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VECTOR BUNDLE EPIMORPHISMS AND SUBMERSIONS

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To the memory of professor José Adem

§1. Introduction

Given two vector bundles ξ over X and ζ over Y, a vector bundle epimorphism (hereafter, we call it an epimorphism) of ξ to ζ is a continuous fiber preserving map of ξ to ζ such that its restriction to each fiber is a linear epimorphism. For a (continuous) map $f: X \to Y$, the set $\mathrm{Epi}(\xi, \zeta)_f$, endowed with compact open topology, denotes the set consisting of all epimorphisms of ξ to ζ covering f, and $\mathrm{Epi}[\xi, \zeta]_f$ denotes its homotopy set, that is, the set of path components of $\mathrm{Epi}(\xi, \zeta)_f$.

In this note we first study the set $\mathrm{Epi}[\xi, \zeta]_f$ and prove the following theorem in §3:

THEOREM (A). *Let ξ be an m-plane bundle over a CW-complex X and ζ an n-plane bundle over Y ($m > n$), both admitting Riemannian metrics and let $f: X \to Y$ be a map. Then there exists a map $F: \mathrm{Epi}[\xi, \zeta]_f \to [X, BO(m-n); \xi - f^*\zeta]$ such that it is surjective if $\dim X < m$ and is bijective if $\dim X < m - 1$.*

Here $\xi - f^*\zeta: X \to BO$ stands for the classifying map of the stable bundle $\xi - f^*\zeta$ and $[X, BO(k); g]$ means the homotopy set of liftings of $g: X \to BO$ to $BO(k)$, where $BO(k) \to BO$ is the universal $O/O(k)$-bundle.

Our interest in the study of epimorphisms of vector bundles is due to the following fact. For a map $f: M \to N$ between connected C^∞-manifolds without boundary, let $S[M, N]_f$ be the regular homotopy set of submersions homotopic to f, and let $\mathrm{Epi}[\tau_M, \tau_N]_{[f]}$ be the homotopy set of epimorphisms of τ_M to τ_N covering maps homotopic to f, where τ_M for a manifold M means its tangent bundle. Then Phillips [2] has shown that if M is open then the differential map d induces a bijection $d_*: S[M, N]_f \to \mathrm{Epi}[\tau_M, \tau_N]_{[f]}$. On the other hand, there exists a $\pi_1(N^M, f)$-action on $\mathrm{Epi}[\tau_M, \tau_N]_f$ such that $\mathrm{Epi}[\tau_M, \tau_N]_f / \pi_1(N^M, f) = \mathrm{Epi}[\tau_M, \tau_N]_{[f]}$ (see Theorem (2.2)). Therefore studying epimorphisms will play an important role in the investigation of submersions of open manifolds. In fact, we will get a result concerning submersions of $P^m - P^{k-1}$ to P^n in the following theorem, where P^r and ξ_r denote the real projective r-space and its canonical real line bundle, respectively.

THEOREM (B). *Assume that $m > k \geq n \geq 1$, $k > 1$ and that if $k = n$ then $m + 1 < 2n$. Then for a map $f: P^m - P^{k-1} \to P^n$, the cardinality of the set*

$S[P^m - P^{k-1}, P^n]_f$ is given by

$$\#S[P^m - P^{k-1}, P^n]_f = \begin{cases} 1 & \text{if } k > n, \\ \infty & \text{if } k = n,\ m \equiv n \equiv 0(2) \text{ and } f^*\xi_n \text{ is trivial}, \\ 2 & \text{otherwise.} \end{cases}$$

