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THE BLOCH SPACE OF A HOMOGENEOUS TREE

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*Dedicated to the Memory of Professor José Adem**

1.Introduction and basic definitions

The classical theory of Bloch functions in the open unit disk Δ and the interesting connections between the Bloch space $\mathcal{B}(\Delta)$ and other function spaces has led to many new areas of research in the past two decades. An analytic function on Δ is called *Bloch* [10] if the growth of the modulus of its derivative is controlled by the density of the hyperbolic metric of the disk (the Bloch condition). There are several equivalent characterizations of Bloch functions. As a consequence, Bloch functions arise in many different contexts. For a comprehensive treatise on Bloch functions cf. [3].

For complex-valued harmonic functions on Δ, analogous definitions yield a Bloch space which is essentially the product of two copies of $\mathcal{B}(\Delta)$, since every harmonic function can be written uniquely up to additive constants as the sum of an analytic function and the conjugate of an analytic function. [8]

In this work we consider a class of complex-valued harmonic functions on a homogeneous tree characterized by the property that the difference between the values of such functions at neighboring vertices (the analogue of derivative divided by density in the classical case) remains bounded throughout the tree. We shall show that this class is a complex Banach space which we shall call the *Bloch space*, in analogy with the classical case of analytic and harmonic functions on the open unit disk. Subsequently we shall study its properties in relation to a proper subspace, the *little Bloch space*. Both of these spaces are very rich. For example, fixing a vertex and a positive integer n, we can preassign any values on the sphere of radius n and get an extention which is in the little Bloch space.

By a *tree* we mean a connected and simply-connected graph which is locally finite and contains more than two vertices. The vertices v and w are called *neighbors* if they are connected by an edge. We use the notation $v \sim w$ for neighboring vertices v and w.

A tree is called *homogeneous* if each vertex has the same number of neighbors. This number is called the *degree* of the tree.

A *path* $[\ldots, v_k, v_{k+1}, \ldots]$ is a finite, infinite or doubly infinite sequence of vertices v_k such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$, for each k. We define the *length* of a finite path $[v_0, \ldots, v_k]$ to be k.

For v and w vertices we define the *tree distance* $d(v,w)$ between v and w to be the length of the path connecting v to w. An *automorphism* of a tree

* The pioneering results of Prof. Adem in algebraic topology were very influential on the first author's work on stable homotopy.

T is an isometry of T, that is, a bijective function from T onto itself which sends edges to edges. When we consider T as a point set, we refer only to the vertices of T.

Let $\simeq$ be the equivalence relation generated by the unit shift: If $[v_0, v_1, \ldots]$ is an infinite path in a tree T, then $[v_0, v_1, \ldots] \simeq [v_1, v_2, \ldots]$. An equivalence class of infinite paths under $\simeq$ is called an *end* of T. Letting Ω denote the set of ends, there is a topology on $\overline{T} = T \cup \Omega$ under which $\overline{T}$ is a compact Hausdorff space with T an open, discrete, and dense subset. We may think of Ω as the *boundary of* T. The main idea of the construction is to endow the set of vertices of T with a metric m defined as follows: fix one vertex v_0, and then define the length of an edge $[v_1, v_2]$ as n^{-2}, where $n = \max_{j=1,2} d(v_j, v_0)$. For any vertices u and v define $m(u, v)$ as the sum of the lengths of the edges joining u to v. It is easy to see that the completion of the resulting bounded metric space is exactly $\overline{T}$, that it is totally bounded, and hence compact. (Cf. [5].) We use this topology to observe that any sequence of edges has a subsequence which lies along a single path, a result which shall be used to prove Theorem (1).

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2. Bounded harmonic functions

In this section we recall the definition of a harmonic function on a homogeneous tree as a complex-valued function whose value at any vertex is the average of its values at all its neighboring vertices. We shall then define Bloch functions and, noting that bounded harmonic functions are Bloch, show that the maximum value of their Bloch constants can be expressed in terms of the degree of homogeneity of the tree. We shall then analyze the extremal functions.

Definition. Let T be a homogeneous tree of degree $s + 1$, with $s \in \mathbb{N}, s \geq 2$.

