrow & SO(n)/T_n & \longrightarrow & BT_n, \end{array}$$

and $H^*(BT;\mathbb{Z}) = \mathbb{Z}[t_1, \cdots, t_n]$, where $t_i \in H^2(BT;\mathbb{Z})$. Clearly $\tau(x_{2n-1}) = c_n = \sigma_n(t_1, \cdots, t_n)$, where σ_i denotes the i-th elementary symmetric function, and $\tau(x'_{2n-1}) = c_n$ (see [5; Chapter 3]). Hence the first statement is obvious. When $n = 2m$, two elements $x'_{4m-1}, x_{4m-1} \in H^{4m-1}(SO(4m);\mathbb{Z})$ have to be dealt with. We may set

$$\tau(x_{4m-1}) = c \cdot f_{4m} \quad \text{for some } c \in \mathbb{Z},$$

where $f_{4m} \in H^{4m}(BT;\mathbb{Z})$ is not divisible by any other elements of $H^{4m}(BT;\mathbb{Z})$ and gives rise to the generator p_m of the $W(SO(4m))$-invariant subalgebra

$$H^*(BT;\mathbb{Q})^{W(SO(4m))} = \mathbb{Q}[p_1, p_2, \cdots, p_{2m-1}, c_{2m}],$$

where $p_i = \sigma_i(t_1^2, \cdots, t_n^2)$. In view of [6; Remark of p. 466], the element $p_m - (-1)^m 2c_{2m}$ which does not contain the term c_{2m} modulo decomposables in

$$H^*(BT;\mathbb{Z})^{W(U(2m))} = \mathbb{Z}[c_1, c_2, \cdots, c_{2m}]$$

must give rise to f_{4m}. (For the case $G = Spin(4m)$ with $m = 2$, see the bottom of [6; p. 476].) This implies the second statement.

3. K-rings

In this section we collect some results on the complex K-theory of our groups G.

Let

$$\beta : R(G) \to K^{-1}(G)$$

be the map of [3]. Then it is natural with respect to group homomorphisms and has the following properties:

(3.1) *For each* $\rho_1, \rho_2 \in R(G)$,

$$\beta(\rho_1 + \rho_2) = \beta(\rho_1) + \beta(\rho_2);$$

(3.2) *If* $n \in R(G)$ *is represented by a trivial* G-$\mathbb{C}$*-module of dimension* n, *then* $\beta(n) = 0$;

(3.3) *For each* $\rho_1, \rho_2 \in R(G)$,

$$\beta(\rho_1\rho_2) = \varepsilon(\rho_2)\beta(\rho_1) + \varepsilon(\rho_1)\beta(\rho_2).$$

Hodgkin [3; Theorem A] says that, if G is a compact connected Lie group with $\pi_1(G)$ torsion-free, the $\mathbb{Z}/(2)$-graded K-ring $K^*(G)$ is torsion-free and has the structure of a Hopf algebra over $\mathbb{Z}$; more precisely, if

$$R(G) = \mathbb{Z}[\rho_1, \rho_2, \cdots, \rho_l],$$

we have

$$K^*(G) = \Lambda_{\mathbb{Z}}(\beta(\rho_1), \beta(\rho_2), \cdots, \beta(\rho_l)),$$

where each $\beta(\rho_i)$ is primitive. This yields the following results. First, if $G = U(n)$, from (1.2) we have elements $\beta(\lambda_1), \cdots, \beta(\lambda_{n-1}), \beta(\lambda_n), \beta(\lambda_n^{-1}) \in K^{-1}(U(n))$. But the relation

$$\beta(\lambda_n^{-1}) = -\beta(\lambda_n) \tag{3.4}$$

holds; in fact, since $\varepsilon(\lambda_n) = \varepsilon(\lambda_n^{-1}) = 1$,

$$\begin{aligned} \beta(\lambda_n) + \beta(\lambda_n^{-1}) &= \beta(\lambda_n\lambda_n^{-1}) \quad \text{by} \quad (3.3) \\ &= \beta(1) = 0 \quad \text{by} \quad (3.2). \end{aligned}$$

Thus

$$K^*(U(n)) = \Lambda_{\mathbb{Z}}(\beta(\lambda_1), \cdots, \beta(\lambda_{n-1}), \beta(\lambda_n)). \tag{3.5}$$