§2. Preliminaries

For an m-plane bundle ξ over a CW-complex X, and an n-plane bundle ζ over Y, and for a map $f: X \to Y$, let $\mathrm{Epi}[\xi,\zeta]_f$ and $\mathrm{Epi}[\xi,\zeta]_{[f]}$ denote the homotopy sets of epimorphisms of ξ to ζ covering, respectively, f and maps homotopic to f, and let $\mathcal{B}(\xi,\zeta;n) = (q : B(\xi,\zeta;n) \to X \times Y)$ denote the bundle with fiber $M^*(n,m;n)$, the space consisting of all real $n \times m$-matrices of rank n, and let $\mathcal{B}_f(\xi,\zeta;n)$ denote the pull-back of $\mathcal{B}(\xi,\zeta;n)$ along $(1_X, f): X \to X \times Y$ (see [5, §1]). We note that the fiber of $\mathcal{B}(\xi,\zeta;n)$ at (x,y) is the space of all epimorphisms of the fiber ξ_x of ξ at x to the fiber ζ_y of ζ at y (see also [5, §1]). Given an epimorphism $g: \xi \to \zeta$ covering f, let $\phi_f(g): X \to B(\xi,\zeta;n)$ be the map defined by $\phi_f(g)(x) = g|\xi_x: \xi_x \to \zeta_{f(x)}$. Then in [5] we have shown the following results:

PROPOSITION (2.1). ([5, Proposition 2.1]). *Let $\Gamma(\mathcal{B}_f(\xi,\zeta;n))$ be the homotopy set of cross sections of $\mathcal{B}_f(\xi,\zeta;n)$. Then, ϕ_f induces a bijection $\phi_{f*}: \mathrm{Epi}[\xi,\zeta]_f \to \Gamma(\mathcal{B}_f(\xi,\zeta;n))$.*

Given an epimorphism $\psi_0: \xi \to \zeta$ covering f and a self-homotopy $f_t: X \to Y$ of f, there exists a homotopy of epimorphisms $\psi_t: \xi \to \zeta$ covering f_t. We define a right $\pi_1(Y^X, f)$–action on $\mathrm{Epi}[\xi,\zeta]_f$ by $[\psi_0][f_t] = [\psi_1]$.

THEOREM (2.2). ([5, Theorem 6.1]). *The natural map $\mathrm{Epi}[\xi,\zeta]_f \to \mathrm{Epi}[\xi,\zeta]_{[f]}$ induces a bijection $\mathrm{Epi}[\xi,\zeta]_f/\pi_1(Y^X, f) \cong \mathrm{Epi}[\xi,\zeta]_{[f]}$.*

Let θ^k_Z denote the trivial k-plane bundle over a space Z. Then the natural inclusion induces a map $I_{f*}: \mathrm{Epi}[\xi,\zeta]_f \to \mathrm{Epi}[\xi \oplus \theta^k_X, \zeta \oplus \theta^k_Y]_f$.

THEOREM (2.3). ([5, Theorem 5.1]). *The map I_{f*} has the following properties:*

1. *The map I_{f*} is $\pi_1(Y^X, f)$-equivariant,*
2. *I_{f*} is surjective if $\dim X < m$ and is bijective if $\dim X < m - 1$.*

§3. Proof of Theorem (A)

We assume that η and ζ are, respectively, an $(m+k)$-plane bundle over a CW-complex X and an n-plane bundle over $Y(m > n)$, both of which admit Riemannian metrics. For a map $f: X \to Y$ and any epimorphism $\tilde{f}: \eta \to \zeta$ covering f, the kernel of $\tilde{f}$ is an $(m - n + k)$-plane subbundle of η, which we denote by $\ker \tilde{f}$. The bundle $(\ker \tilde{f})_k$ means the orthonomal k-frame one associated with $\ker \tilde{f}$.

We now define a bundle map $J_f: (\ker \tilde{f})_k \to \mathcal{B}_f(\eta, \zeta \oplus \theta_Y^k; n + k)$ as follows: For an element $v = (v_1, v_2, \ldots, v_k) \in (\ker \tilde{f})_k$ at $x \in X$, we denote by $L(v)$ and $L(v)^\perp$, respectively, the vector subspace spanned by v and its orthogonal complement in the fiber $(\ker \tilde{f})_x$ of $\ker \tilde{f}$ at x. Since $\eta = \ker \tilde{f} \oplus (\ker \tilde{f})^\perp$, where the bundle $(\ker \tilde{f})^\perp$ is the orthogonal complement of $\ker \tilde{f}$ in η, the fiber η_x is expressed in the form $\eta_x = L(v) \oplus L(v)^\perp \oplus (\ker \tilde{f})_x^\perp$. Let