1) A function $f : T \to \mathbb{C}$ is *harmonic* if for every vertex v of T, $f(v) = \frac{1}{s+1} \sum_{w \sim v} f(w)$.

2) A function $f : T \to \mathbb{C}$ is *Bloch* if $\beta_f = \sup_{w \sim v} |f(w) - f(v)| < \infty$.

The number β_f, called the *Bloch constant* of the function f, measures its maximum "stretch". We use this terminology because in the classical case it is related to the universal Bloch constant β (cf. [13], p. 133). It is the same as the Lipschitz number of f, where the function is thought of as a map between the metric spaces (T, d) and the Euclidean complex plane. Bloch functions have been studied in a variety of contexts, generally as analytic or harmonic functions on some complex manifold satisfying a growth condition on the derivative. This Bloch condition in the case of a bounded homogeneous domain in $\mathbb{C}^n$ is analogous to the Lipschitz condition with respect to the Bergman metric on the domain and the Euclidean metric on $\mathbb{C}$ (cf. e.g. [7,8,4]).

In the case of the open unit disk Δ, if f is analytic on Δ, then f is a *Bloch function* if $\beta_f = \sup_{z \in \Delta} (1 - |z|^2)|f'(z)|$ is finite. Let $\mathcal{B}(\Delta)$ be the *Bloch space* defined as the set of all Bloch functions on Δ. Then $\mathcal{B}(\Delta)$ is a complex Banach space under the norm $\|f\| = |f(0)| + \beta_f$ (cf. [1]) and is the double dual of its closed subspace $\mathcal{B}_0(\Delta)$, the *little Bloch space*, which is the set of all $f \in \mathcal{B}(\Delta)$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$ (cf. [11] or [14]).

In [8] it is shown that a bounded harmonic function on the open unit disk has a maximum Bloch constant of $4/\pi$ times its supremum norm. In [4] there are corresponding results for bounded holomorphic functions on bounded symmetric domains. These provide the impetus for the study of the Bloch constant of bounded harmonic functions on a homogeneous tree.

Before stating the main result of this section, we introduce some notation and definitions.

Let T be a homogeneous tree of degree $s+1$. For neighboring vertices v_0 and v_1 and for a positive integer n define the set

$$V_n(v_0, v_1) = \{v \in T | d(v, v_0) = n,\ d(v, v_1) = n-1\}.$$

Note that the collection $\{V_n(v_0,v_1), V_n(v_1,v_0)\}_{n=1}^{\infty}$ is a partition of T, and that $V_n(v_0,v_1)$ has cardinality s^{n-1}.

We now give an example of a bounded harmonic function on T which will be used frequently throughout the paper.

Example (1). Fix an edge $[v_0, v_1]$. Define $F : T \to \mathbb{C}$ as follows.

$$F(v) = \begin{cases} 1 - \dfrac{2}{s^{n-1}(s+1)}, & \text{for } v \in V_n(v_0,v_1), n \in \mathbb{N} \\ -1 + \dfrac{2}{s^{n-1}(s+1)}, & \text{for } v \in V_n(v_1,v_0), n \in \mathbb{N}. \end{cases}$$

Clearly F has image contained in $(-1,1) \subset \Delta$. It is straightforward to verify that F is a harmonic function on T. Observe that if $v \in V_n(v_0,v_1)$ and $u \in V_{n+1}(v_0,v_1)$ (or $v \in V_n(v_1,v_0)$ and $u \in V_{n+1}(v_1,v_0)$), then $|F(v) - F(u)| = \dfrac{2(s-1)}{s^n(s+1)}$. Furthermore $\beta_F = \dfrac{2(s-1)}{s+1}$ and this value is attained inside T. Indeed $|F(v_1) - F(v_0)| = \dfrac{2(s-1)}{s+1}$.

Our main result, Theorem (1), is a direct analogue of the classical case for the unit disk.