If $G = Spin(2n+1)$, it follows from (1.9) that

$$K^*(Spin(2n+1)) = \Lambda_{\mathbb{Z}}(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \beta(\Delta_{2n+1})). \tag{3.6}$$

Finally, if $G = Spin(2n)$, it follows from (1.15) that

$$K^*(Spin(2n)) = \Lambda_{\mathbb{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \beta(\Delta_{2n}^+), \beta(\Delta_{2n}^-)). \tag{3.7}$$

If $G = SO(2n+1)$, from (1.5) we have elements $\beta(\mu_1'), \cdots, \beta(\mu_n') \in K^{-1}(SO(2n+1))$. According to [2], there exist other two elements $\epsilon_{2n+1} \in K^{-1}(SO(2n+1))$ and $\xi_{2n+1} \in K^0(SO(2n+1))$ such that

$$\begin{aligned} K^*(SO(2n+1)) = & \Lambda_{\mathbb{Z}}(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \epsilon_{2n+1}) \otimes \\ & (\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^n)\{\xi_{2n+1}\})/(\epsilon_{2n+1} \otimes \xi_{2n+1}) \end{aligned}$$

and

$$p_{2n+1}{}^*(\epsilon_{2n+1}) = 2\beta(\Delta_{2n+1}). \tag{3.8}$$

Therefore,

$$K^*(SO(2n+1))/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \epsilon_{2n+1}). \tag{3.9}$$

If $G = SO(2n)$, from (1.13) we have elements $\beta(\mu_1), \cdots, \beta(\mu_{n-1}), \beta(\mu_n^+), \beta(\mu_n^-) \in K^{-1}(SO(2n))$. According to [2], there exist other three elements $\delta_{2n}, \epsilon_{2n} \in K^{-1}(SO(2n))$ and $\xi_{2n} \in K^0(SO(2n))$ such that

$$\begin{aligned} K^*(SO(2n)) = & \Lambda_{\mathbb{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \delta_{2n}, \epsilon_{2n}) \otimes \\ & (\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^{n-1})\{\xi_{2n}\})/(\epsilon_{2n} \otimes \xi_{2n}) \end{aligned}$$

and

$$p_{2n}{}^*(\delta_{2n}) = \beta(\Delta_{2n}^+) - \beta(\Delta_{2n}^-), \quad p_{2n}{}^*(\epsilon_{2n}) = 2\beta(\Delta_{2n}^+). \tag{3.10}$$

Therefore,

$$K^*(SO(2n))/\mathrm{Tor} = \Lambda_{\mathbb{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \delta_{2n}, \epsilon_{2n}). \tag{3.11}$$

PROPOSITION (3.12). (1) *In $K^*(Spin(2n+1))$ the relation*

$$\beta(\mu_n') = 2^{n+1}\beta(\Delta_{2n+1}) - \sum_{k=1}^{n-1} \beta(\mu_k')$$

holds.

(2) *In $K^*(Spin(2n))$ the relation*

$$\beta(\mu_{n-1}) = 2^{n-1}\beta(\Delta_{2n}^+) + 2^{n-1}\beta(\Delta_{2n}^-) - \sum_{k=1}^{[(n-2)/2]} \beta(\mu_{n-1-2k})$$

holds (where $[x]$ is the Gauss' notation of $x \in \mathbb{R}$).

Proof. By using (3.1)-(3.3), (1) follows from (1.10) and (2) follows from the first relation of (1.16).

PROPOSITION (3.13). (1) *In $K^*(SO(2n+1))/\mathrm{Tor}$ the relation*

$$\beta(\mu_n') = 2^n \epsilon_{2n+1} - \sum_{k=1}^{n-1} \beta(\mu_k')$$

holds.

(2) *In $K^*(SO(2n))/\mathrm{Tor}$ the relations*

$$\beta(\mu_{n-1}) = -2^{n-1}\delta_{2n} + 2^{n-1}\epsilon_{2n} - \sum_{k=1}^{[(n-2)/2]} \beta(\mu_{n-1-2k}),$$

$$\beta(\mu_n^+) = 2^{n-1}\epsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \beta(\mu_{n-2k}),$$

$$\beta(\mu_n^-) = -2^n\delta_{2n} + 2^{n-1}\epsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \beta(\mu_{n-2k})$$

hold.