$$J(v): \eta_x \to \zeta_{f(x)} \oplus R^k$$

be the linear map given by

$$J(v)\left(\sum_{i=1}^k a_i v_i + a + b\right) = (\tilde{f}(a), a_1, \ldots, a_k) \tag{3.1}$$

for $a \in (\ker \tilde{f})_k^\perp, b \in L(v)^\perp$ and $a_i \in R (1 \le i \le k)$.

The map $J(v)$ is clearly an epimorphism and hence $J(v) \in B(\eta, \zeta \oplus \theta_Y^k; n + k)$. Thus the following map J can be defined:

$$J: (\ker \tilde{f})_k \to \mathcal{B}(\eta, \zeta \oplus \theta_Y^k; n + k) \qquad \text{covering} \quad 1_X \times f.$$

This map J is continuous, for J can be expressed locally as

$$J: U \cap f^{-1}(V) \times V_{m-n+k,k} \to U \times V \times M^*(n + k, m + k; n + k) \tag{3.2}$$

given by

$$J(x, v) = \left((x, f(x)), \begin{pmatrix} hfg^{-1}(e_1), \ldots, hfg^{-1}(e_n) & 0 \\ 0 & v^t \end{pmatrix}\right), \tag{3.3}$$

where (U, g) and (V, h) are charts of $(\ker \tilde{f})^\perp$ at x and of ζ at $f(x)$, respectively, $\{e_1, \ldots, e_n\}$ is the standard basis of R^n and v^t is the transpose of a matrix v. Therefore we have a bundle map

$$J_f: (\ker \tilde{f})_k \to \mathcal{B}_f(\eta, \zeta \oplus \theta_Y^k; n + k) \qquad \text{covering } 1_X,$$

defined by (3.1), or equivalently by (3.3).

From (3.2)–(3.3), the restriction $J_{f,x}$ of J_f to the fiber at x is a map

$$J_{f,x}: V_{m-n+k,k} \to M^*(n + k, m + k; n + k)$$

given by

$$J_{f,x}(v) = \begin{pmatrix} A & 0 \\ 0 & v^t \end{pmatrix},$$

where A is nonsingular, because it is the matrix of the linear map $\bar{f}: (\ker \bar{f})_x^\perp \to \zeta_{f(x)}$ relative to some bases of $(\ker \bar{f})_x^\perp$ and $\zeta_{f(x)}$. Therefore $J_{f,x}$, and hence J_f, is an $(m-n+k-1)$-equivalence. If we write the homotopy set of cross sections of $(\ker \bar{f})_k$ as $\Gamma((\ker \bar{f})_k)$, then we get a map $J_{f_*}: \Gamma((\ker \bar{f})_k) \to \Gamma(\mathcal{B}_f(\eta, \zeta \oplus \theta_Y^k; n+k))$ which is a bijection if $\dim X < m-n+k-1$. Regarding the bijection $\phi_{f_*}: \mathrm{Epi}[\eta, \zeta \oplus \theta_Y^k]_f \to \Gamma(\mathcal{B}_f(\eta, \zeta \oplus \theta_Y^k; n+k))$ of Proposition (2.1) as an identity, we have the following

LEMMA (3.4). *Let η and ζ be, respectively, an $(m+k)$-plane bundle over a CW-complex X and an n-plane bundle over $Y (m > n)$, both of which admit Riemannian metrics. Then for an epimorphism $\bar{f}: \eta \to \zeta$ covering $f: X \to Y$, the map $J_{f_*}: \Gamma((\ker \bar{f})_k) \to \mathrm{Epi}[\eta, \zeta \oplus \theta_Y^k]_f$ is a bijection if* $\dim X < m-n+k-1$.