THEOREM (1). *Let T be a homogeneous tree of degree $s+1$ and let $f : T \to \mathbb{C}$ be a bounded harmonic function with supremum norm $\|f\|_\infty$. Then f is Bloch and $\beta_f \leq \dfrac{2(s-1)}{s+1}\|f\|_\infty$. Furthermore, if $\beta_f = \dfrac{2(s-1)}{s+1}\|f\|_\infty$, then this value is attained inside the tree (i.e. $\beta_f = \max_{v\sim w}|f(v) - f(w)|$) if and only if there exist $\lambda \in \mathbb{C}$ of modulus one and an edge $[v_0, v_1]$ such that $f = \lambda\|f\|_\infty F$, where F is the function in Example (1) for this edge.*

If $\beta_f = \dfrac{2(s-1)}{s+1}\|f\|_\infty$ and this value is not attained inside the tree, then for any edge $[v_0, v_1]$ there exist an automorphism S of T, a sequence $\{n_k\}$ of *positive integers, and* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$, *such that*

$$\lim_{k\to\infty} f \circ S^{n_k} = \lambda \|f\|_\infty F,$$

where the limit is pointwise in T. *(Here* S^j *is the* j*-fold composition product of* S.*)*

By normalizing the function f, we may assume that $\|f\|_\infty = 1$, so throughout this proof, we assume that the image of f is in the unit disk.

To prove Theorem (1) we introduce several lemmas.

If f is a function from T to $\mathbb{C}$, and X is a finite subset of T, denote by $\mu(f, X)$ the *mean value of* f *on* X, that is,

$$\mu(f, X) = \frac{1}{|X|} \sum_{v\in X} f(v),$$

where $|X|$ denotes the cardinality of X.

LEMMA (1). *Let* T *be a homogeneous tree of degree* $s+1$ *and let* $f : T \to \mathbb{C}$ *be a harmonic function. Then for any neighboring vertices* v_0, v_1 *of* T *and* $n \in \mathbb{N}$ *we have*

$$\begin{aligned}\mu(f, V_n(v_0, v_1)) &= f(v_0) + (f(v_1) - f(v_0)) \frac{s^n - 1}{s^{n-1}(s-1)}\\ &= f(v_1) + (f(v_1) - f(v_0)) \frac{s^{n-1} - 1}{s^{n-1}(s-1)}.\end{aligned}$$

Proof. Let V_n denote $V_n(v_0, v_1)$. For any $n \geq 2$, a vertex $v \in V_n$ has exactly s neighbors in V_{n+1} and one neighbor in V_{n-1}. So by the harmonicity of f, $(s+1)f(v)$ is the sum of the values of f on a set of s elements in V_{n+1}, together with one element of V_{n-1}. The elements of V_{n+1} are uniquely determined by v, while each element of V_{n-1} corresponds to s different values of v. So when we add all these up we get the equality

$$(s+1)\sum_{V_n} f(v) = \sum_{V_{n+1}} f(v) + s \sum_{V_{n-1}} f(v), \text{ for } n \geq 2.$$

For $n = 1$, instead, we get that $(s+1)f(v_1) = \sum_{V_2} f(v) + f(v_0)$.

Let us define $a_n = \sum_{V_n} f(v)$, and $a_0 = f(v_0)$. Then we obtain the relations

$$\begin{aligned}&a_0 = f(v_0),\ a_1 = f(v_1),\ a_2 = (s+1)a_1 - a_0,\\ &a_{n+1} = (s+1)a_n - s a_{n-1}, \text{ for } n \geq 2.\end{aligned}$$

First let us consider the case $f(v_0) = 0$ and $f(v_1) = 1$. Using induction, it is easy to show that $a_n = \dfrac{s^n - 1}{s - 1}$. Since there are s^{n-1} elements in the set V_n, it follows that $\mu(f, V_n) = \dfrac{a_n}{s^{n-1}} = \dfrac{s^n - 1}{s^{n-1}(s-1)}$.

Now let f be any harmonic function on T such that $f(v_0) \neq f(v_1)$. We may normalize f by setting $g(v) = \dfrac{f(v) - f(v_0)}{f(v_1) - f(v_0)}$, for all $v \in T$. We see that g is harmonic, $g(v_0) = 0$ and $g(v_1) = 1$. From the preceding case it follows that $\mu(g, V_n) = \dfrac{s^n - 1}{s^{n-1}(s-1)}$. Since $f = f(v_0) + (f(v_1) - f(v_0))g$, it then follows that $\mu(f, V_n) = f(v_0) + (f(v_1) - f(v_0))\mu(g, V_n)$, which yields the conclusion of the lemma.