Proof. Since $p_n{}^* : K^*(SO(n))/\mathrm{Tor} \to K^*(Spin(n))$ is injective for all n, it suffices to verify the stated relations by applying $p_n{}^*$ to them. Then (1) follows from (3.8) and Proposition (3.12)(1). Similarly, by using (3.1)-(3.3), (2) follows from (1.16) and (3.10).

PROPOSITION (3.14). (1) $\tilde{i}_{2n}{}^* : K^*(Spin(2n+1)) \to K^*(Spin(2n))$ *satisfies*

$$\tilde{i}_{2n}{}^*(\beta(\mu_1')) = \beta(\mu_1),$$
$$\tilde{i}_{2n}{}^*(\beta(\Delta_{2n+1})) = \beta(\Delta_{2n}^+) + \beta(\Delta_{2n}^-).$$

(2) $\tilde{i}_{2n-1}{}^* : K^*(Spin(2n)) \to K^*(Spin(2n-1))$ *satisfies*

$$\tilde{i}_{2n-1}{}^*(\beta(\mu_1)) = \beta(\mu_1'),$$
$$\tilde{i}_{2n-1}{}^*(\beta(\Delta_{2n}^+)) = \tilde{i}_{2n-1}{}^*(\beta(\Delta_{2n}^-)) = \beta(\Delta_{2n-1}).$$

Proof. By using (3.1) and (3.2), this follows from Proposition (1.20).

PROPOSITION (3.15). (1) $i_{2n}{}^* : K^*(SO(2n+1))/\mathrm{Tor} \to K^*(SO(2n))/\mathrm{Tor}$ *satisfies*

$$i_{2n}{}^*(\beta(\mu_1')) = \beta(\mu_1),$$
$$i_{2n}{}^*(\epsilon_{2n+1}) = -2\delta_{2n} + 2\epsilon_{2n}.$$

(2) ${i_{2n-1}}^* : K^*(SO(2n))/\mathrm{Tor} \to K^*(SO(2n-1))/\mathrm{Tor}$ *satisfies*

$$\begin{aligned}
{i_{2n-1}}^*(\beta(\mu_1)) &= \beta(\mu_1'),\\
{i_{2n-1}}^*(\delta_{2n}) &= 0,\\
{i_{2n-1}}^*(\epsilon_{2n}) &= \epsilon_{2n-1}.
\end{aligned}$$

Proof. This is similar to the proof of Proposition (3.13).

PROPOSITION (3.16). ${i_n'}^* : K^*(SO(2n))/\mathrm{Tor} \to K^*(U(n))$ *satisfies*

$$\begin{aligned}
{i_n'}^*(\beta(\mu_1)) &= \beta(\lambda_1) + \beta(\lambda_{n-1}) - n\beta(\lambda_n),\\
{i_n'}^*(\delta_{2n}) &= \sum_{k=1}^{n}(-1)^{n-k}\beta(\lambda_k),\\
{i_n'}^*(\epsilon_{2n}) &= \sum_{k=1}^{n-1}(1+(-1)^{n-k})\beta(\lambda_k) - 2(2^{n-2}-1)\beta(\lambda_n).
\end{aligned}$$

Proof. The first relation follows from:

$$\begin{aligned}
{i_n'}^*(\beta(\mu_1)) &= \beta({i_n'}^*(\mu_1)),\\
&= \beta(\lambda_1 + \lambda_{n-1}\lambda_n^{-1}) \quad \text{by Proposition (1.21)}\\
&= \beta(\lambda_1) + \beta(\lambda_{n-1}) + n\beta(\lambda_n^{-1}) \quad \text{by (3.1) and (3.3)}\\
&= \beta(\lambda_1) + \beta(\lambda_{n-1}) - n\beta(\lambda_n) \quad \text{by (3.4).}
\end{aligned}$$