Proof of Theorem (A). Take η to be $\xi \oplus \theta_X^k$, where ξ is an m-plane bundle over X admitting a Riemannian metric. For large enough $k (k+m-n-1 > \dim X)$, the bundle $\mathcal{B}_f(\xi \oplus \theta_X^k, \zeta; n)$ with fiber $M^*(n, m+k; n)$ has a cross section, because $M^*(n, m+k; n) (\simeq V_{m+k,n})$ is $(m-n+k-1)$-connected. Hence there is an epimorphism $\bar{f}: \xi \oplus \theta_X^k \to \zeta$ covering f by Proposition (2.1). By means of the epimorphism $\bar{f}$, we have a bijection $J_{f_*}: \Gamma((\ker \bar{f})_k) \to \mathrm{Epi}[\xi \oplus \theta_X^k, \zeta \oplus \theta_Y^k]_f$ according to Lemma (3.4). It is obvious that there is a bijection $\Gamma((\ker \bar{f})_k) = [X, BO(m-n); \ker \bar{f}]$, the homotopy set of liftings, to $BO(m-n)$, of the classifying map $X \to BO(m-n+k)$ of the bundle $\ker \bar{f}$. We may describe as $\xi - f^*\zeta$ both the stable class of the bundle $\ker \bar{f}$ and its classifying map $X \to BO$. Then for large enough k, the natural inclusion $BO(m-n+k) \to BO$ induces a bijection $[X, BO(m-n); \ker \bar{f}] \to [X, BO(m-n); \xi - f^*\zeta]$, by means of which we regard these two sets as identical. Thus we have a bijection $J_{f_*}: [X, BO(m-n); \xi - f^*\zeta] \to \mathrm{Epi}[\xi \oplus \theta_X^k, \zeta \oplus \theta_Y^k]_f$. The argument made above, together with Theorem (2.3), shows Theorem (A).

§4. Submersions of open manifolds

Throughout this section, manifolds mean connected C^∞-manifolds without boundary, and τ_M for a manifold M stands for its tangent bundle. For two manifolds M and N, and a map $f: M \to N$, we denote by $S[M, N]_f$ the set of regular homotopy classes of submersions homotopic to f. Then Phillips [2, Theorem A] has proved that the differential map d leads to a bijection

$$(4.1) \qquad d_*: S[M, N]_f \cong \mathrm{Epi}[\tau_M, \tau_N]_{[f]} \qquad \text{if } M \text{ is open.}$$

This, together with Theorems (A) and (2.2), leads to the theorem of Phillips, reworded by Thomas [3].

THEOREM (4.2) (Phillips and Thomas). *Let M be an open manifold and N a manifold, where* $\dim M > \dim N$. *Then a map $f: M \to N$ is homotopic to a submersion if and only if $\tau_M - f^*\tau_N$ has geometric dimension* $\leq \dim M - \dim N$.

Further we have the following

PROPOSITION (4.3). *Let M be an open manifold of dimension m and of homotopy* $\dim M < m-1$ *and N a manifold of dimension n $(m > n)$. If the $\pi_1(N^M, f)$-action is trivial on* $\mathrm{Epi}[\tau_M, \tau_N]_f$ *for a map $f: M \to N$, then* $S[M,N]_f = [M, BO(m-n); \tau_M - f^*\tau_N]$.

In particular, if N is a π-manifold, then the action of $\pi_1(N^M, f)$ is trivial on $\mathrm{Epi}[\tau_M, \tau_N]_f$ and so we get $S[M,N]_f = [M, BO(m-n); \tau_M] = S[M, R^n]$, where $\tau_M: M \to BO$ is the classifying map of the stable tangent bundle of M (see [5, §7]). This is an extension of Phillips' result [2, Proposition 10. 4(a)].