Next, in the case that $f(v_0) = f(v_1)$, we need only show that for each $n \in \mathbb{N}$, $a_n = s^{n-1} f(v_0)$. This follows by an easy inductive proof from the relations among the a_n given above. This proves that $\mu(f, V_n) = f(v_0)$ for all $n \in \mathbb{N}$. □

LEMMA (2). *Let $f : T \to \Delta$ be a harmonic function. If v_0 and v_1 are neighboring vertices, then*

$$\left| f(v_0) + (f(v_1) - f(v_0)) \frac{s}{s-1} \right| \leq 1, \tag{1}$$

$$\left| f(v_0) - (f(v_1) - f(v_0)) \frac{1}{s-1} \right| \leq 1. \tag{2}$$

Proof. Since the image of f is contained in the unit disk, the mean value of f on any finite subset of T must also be inside the unit disk. From Lemma (1) we obtain

$$\left| f(v_0) + (f(v_1) - f(v_0)) \frac{s^n - 1}{s^{n-1}(s-1)} \right| = \left| f(v_1) + (f(v_1) - f(v_0)) \frac{s^{n-1} - 1}{s^{n-1}(s-1)} \right| < 1.$$

Since this must hold for all n, passing to the limit we obtain inequality (1). The second inequality can be derived from the first by interchanging the roles of v_1 and v_0. □

In keeping with the notation established previously for mean values of functions defined on finite sets, we use the following notation when there is no danger of confusion: if θ is a function with a domain of finite cardinality, then $\mu(\theta)$ is the mean value of θ on that domain.

LEMMA (3). *Let $\theta : \{1, \ldots, N\} \to \mathbb{C}$ have image in the closure of the disk of radius 1 centered at 1, where $N \in \mathbb{N}$. Then for all $k = 1, \ldots, N$, we have $|\theta(k)| < \sqrt{2N|\mu(\theta)|}$.*

Proof. Since $|\theta(k) - 1| \leq 1$ for all $k = 1, \ldots, N$, we have $\mathrm{Re}(\theta) \geq 0$ and thus

$$|\theta(k)|^2 \leq 2\mathrm{Re}\theta(k) \leq 2 \sum_{t=1}^{N} \mathrm{Re}\theta(t) = 2N\mathrm{Re}\mu(\theta) \leq 2N|\mu(\theta)|,$$

proving the assertion. □

LEMMA (4). *Let n be a positive integer, and let $f : T \to \Delta$ be a harmonic function satisfying the inequality*

$$\left|(f(v_1) - f(v_0)) - \frac{2(s-1)}{s+1}\right| < \epsilon,$$

for some $v_0, v_1 \in T$ with $v_0 \sim v_1$. Assume that $0 < \epsilon < 4$. *Then for all $v \in V_n(v_0, v_1) \cup V_n(v_1, v_0)$ we have*

$$|f(v) - F(v)| < 10s^{(n-1)/2}\sqrt{\epsilon},$$

where F is the extremal function of Example (1) for the edge $[v_0, v_1]$.

Proof. By symmetry, it suffices to prove the inequality for $v \in V_n = V_n(v_0, v_1)$. Set $z = f(v_1) - f(v_0) - \dfrac{2(s-1)}{s+1}$, so that $|z| < \epsilon$. Then from (1) and (2), respectively, we get the inequalities

$$\left|\left(f(v_0) + \frac{s-1}{s+1}\right) + 1 + z\frac{s}{s-1}\right| \le 1,$$

$$\left|-\left(f(v_0) + \frac{s-1}{s+1}\right) + 1 + z\frac{1}{s-1}\right| \le 1.$$

So $f(v_0) + \dfrac{s-1}{s+1}$ is in the intersection of the closed disks of radius 1 centered at the points $-1 - z\dfrac{s}{s-1}$ and $1 + z\dfrac{1}{s-1}$. Hence, by easy geometric considerations, we obtain

$$\left|f(v_0) + \frac{s-1}{s+1}\right| \le 2|z|\frac{s}{s-1} < 2\epsilon\frac{s}{s-1} \le 4\epsilon. \tag{3}$$

Now $f(v_1) = f(v_0) + \dfrac{2(s-1)}{s+1} + z$, so letting $\varsigma_1 = f(v_1) - \dfrac{s-1}{s+1}$, we obtain

$$|\varsigma_1| = \left|f(v_0) + \frac{s-1}{s+1} + z\right| < 5\epsilon < 10\sqrt{\epsilon}. \tag{4}$$

Notice that $|f(v_1) - F(v_1)| = |\varsigma_1| < 10\sqrt{\epsilon}$, so the lemma is true for $n = 1$.