Since

$$\varepsilon(\sum_{i=0}^{n}\lambda_i) = \sum_{i=0}^{n}\binom{n}{i} = (1+1)^n = 2^n$$

and

$$\varepsilon(\sum_{j=0}^{n}(-1)^{n-j}\lambda_j) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} = (-1+1)^n = 0,$$

we have

$$\begin{aligned}
i_n'^*(\delta_{2n}) &= i_n'^*((p_{2n}{}^*)^{-1}(\beta(\Delta_{2n}^+) - \beta(\Delta_{2n}^-))) \quad \text{by (3.10)} \\
&= i_n'^*((p_{2n}{}^*)^{-1}(2^{-n}\beta((\Delta_{2n}^+)^2 - (\Delta_{2n}^-)^2))) \quad \text{by (3.3)} \\
&= 2^{-n}\beta(\lambda_n^{-1}(\sum_{i=0}^{n}\lambda_i)(\sum_{j=0}^{n}(-1)^{n-j}\lambda_j)) \quad \text{by Lemma (1.22)} \\
&= 2^{-n}\beta((\sum_{i=0}^{n}\lambda_i)(\sum_{j=0}^{n}(-1)^{n-j}\lambda_j)) \quad \text{by (3.3)} \\
&= 2^{-n}2^{n}\beta(\sum_{j=0}^{n}(-1)^{n-j}\lambda_j) \quad \text{by (3.3)} \\
&= \sum_{j=1}^{n}(-1)^{n-j}\beta(\lambda_j) \quad \text{by (3.1) and (3.2).}
\end{aligned}$$

The third relation will not be used later, so we omit its proof.

4. The Chern character homomorphisms

In this section we prove our main results.

Put $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ and let $\phi : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{Z}$ be the function defined by

$$\phi(n, k, q) = \sum_{i=1}^{k}(-1)^{i-1}\binom{n}{k-i}i^{q-1} \tag{4.1}$$

for $n, k, q \in \mathbb{N}^+$. Then, by an argument analogous to that of [6; p. 470] in which the Chern character of $SU(n+1)$ is calculated, we have

PROPOSITION (4.2). *With the notations of* (3.5) *and* (2.1), $ch : K^*(U(n)) \to H^{**}(U(n); \mathbb{Q})$ *is given by*

$$ch(\beta(\lambda_k)) = \sum_{i=1}^{n}\frac{(-1)^{i-1}}{(i-1)!}\phi(n, k, i)x_{2i-1} \quad for \ k \geq 1.$$

Let $P_2 = \{2^i \mid i = 0, 1, 2, \ldots\}$. For each $n \in \mathbb{N}^+$ there is a unique integer $s(n)$ so that

$$2^{s(n)-1} < n \leq 2^{s(n)}.$$

Let

$$r(n, i) = \begin{cases} 2 & \text{if } n \notin P_2 \text{ and } i = 2^{s(n)-1} \\ & \quad \text{or if } n \in P_2 \text{ and } i = 2^{s(n)} \\ 1 & \text{otherwise.} \end{cases}$$

For convenience we introduce a function $\phi_1 : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{Z}$ defined by

$$\phi_1(n,q) = 2^{-n} \sum_{k=1}^{n} \phi(2n+1,k,q).$$

The following result was shown in [7].

THEOREM (4.3). *With the notations of* (3.6) *and* (2.4), $ch : K^*(Spin(2n+1)) \to H^{**}(Spin(2n+1);\mathbb{Q})$ *is given by*

$$ch(\beta(\mu_k')) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2^{r(n,i)}}{(2i-1)!}\phi(2n+1,k,2i)\tilde{x}_{4i-1},$$

$$ch(\beta(\Delta_{2n+1})) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2^{r(n,i)-1}}{(2i-1)!}\phi_1(n,2i)\tilde{x}_{4i-1}.$$

The following is quoted from [8; Theorem 4.4].

THEOREM (4.4). *With the notations of* (3.9) *and* (2.2), $ch : K^*(SO(2n+1))/\mathrm{Tor} \to H^{**}(SO(2n+1);\mathbb{Q})$ *is given by*

$$ch(\beta(\mu_k')) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!}\phi(2n+1,k,2i)x_{4i-1},$$

$$ch(\epsilon_{2n+1}) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!}\phi_1(n,2i)x_{4i-1}.$$

COROLLARY (4.5). (1) $p_{2n+1}{}^* : H^*(SO(2n+1);\mathbb{Z}) \to H^*(Spin(2n+1);\mathbb{Z})$ *satisfies*

$$p_{2n+1}{}^*(x_{4i-1}) = 2^{r(n,i)-1}\tilde{x}_{4i-1} \quad for\ i = 1,\ldots,n.$$