In the rest of this note, we consider the submersions of $P^m - P^{k-1}$ to P^n, where P^r means the real projective r-space and $P^m - P^{k-1} = \{ [x_0, x_1, \ldots, x_m] \in P^m \mid [x_{m-k+1}, \ldots, x_m] \notin P^{k-1}\}$. Let ξ_r be the canonical real line bundle of P^r and denote $P^m - P^{k-1}$ by M sometimes for a typographical reason. Then $\tau_M = \tau_{P^m}|M$ and hence $\tau_M \oplus \theta^1_M = (m+1)\xi_m|M$. The natural inclusion $i: P^{m-k} \to M$ is a homotopy equivalence, whose homotopy inverse $r: M \to P^{m-k}$ is given by $r[x_0, \ldots, x_m] = [ax_0, \ldots, ax_{m-k}]$ where $a = \left(\sum_{i=0}^{m-k} x_i^2\right)^{-1/2}$ (cf. [2, p. 200]).

For a map $f: M \to P^n$, the map $I_{f_*}: \mathrm{Epi}[\tau_M, \tau_{P^n}]_f \to \mathrm{Epi}[(m+1)\xi_m|M, (n+1)\xi_n]_f$ is surjection if $k \geq 1$ by Theorem (2.3). On the other hand, $f^*\xi_n$ is isomorphic to $\xi_m|M$ or the trivial bundle, because $i: P^{m-k} \to M$ is a homotopy equivalence. Using these facts, (4.1), Theorems (2.2–3), and Theorem (A), we have the following

PROPOSITION (4.4). *Assume that $k \geq 1$, and $m > n \geq 1$, and let $f: P^m - P^{k-1} \to P^n$ be a map.*

1. *If $f^*\xi_n$ is trivial, then f is homotopic to a submersion if and only if there exists a submersion of $P^m - P^{k-1}$ to R^n.*
2. *Otherwise, f is always homotopic to a submersion.*

EXAMPLE. In case $m-k \leq n$, let $r: M (= P^m - P^{k-1}) \to P^n$ be a map defined by

$$(4.5) \qquad r[x_0, x_1, \ldots, x_m] = [ax_0, ax_1, \ldots, ax_n] \qquad \left(a = \left(\sum_{i=0}^{n} x_i^2\right)^{-1/2}\right).$$

Then r is a submersion. This map r is not homotopic to a constant map, because $r|P^1: P^1 \to P^n$ is the natural inclusion. In particular, r is the retraction mentioned above if $n = m-k$.

Proof of Theorem (B). Because $n \leq k$, any map $f: M \to P^n$ is homotopic to a submersion. This follows immediately from Theorem (A), the obstruction theory and the fact that $i: P^{m-k} \subset M$ is a homotopy equivalence. Since $m-k < m-1$, the map $i^* I_{f_*}: \mathrm{Epi}[\tau_M, \tau_{P^n}]_f \to \mathrm{Epi}[(m+1)\xi_{m-k}, (n+1)\xi_n]_{fi}$ is an $i''_\#$-equivariant bijection for any map $f: M \to P^n$, where $i''_\#: \pi_1((P^n)^M, f) \to \pi_1((P^n)^{P^{m-k}}, fi)$ is an isomorphism (see [5, §3, §5]). The fiber $O/O(m-n)$

of $BO(m-n) \to BO$ is $(m-n-1)$-connected and $\pi_{m-n}(O/O(m-n)) = Z$ or Z_2 according as $m-n$ is even or odd. The theorem A, together with the classification theorem of liftings (see [4, p. 302]), implies that

$$\begin{aligned}\mathrm{Epi}[\tau_M, \tau_{P^n}]_f &= \mathrm{Epi}[(m+1)\xi_{m-k}, (n+1)\xi_n]_{fi} \\ &= \begin{cases} 0 & \text{if } n < k, \\ H^{m-n}(P^{m-n}; Z[w_1((m+1)\xi_{m-n} - (n+1)(fi)^*\xi_n)]) & \\ & \text{if } n = k,\ m-n \equiv 0(2), \\ H^{m-n}(P^{m-n}; Z_2) & \text{if } n = k,\ m-n \equiv 1(2). \end{cases}\end{aligned}$$