Let us assume inductively that for all $w \in V_k$ we have $|f(w) - F(w)| \le 10s^{\frac{k-1}{2}}\sqrt{\epsilon}$. First from Lemma (1) we see that for any positive integer n

$$\begin{aligned}\mu(f, V_n) - \left(1 - \frac{2}{s^{n-1}(s+1)}\right) &= f(v_0) + \left(\frac{2(s-1)}{s+1} + z\right)\frac{s^n - 1}{s^{n-1}(s-1)} - 1 + \frac{2}{s^{n-1}(s+1)} \\ &= \left(f(v_0) + \frac{s-1}{s+1}\right) + z\frac{s^n - 1}{s^{n-1}(s-1)}.\end{aligned}$$

So, using (3) we obtain

$$\left|\mu(f,V_n)-\left(1-\frac{2}{s^{n-1}(s+1)}\right)\right|<4\epsilon+\epsilon\frac{s^n-1}{s^{n-1}(s-1)}<6\epsilon. \tag{5}$$

Now let $a_1\in V_{k+1}$ and $a_0\in V_k$ be neighbors. Define ς_k and ς_{k+1} by

$$\varsigma_k=f(a_0)-1+\frac{2}{s^{k-1}(s+1)} \text{ and } \varsigma_{k+1}=f(a_1)-1+\frac{2}{s^k(s+1)}.$$

Then $f(a_1)-f(a_0)=\dfrac{2(s-1)}{s^k(s+1)}+\varsigma_{k+1}-\varsigma_k$. Applying (1) to the vertices a_0,a_1 we get $\left|1+\frac{s\varsigma_{k+1}-\varsigma_k}{s-1}\right|\le 1$. Thus $-\frac{s\varsigma_{k+1}-\varsigma_k}{s-1}$ is in the closed disk of radius 1 centered at 1. Now observe that for $n=k+j$ we have $\varsigma_n=f(a_j)-\left(1-\dfrac{2}{s^{n-1}(s+1)}\right)$ where $j=0,1$. Thus by (5), the mean values of ς_k and ς_{k+1} over the sets V_k and V_{k+1}, respectively, have modulus less than 6ϵ. Therefore for $\theta=\dfrac{s\varsigma_{k+1}-\varsigma_k}{s-1}$ we obtain $|\mu(\theta)|<6\epsilon\dfrac{s+1}{s-1}\le 18\epsilon$. So applying Lemma (3), and noting that V_{k+1} has cardinality s^k, we get $\left|\dfrac{s\varsigma_{k+1}-\varsigma_k}{s-1}\right|<\sqrt{2s^k18\epsilon}=6s^{k/2}\sqrt{\epsilon}$ and so

$$|\varsigma_{k+1}|<6s^{k/2}\sqrt{\epsilon}+\frac{1}{s}|\varsigma_k|.$$

By our inductive assumption, we have $|\varsigma_k|<10s^{\frac{k-1}{2}}\sqrt{\epsilon}$. We then get

$$|\varsigma_{k+1}|<6s^{k/2}\sqrt{\epsilon}+\frac{1}{s}10s^{\frac{k-1}{2}}\sqrt{\epsilon}=6s^{k/2}\sqrt{\epsilon}\left(1+\frac{5}{3}s^{-3/2}\right)<10s^{\frac{k}{2}}\sqrt{\epsilon}.$$

This completes the proof. □

LEMMA (5). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of harmonic functions from T to Δ such that*

$$\lim_{n\to\infty}|f_n(v_1)-f_n(v_0)|=\frac{2(s-1)}{s+1},$$

where v_0 and v_1 are neighboring vertices of T. Then, for some $\lambda\in\mathbb{C}$ of modulus one, there exists a subsequence $\{f_{n_k}\}$ converging pointwise to λF, where F is the harmonic function of Example (1) for the edge $[v_0,v_1]$.