(2) $p_{2n}{}^* : H^*(SO(2n);\mathbb{Z}) \to H^*(Spin(2n);\mathbb{Z})$ *satisfies*

$$p_{2n}{}^*(x_{4i-1}) = 2^{r(n-1,i)-1}\tilde{x}_{4i-1} \quad for\ i = 1,\ldots,n-1,$$
$$p_{2n}{}^*(x'_{2n-1}) = \tilde{x}'_{2n-1}.$$

Proof. Since $\phi(n,1,q) = 1$ for all $n, q \in \mathbb{N}^+$, we have

$$\begin{aligned}\sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!}p_{2n+1}{}^*(x_{4i-1}) &= p_{2n+1}{}^*(ch(\beta(\mu_1'))) \quad \text{by Theorem (4.4)}\\ &= ch(p_{2n+1}{}^*(\beta(\mu_1'))) = ch(\beta(\mu_1'))\\ &= \sum_{i=1}^{n} \frac{(-1)^{i-1}2^{r(n,i)}}{(2i-1)!}\tilde{x}_{4i-1} \quad \text{by Theorem (4.3)}\end{aligned}$$

and by comparing both sides, (1) follows.

To prove (2), we use the following commutative diagram induced from (1.7):

$$\begin{array}{ccccc} H^*(SO(2n-1);\mathbb{Z}) & \overset{i_{2n-1}^*}{\longleftarrow} & H^*(SO(2n);\mathbb{Z}) & \overset{q_{2n-1}^*}{\longleftarrow} & H^*(S^{2n-1};\mathbb{Z}) \\ \downarrow p_{2n-1}^* & & \downarrow p_{2n}^* & & \downarrow = \\ H^*(Spin(2n-1);\mathbb{Z}) & \overset{\tilde{i}_{2n-1}^*}{\longleftarrow} & H^*(Spin(2n);\mathbb{Z}) & \overset{\tilde{q}_{2n-1}^*}{\longleftarrow} & H^*(S^{2n-1};\mathbb{Z}). \end{array}$$

Since

$$q_{2n-1}^*(u_{2n-1}) = x'_{2n-1} \quad \text{and} \quad \tilde{q}_{2n-1}^*(u_{2n-1}) = \tilde{x}'_{2n-1},$$

where $H^{2n-1}(S^{2n-1};\mathbb{Z}) = \mathbb{Z}\{u_{2n-1}\}$, the second relation of (2) follows. By the choice of x_{4i-1} and $\tilde{x}_{4i-1}$ (cf. the proof of Proposition (2.8)), we may set

$$p_{2n}^*(x_{4i-1}) = a \cdot \tilde{x}_{4i-1} \quad \text{for some } a \in \mathbb{Z}.$$

Then the first relation of (2) follows from (1) and Propositions (2.6), (2.7).

For convenience we introduce a function $\phi_0 : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{Z}$ defined by

$$\phi_0(n,q) = 2^{-(n-1)} \sum_{k=0}^{[(n-2)/2]} \phi(2n, n-1-2k, q).$$

THEOREM (4.6). *With the notations of* (3.11) *and* (2.3), $ch : K^*(SO(2n))/\mathrm{Tor} \to H^{**}(SO(2n);\mathbb{Q})$ *is given by*

$$\begin{aligned} ch(\beta(\mu_k)) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\phi(2n,k,2i)x_{4i-1} \\ &\quad + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi(2n,k,n)x'_{2n-1}, \\ ch(\delta_{2n}) &= x'_{2n-1}, \\ ch(\epsilon_{2n}) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\phi_0(n,2i)x_{4i-1} \\ &\quad + \left(1 + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi_0(n,n)\right)x'_{2n-1}. \end{aligned}$$