Hence if $n < k$ then the cardinality $\#S[M, P^n]_f = \#\,\mathrm{Epi}[\tau_M, \tau_{P^n}]_{[f]} = 1$ by Theorem (2.2) and (4.1). Here $\#S$ denotes the cardinality of the set S. On the other hand, if $k = n$ then

$$\begin{aligned} i^* I_{f_*}\colon \mathrm{Epi}[\tau_M, \tau_{P^n}]_f &= \mathrm{Epi}[(m+1)\xi_{m-n}, (n+1)\xi_n]_{fi} \\ &= \begin{cases} Z & \text{for } m \equiv n \equiv 0(2),\ f^*\xi_n = \theta_M, \\ Z_2 & \text{otherwise.} \end{cases}\end{aligned}$$

Here $i^* I_{f_*}$ is an $i''_{\#}$-equivariant, where $i''_{\#}\colon \pi_1((P^n)^M, f) \to \pi_1((P^n)^{P^{m-n}}, fi)$ is an isomorphism.

From now on, we assume that $n = k$ and that $1 \le m-k < n-1$. Then it follows from [1, Lemma 1](cf. [5,§4]) that $\pi_1((P^n)^M, f) = \pi_1((P^n)^{P^{m-k}}, fi) = Z_2$.

If $m \equiv n \equiv 0(2)$ and $f^*\xi_n = \theta^1_M$, then obviously $\#\,\mathrm{Epi}[\tau_M, \tau_{P^n}]_f = \infty$ and hence $\#S[M, P^n]_f = \infty$.

To prove that the action of $\pi_1((P^n)^M, f)$ is trivial on $\mathrm{Epi}[\tau_M, \tau_{P^n}]_f$ in the other cases, it is enough to show the fact that $[\psi][f_t] = [\psi]$ for some epimorphism $\psi\colon \tau_M \to \tau_{P^n}$ covering f, or $\psi\colon (m+1)\xi_{m-n} \to (n+1)\xi_n$ covering fi, and a generator $[f_t] \in \pi_1((P^n)^M, f)$, or $[f_t] \in \pi_1((P^n)^{P^{m-n}})$, respectively, for the set $\mathrm{Epi}[\tau_M, \tau_{P^n}]_f = Z_2$ and $i^* I_{f_*}$ is $i''_{\#}$-equivariant.

If $n \equiv 1(2)$, it follows from [5, Proposition 4.2] (cf. [1, Theorem 1]) that the $\pi_1((P^n)^M, f)$-action is trivial on $\mathrm{Epi}[\tau_M, \tau_{P^n}]_f$. Therefore we have $\#S[M, P^n]_f = 2$ for $n \equiv 1(2)$.

If $m \equiv 1(2)$, $n \equiv 0(2)$, and if $f^*\xi_n = \xi_m|M$, then we may assume that $f|P^1\colon P^1 \to P^n$ represents a generator of $\pi_1(P^n)$ (see [5, Lemma 4.1]). Because $m \equiv 1(2)$, there is a flow Φ_t on P^m such that $\Phi_t = \Phi_{t+1}$ which is defined in [5, §4] (cf.[1]). Its restriction $\Phi_t|M$ to M is also a flow on M. Hence its differential $d\Phi_t|\tau_M$ is a flow on τ_M. For any epimorphism $\psi\colon \tau_M \to \tau_{P^n}$ covering f, the composition $\psi d\Phi|\tau_M\colon \tau_M \to \tau_{P^n}$ is a homotopy of epimorphism covering $f\Phi_t|M$, while the homotopy $f\Phi_t|M$ represents the generator of $\pi_1((P^n)^M, f)$. Hence $[\psi][f\Phi_t|M] = [\psi d\Phi_1|M] = [\psi]$. This implies that the action is trivial and therefore $\#S[M, P^n]_f = 2$.