Proof. For sufficiently large integers n we may choose a complex number λ_n of modulus one such that $\overline{\lambda_n}(f_n(v_1)-f_n(v_0))$ is positive. Let $\{\lambda_{n_k}\}$ be a convergent subsequence with limit λ. Thus

$$\lim_{k\to\infty}\overline{\lambda}\left(f_{n_k}(v_1)-f_{n_k}(v_0)\right)=\frac{2(s-1)}{s+1}.$$

By Lemma (4), for all $v \in T$ we obtain

$$\lim_{k\to\infty} \overline{\lambda} f_{n_k}(v) = F(v),$$

and so $\{f_{n_k}\}$ converges pointwise to λF. □

We are now ready to prove Theorem (1).

Proof. Let us assume that $f : T \to \Delta$ is a harmonic function. Choose any neighboring vertices $v_0, v_1 \in T$. Putting together relations (1) and (2) yields the inequality

$$\left|(f(v_1) - f(v_0)) \frac{s}{s-1} + (f(v_1) - f(v_0)) \frac{1}{s-1}\right| \leq 2.$$

Thus $\frac{s+1}{s-1}|f(v_1) - f(v_0)| \leq 2$ yielding $\beta_f \leq \frac{2(s-1)}{s+1}$.

Next suppose that $\beta_f = \frac{2(s-1)}{s+1}$ and that this value is attained in the tree. That is, there exist neighboring vertices v_0, v_1 such that $|f(v_1) - f(v_0)| = \frac{2(s-1)}{s+1}$. Hence $f(v_1) - f(v_0) = \lambda \frac{2(s-1)}{s+1}$, for some complex number λ of modulus one. Letting $g = \overline{\lambda} f$ we see that $g : T \to \Delta$ is harmonic and $g(v_1) - g(v_0) = \frac{2(s-1)}{s+1}$. We are going to show that $g = F$.

Given $n \in \mathbb{N}$ we prove that $g(v) = F(v)$ for all $v \in V_n(v_0, v_1) \cup V_n(v_1, v_0)$: Letting ϵ be any positive constant with $\epsilon < 4$, and applying Lemma (4) to the function g, we obtain $|g(v) - F(v)| < 10s^{(n-1)/2}\sqrt{\epsilon}$, for all $v \in V_n(v_0, v_1) \cup V_n(v_1, v_0)$. Taking the limit as ϵ goes to 0, we obtain $g(v) = F(v)$, completing the proof in the attainment case.

Now assume that for each $n \in \mathbb{N}$ there exist neighboring vertices w_n and u_n such that $\lim_{n\to\infty} |f(w_n) - f(u_n)| = \frac{2(s-1)}{s+1}$. Without loss of generality we may assume that $d(w_n, v_0) = d(u_n, v_0) + 1$. Since $\overline{T} = T \cup \Omega$ is compact, there exists a subsequence $\{w_{\nu_k}\}$ converging to some end ω. Similarly corresponding to $\{u_{\nu_k}\}$ there exists a subsequence $\{u_{\nu_{k_l}}\}$ converging to some end ω'. But if these ends were distinct, then for all l sufficiently large, the vertices $w_{\nu_{k_l}}$ and $u_{\nu_{k_l}}$ could not be neighbors. So ω must equal ω'.

There are two cases to consider. First assume that the path p from v_0 representing ω contains v_1: $p = [v_0, v_1, v_2, \ldots]$. Then there exists a sequence $\{m_j\}$ of positive integers such that $[v_{m_j}, v_{m_j+1}] = [u_{\nu_{k_j}}, w_{\nu_{k_j}}]$. Thus

$$\lim_{j\to\infty} |f(v_{m_j+1}) - f(v_{m_j})| = \frac{2(s-1)}{s+1}.$$

Let S be any automorphism of T that moves right on the path p, that is, such that $S(v_{n-1}) = v_n$, $n \in \mathbb{N}$. Thus the preceding limit can be written as

$$\lim_{j\to\infty} |f \circ S^{m_j}(v_1) - f \circ S^{m_j}(v_0)| = \frac{2(s-1)}{s+1}.$$

Thus by Lemma (5) there is a constant λ of modulus one and a subsequence $\{m_{j(k)}\}_{k\in\mathbb{N}}$ such that the sequence $\{f \circ S^{m_{j(k)}}\}$ converges pointwise to λF.