Proof. Since $\beta(\mu_1)$ is primitive in the Hopf algebra $K^*(SO(2n))/\mathrm{Tor}$ (see [2]) and ch is a homomorphism of Hopf algebras, we may set

$$ch(\beta(\mu_1)) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}$$

for some $a_i, a' \in \mathbb{Q}$. First, apply ${i_{2n-1}}^* : H^*(SO(2n);\mathbb{Q}) \to H^*(SO(2n-1);\mathbb{Q})$ to this equation. Then the left hand side is

$$\begin{aligned} {i_{2n-1}}^*(ch(\beta(\mu_1))) &= ch({i_{2n-1}}^*(\beta(\mu_1))) \\ &= ch(\beta(\mu_1')) \quad \text{by Proposition (3.15)(2)} \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!} x_{4i-1} \quad \text{by Theorem (4.4)} \end{aligned}$$

and the right hand side is

$${i_{2n-1}}^*\Big(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}\Big) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition (2.6). Hence $a_i = (-1)^{i-1}2/(2i-1)!$ for $i = 1, \dots, n-1$. Next, apply ${i'_n}^* : H^*(SO(2n);\mathbb{Q}) \to H^*(U(n);\mathbb{Q})$ to the above equation. Then the left hand side is

$$\begin{aligned} {i'_n}^*(ch(\beta(\mu_1))) &= ch({i'_n}^*(\beta(\mu_1))) \\ &= ch(\beta(\mu_1) + \beta(\mu_{n-1}) - n\beta(\mu_n)) \quad \text{by Proposition (3.16)} \\ &= \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!}(1 + \phi(n, n-1, i) - n\phi(n, n, i))x_{2i-1} \end{aligned}$$

by Proposition (4.2). Since $\phi(n, k, 1) = \binom{n-1}{k-1}$ for all $n, k \in \mathbb{N}^+$, we have

$$1 + \phi(n, n-1, 1) - n\phi(n, n, 1) = 1 + (n-1) - n = 0.$$

If $2 \le i \le n$, since $\phi(n, n-k, i) = (-1)^i\phi(n, k, i)$ by [8; (3.9)] and $\phi(n, k, i) = 0$ for all $k \ge n$ by [8; (3.10)], we have

$$\begin{aligned} 1 + \phi(n, n-1, i) - n\phi(n, n, i) &= 1 + (-1)^i\phi(n, 1, i) \\ &= 1 + (-1)^i. \end{aligned}$$

Consequently, for $i = 1, \dots, n$,

$$1 + \phi(n, n-1, i) - n\phi(n, n, i) = 1 + (-1)^i.$$

Thus

$$\text{(4.7)} \qquad {i'_n}^*(ch(\beta(\mu_1))) = \sum_{i=1}^{n} \frac{(-1)^{i-1}(1 + (-1)^i)}{(i-1)!} x_{2i-1}.$$

On the other hand, the right hand side is

$$i_n'^*(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1}') = \cdots + a' x_{2n-1}$$

by Proposition (2.8). Hence $a' = (-1)^{n-1}(1+(-1)^n)/(n-1)!$. This proves the first relation for $k = 1$ and that for $k > 1$ is obtained by applying [6; Lemma 1] to it.

Similarly we may set

$$ch(\delta_{2n}) = \sum_{i=1}^{n-1} b_i x_{4i-1} + b' x_{2n-1}'$$

for some $b_i, b' \in \mathbb{Q}$. First, apply $i_{2n-1}{}^*$ to this equation. Then the left hand side is

$$\begin{aligned} i_{2n-1}{}^*(ch(\delta_{2n})) &= ch(i_{2n-1}{}^*(\delta_{2n})) \\ &= 0 \quad \text{by Proposition (3.15)(2)} \end{aligned}$$

and the right hand side is

$$i_{2n-1}{}^*(\sum_{i=1}^{n-1} b_i x_{4i-1} + b' x_{2n-1}') = \sum_{i=1}^{n-1} b_i x_{4i-1}$$

by Proposition (2.6). Hence $b_i = 0$ for $i = 1, \ldots, n-1$. Next, apply $i_n'^*$ to the above equation. Then the left hand side is

$$\begin{aligned} i_n'^*(ch(\delta_{2n})) &= ch(i_n'^*(\delta_{2n})) \\ &= ch(\sum_{k=1}^{n}(-1)^{n-k}\beta(\lambda_k)) \quad \text{by Proposition (3.16)} \\ &= \sum_{k=1}^{n}(-1)^{n-k} ch(\beta(\lambda_k)) \\ &= \sum_{k=1}^{n}(-1)^{n-k}(\sum_{i=1}^{n}\frac{(-1)^{i-1}}{(i-1)!}\phi(n,k,i)x_{2i-1}) \quad \text{by Proposition (4.2)} \\ &= \sum_{i=1}^{n}\frac{(-1)^{i-1}}{(i-1)!}(\sum_{k=1}^{n}(-1)^{n-k}\phi(n,k,i))x_{2i-1}. \end{aligned}$$