Next we consider the case where $m \equiv n \equiv 0(2)$ and $f^*\xi_n = \xi_m|M$. Then because of the assumption $m - n < n - 1$, we may assume that $f\colon M \to P^n$ is a submersion defined by (4.5) and hence $fi\colon P^{m-n} \to P^n$ is the natural inclusion. Let $N = P^{n+1} - P^1$ and let $j: P^{m-n} \subset N$ be the natural inclusion. Then $I_{j_*}: \mathrm{Epi}[\tau_M|P^{m-n}, \tau_N]_j \to \mathrm{Epi}[(m+1)\xi_{m-n}, (n+2)\xi_{n+1}|N]_j$ is surjective by Theorem (2.3), and further it follows easily that the latter set is not empty. Therefore there exists an epimorphism $\psi: \tau_M|P^{m-n} \to \tau_N$ covering $j: P^{m-n} \subset N$. Because $n+1 \equiv 1(2)$, there exists a flow Φ_t on P^{n+1}, defined in [5, §4] (cf.[1]), such that $\Phi_t = \Phi_{t+1}$. The restriction of this flow to N becomes a flow on N. Here, let $r: N \to P^n$ be a submersion defined in (4.5). Then $dr\, d\Phi_t\, \psi: \tau_M|P^{m-n} \to \tau_{P^n}$ is a homotopy of epimorphisms covering the homotopy $r\Phi_t j$. Now the class $[r\Phi_t j]$ is a generator of $\pi_1((P^n)^{P^{m-n}}, fi)$ by [5, Lemma 4. 1]. Hence we have $[dr\,\psi][r\Phi_t j] = [dr\, d\Phi_1 \psi] = [dr\,\psi]$ and hence $\#\,\mathrm{Epi}[\tau_M|P^{m-n}, \tau_{P^n}]_{[fi]} = \#S[M, P^n]_f = 2$.

The investigation in the case where $n \equiv 0(2)$, $m \equiv 1(2)$ and $f^*\xi_n = \theta^1_M$ remains. In this case, f can be regarded as a constant map. We notice that there is an epimorphism $\psi: (m+1)\xi_{m-n} \to (n+2)\xi_n$ covering constant map c. In fact, the classifying map $P^{m-n} \to BO$ of the stable class of $(m+1)\xi_{m-n}$ has a lift to $BO(m-n)$, because $\beta_2 w_{m-n-1}((m+1)\xi_{m-n}) \in H^{m-n}(P^{m-n}; Z[w_1((m+1)\xi_{m-n})])$ is a unique obstruction to lifting this map to $BO(m-n)$, where β_2 is a Bockstein operator, and $\beta_2 = 0$ in this case. Let $f_t: P^{m-n} \to P^n$ be a homotopy representing a generator of $\pi_1((P^n)^{P^{m-n}}, c)$. Since the action of this group on $\mathrm{Epi}[(m+1)\xi_{m-n}, (n+2)\xi_n]_c$ is trivial by [5, Lemma 4.2] because $n+2 \equiv 0(2)$, we have a homotopy of epimorphisms ψ_t covering f_t such that $\psi_0 = \psi$ and $[\psi_1] = [\psi] \in \mathrm{Epi}[(m+1)\xi_{m-n}, (n+2)\xi_n]_c$. Let $\pi: (n+2)\xi_n \to (n+1)\xi_n$ be a natural projection to the first $(n+1)$-components. Then $\pi\psi_t: (m+1)\xi_{m-n} \to (n+1)\xi_n$ is a homotopy of epimorphism covering f_t such that $\pi\psi_0 = \pi\psi$ and $[\pi\psi_1] = [\pi\psi] \in \mathrm{Epi}[(m+1)\xi_{m-n}, (n+1)\xi_n]_c$. This shows that $[\pi\psi][f_t] = [\pi\psi]$ and that $\pi_1((P^n)^{P^{m-n}}, c)$-action on $\mathrm{Epi}[(m+1)\xi_{m-n}, (n+1)\xi_n]_c$ is trivial. Therefore we have $\#S[M, P^n]_c = \#\,\mathrm{Epi}[(m+1)\xi_{m-n}, (n+1)\xi_n]_c = 2$. Summing up the above calculation, we get the Theorem (B).

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