In the case that the path p representing ω is of the form $[v_1, v_0, v_2, \ldots]$ we may apply the same argument and get a sequence $\{f \circ S^{m_{j(k)}}\}$ converging pointwise to $-\lambda F$. Letting $n_k = m_{j(k)}$, we obtain the assertion. □

There is an alternate approach leading to the inequality $\beta_f \leq \frac{2(s-1)}{s+1}\|f\|_\infty$ in Theorem (1), which was pointed out to us by Mitchell Taibleson. It is a more elegant point of view which uses the Poisson integral representation of bounded harmonic functions on a homogeneous tree. Here is a sketch of his basic idea: For any vertex v, there is a measure $d\mu_v$ on the boundary Ω of the tree which may be thought of as the hitting measure, that is, the probability that an infinite path beginning at v ends in a subset A of Ω is the integral $\int_A d\mu_v(\omega)$. Fixing a vertex v_0, let $d\mu$ be $d\mu_{v_0}$. All the measures $d\mu_v$ are absolutely continuous with respect to one another, so for each vertex v,there exist functions $K(v,\omega)$ defined for $\omega \in \Omega$, such that $d\mu_v(\omega) = K(v,\omega)d\mu$. This is the *Poisson kernel*, and $K(v,\omega)d\mu(\omega)$ is the harmonic measure on Ω relative to the point v. The value of $K(v,\omega)$ is gotten as follows: let u be the vertex where the paths from v to ω and v_0 to ω cross. Then $K(v,\omega)d\mu(\omega) = s^{d(v_0,u)-d(v,u)}$. In particular, every bounded harmonic function f gives rise to a bounded measurable function $\widehat{f}$ on Ω such that for each vertex v

$$f(v) = \int_\Omega \widehat{f}(\omega)K(v,\omega)d\mu(\omega).$$

Then for vertices $u, v \in T$ we have

$$f(u) - f(v) = \int_\Omega \widehat{f}(\omega)[K(u,\omega) - K(v,\omega)]d\mu(\omega),$$

hence $|f(u) - f(v)| \leq \|\widehat{f}\|_\infty \|K(u,-) - K(v,-)\|_{L^1(\Omega)}$. But for u, v neighbors, a direct calculation shows that $\|K(u,-) - K(v,-)\|_{L^1(\Omega)} = \frac{2(s-1)}{s+1}\mu(\Omega)$ and $\|f\|_\infty = \mu(\Omega)\|\widehat{f}\|_\infty$. (Different authors use different scaling factors for the total measure of Ω.) Thus we get

$$\beta_f = \sup_{u\sim v} |f(u) - f(v)| \leq \frac{2(s-1)}{s+1}\|f\|_\infty.$$

3. The Bloch space

Let $\mathcal{B}$ be the space of all Bloch harmonic functions on a homogeneous tree T of degree $s+1$. We first show that, in analogy with the classical case of the Bloch analytic functions on the hyperbolic disk, $\mathcal{B}$ is a complex Banach space, called the *Bloch space*. Fix a vertex $v_0 \in T$. For $f \in \mathcal{B}$, define

$$\|f\| = |f(v_0)| + \beta_f.$$

THEOREM (2). *$\mathcal{B}$ is a complex Banach space under the norm $\| \ \|$.*

Proof. Write $\alpha_f = |f(v_0)|$. Notice that $\| \ \|$ is the sum of two semi-norms α and β. So $\| \ \|$ is automatically a semi-norm which is actually a norm on $\mathcal{B}$, because $\beta_f = 0$ implies that f is a constant and $|f(v_0)| = 0$ means that this constant is zero. Thus to prove the theorem we need only show the completeness property.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}$. This implies that $\{f_n\}$ is a Cauchy sequence with respect to α (which means that $\{f_n(v_0)\}$ is a Cauchy sequence in $\mathbb{C}$) and is also a Cauchy sequence with respect to β. We must prove that the sequence $\{f_n\}$ converges in norm to a function $f \in \mathcal{B}$. First we are going to show that the sequence is pointwise convergent. We use induction on the distance $d(v, v_0)$ to show that $\{f_n(v)\}$ is a convergent sequence for any vertex $v \in T$. To begin the induction, we observe that since the sequence $\{f_n(v_0)\}$ is Cauchy, it converges to some complex number which we denote as $f(v_0)$.