But

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^{n-k}\phi(n,k,i)\\
&=\sum_{k=1}^{n}(-1)^{n-k}\Big(\sum_{j=1}^{k}(-1)^{j-1}\binom{n}{k-j}j^{i-1}\Big) \quad \text{by (4.1)}\\
&=\sum_{j=1}^{n}\sum_{k=j}^{n}(-1)^{n-k+j-1}\binom{n}{k-j}j^{i-1}=\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^{n-m-1}\binom{n}{m}j^{i-1}\\
&=(-1)^{n-1}\sum_{j=1}^{n}\Big(\sum_{m=0}^{n-j}(-1)^{m}\binom{n}{m}\Big)j^{i-1}=(-1)^{n-1}\sum_{j=1}^{n}(-1)^{n-j}\binom{n-1}{n-j}j^{i-1}\\
&=\sum_{j=1}^{n}(-1)^{j-1}\binom{n-1}{n-j}j^{i-1}=\phi(n-1,n,i)\\
&=\begin{cases}0 & \text{for } i=1,\dots,n-1 \text{ by [8; (3.10)]}\\ (-1)^{n-1}(n-1)! & \text{for } i=n \text{ by (4.1)}\end{cases}
\end{aligned}
$$

Thus

$$
i_n'^{*}(ch(\delta_{2n}))=\frac{(-1)^{n-1}}{(n-1)!}(-1)^{n-1}(n-1)!x_{2n-1}=x_{2n-1}.
$$

On the other hand, the right hand side is

$$
i_n'^{*}\Big(\sum_{i=1}^{n-1}b_i x_{4i-1}+b'x'_{2n-1}\Big)=b'x_{2n-1}
$$

by Proposition (2.8). Hence $b'=1$. This proves the second relation.

The third relation is obtained from the first and second relations by using the first relation of Proposition (3.13)(2).

The second relation of this theorem reflects the fact that the image of $q_{2n-1}{}^{*}: K^{*}(S^{2n-1})\to K^{*}(SO(2n))$ equals $\Lambda_{\mathbb{Z}}(\delta_{2n})$.

The following corollaries are probably known.

COROLLARY (4.8). *$i_n'^{*}: H^{*}(SO(2n);\mathbb{Z})\to H^{*}(U(n);\mathbb{Z})$ satisfies*

$$
i_n'^{*}(x_{4i-1})=\begin{cases}(-1)^{i}x_{4i-1} & \textit{for } i=1,\dots,[(n-1)/2]\\ 0 & \textit{for } i=[(n-1)/2]+1,\dots,n-1.\end{cases}
$$

Proof. We have

$$
\sum_{i=1}^{n-1}\frac{(-1)^{i-1}2}{(2i-1)!}i_n'^{*}(x_{4i-1})+\frac{(-1)^{n-1}(1+(-1)^{n})}{(n-1)!}i_n'^{*}(x'_{2n-1})
$$

$$= i_n'^*(ch(\beta(\mu_1))) \quad \text{by Theorem (4.6)}$$
$$= \sum_{i=1}^{n} \frac{(-1)^{i-1}(1+(-1)^i)}{(i-1)!} x_{2i-1} \quad \text{by (4.7)}$$

and therefore, by comparing both sides, the result follows.

COROLLARY (4.9). *$i_{2n}^* : H^*(SO(2n+1);\mathbb{Z}) \to H^*(SO(2n);\mathbb{Z})$ satisfies:*
(i) *If $n = 2m+1$,*

$$i_{4m+2}^*(x_{4i-1}) = \begin{cases} x_{4i-1} & \text{for } i = 1, \dots, 2m \\ 0 & \text{for } i = 2m+1; \end{cases}$$

(ii) *If $n = 2m$,*

$$i_{4m}^*(x_{4i-1}) = \begin{cases} x_{4i-1} & \text{for } i = 1, \dots, 2m-1 \text{ and } i \neq m \\ x_{4m-1} + (-1)^m x'_{4m-1} & \text{for } i = m \\ 0 & \text{for } i = 2m. \end{cases}$$

Proof. We have

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!} i_{2n}^*(x_{4i-1}) = i_{2n}^*(ch(\beta(\mu_1'))) \quad \text{by Theorem (4.4)}$$
$$= ch(i_{2n}^*(\beta(\mu_1'))) = ch(\beta(\mu_1)) \quad \text{by Proposition (3.15)(1)}$$
$$= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!} x_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!} x'_{2n-1} \quad \text{by Theorem (4.6)}$$

and therefore, by comparing both sides, the result follows.

THEOREM (4.10). *With the notations of* (3.7) *and* (2.5), $ch : K^*(Spin(2n)) \to$

$H^{**}(Spin(2n);\mathbb{Q})$ *is given by*

$$ch(\beta(\mu_k)) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{r(n-1,i)}}{(2i-1)!}\phi(2n,k,2i)\widetilde{x}_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi(2n,k,n)\widetilde{x}'_{2n-1},$$

$$ch(\beta(\Delta_{2n}^+)) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{r(n-1,i)-1}}{(2i-1)!}\phi_0(n,2i)\widetilde{x}_{4i-1} + \frac{1}{2}(1 + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi_0(n,n))\widetilde{x}'_{2n-1},$$

$$ch(\beta(\Delta_{2n}^-)) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{r(n-1,i)-1}}{(2i-1)!}\phi_0(n,2i)\widetilde{x}_{4i-1} + \frac{1}{2}(-1 + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi_0(n,n))\widetilde{x}'_{2n-1}.$$

Proof. We have

$$\begin{aligned} ch(\beta(\mu_k)) &= ch(\beta(p_{2n}{}^*(\mu_k))) = p_{2n}{}^*(ch(\beta(\mu_k))) \\ &= p_{2n}{}^*(\sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\phi(2n,k,2i)x_{4i-1} \\ &\qquad + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi(2n,k,n)x'_{2n-1}) \\ &\qquad \text{by Theorem (4.6)} \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\phi(2n,k,2i)2^{r(n-1,i)-1}\widetilde{x}_{4i-1} \\ &\qquad + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi(2n,k,n)\widetilde{x}'_{2n-1} \\ &\qquad \text{by Corollary (4.5)(2)} \end{aligned}$$

$$= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{r(n-1,i)}}{(2i-1)!}\phi(2n,k,2i)\widetilde{x}_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\phi(2n,k,n)\widetilde{x}'_{2n-1}.$$

Similarly, the second and third relations follow from (3.10), Theorem (4.6) and Corollary (4.5)(2).

COROLLARY (4.11). $\tilde{i}_{2n}{}^* : H^*(Spin(2n+1);\mathbb{Z}) \to H^*(Spin(2n);\mathbb{Z})$ *satisfies*:
(i) *If* $n = 2m+1$,

$$\tilde{i}_{4m+2}{}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & \text{for } i = 1,\dots,2m \\ 0 & \text{for } i = 2m+1; \end{cases}$$

(ii) *If* $n = 2m$,

$$\tilde{i}_{4m}{}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & \text{for } i = 1,\dots,2m-1 \text{ and } i \neq m \\ \tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & \text{for } i = m \text{ with } m \notin P_2 \\ 2\tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & \text{for } i = m \text{ with } m \in P_2 \\ 0 & \text{for } i = 2m. \end{cases}$$

Proof. We have

$$\sum_{i=1}^{n} \frac{(-1)^{i-1} 2^{r(n,i)}}{(2i-1)!} \tilde{i}_{2n}{}^*(\tilde{x}_{4i-1}) = \tilde{i}_{2n}{}^*(ch(\beta(\mu'_1))) \quad \text{by Theorem (4.3)}$$
$$= ch(\tilde{i}_{2n}{}^*(\beta(\mu'_1)) = ch(\beta(\mu_1)) \quad \text{by Proposition (3.14)(1)}$$
$$= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{r(n-1,i)}}{(2i-1)!} \tilde{x}_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!} \tilde{x}'_{2n-1}$$
$$\text{by Theorem (4.10)}$$

and therefore, by comparing both sides, the result follows.

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