Fix a non-negative integer k and assume that $\{f_n(w)\}$ is convergent for all $w \in T$ with $d(w, v_0) \leq k$. Let v be a vertex with $d(v, v_0) = k + 1$ and w the neighbor of v closer to v_0. By the inductive hypothesis, $\{f_n(w)\}$ is convergent. Then

$$\begin{aligned}|f_n(v) - f_m(v)| \leq |(f_n(v) - f_m(v)) - (f_n(w) - f_m(w))|\\ +|f_n(w) - f_m(w)| \leq \beta_{f_n - f_m} + |f_n(w) - f_m(w)|.\end{aligned}$$

The first term of this last sum converges to zero because $\{f_n\}$ is a Cauchy sequence with respect to β, and the second approaches zero by the inductive hypothesis. Thus the sequence $\{f_n(v)\}$ is Cauchy in $\mathbb{C}$ and hence has a limit, $f(v)$.

We now show that the limit function f is in $\mathcal{B}$. To see that it is Bloch, observe that if v and w are neighbors and $n \in \mathbb{N}$, then

$$(6) \qquad |f(v) - f(w)| \leq |f(v) - f_n(v)| + |f_n(v) - f_n(w)| + |f_n(w) - f(w)|.$$

The first and third terms go to zero and the second term is bounded above by β_{f_n}, which is Cauchy. We now show that the sequence $\{\beta_{f_n}\}$ is bounded. Since $\{f_n\}$ is Cauchy with respect to β, there exists $N \in \mathbb{N}$ such that $\beta_{f_m - f_N} < 1$, for all $m \geq N$. In particular, $|(f_m(u) - f_N(u)) - (f_m(v) - f_N(v))| \leq 1$, for any pair of neighboring vertices u and v, so that

$$|f_m(u) - f_m(v)| \leq 1 + |f_N(u) - f_N(v)| \leq 1 + \beta_{f_N}.$$

Since u, v were arbitrary neighbors, it follows that $\beta_{f_m} \leq 1 + \beta_{f_N}$. Thus the sequence $\{\beta_{f_n}\}$ is bounded.

From (6) we then obtain that $|f(v) - f(w)| \leq \liminf_{n\to\infty} \beta_{f_n}$. Hence f is Bloch and $\beta_f \leq \liminf_{n\to\infty} \beta_{f_n}$.

Next we show that $\lim_{n\to\infty} \|f_n - f\| = 0$. We recall that $\|f_n - f\| = |f_n(v_0) - f(v_0)| + \beta_{f_n - f}$. Since $f_n(v_0) \to f(v_0)$, we only need to prove that

$\beta_{f_n - f}$ approaches 0. Arguing by contradiction, assume that there exist $\epsilon > 0$ and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\beta_{f_{n_k}-f} > \epsilon$ for $k \in \mathbb{N}$.

For notational convenience we pass to the subsequence and assume that $\beta_{f_n - f} > \epsilon$ for $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we may choose $v_n \in T$ and a neighbor u_n of v_n such that

$$|(f_n(v_n) - f(v_n)) - (f_n(u_n) - f(u_n))| \geq \epsilon.$$

Since $\{f_n\}$ is Cauchy with respect to β, there exists $N \in \mathbb{N}$ such that

$$\left|\left(f_n(v) - f_{n+p}(v)\right) - \left(f_n(u) - f_{n+p}(u)\right)\right| \leq \beta_{f_n - f_{n+p}} < \epsilon/2,$$

for all $n \geq N$, $p \in \mathbb{N}$, $v, u \in T$ with $v \sim u$.

From the pointwise convergence of f_n to f we have that

$$\left|\left(f_{N+p}(v_N) - f(v_N)\right) - \left(f_{N+p}(u_N) - f(u_N)\right)\right| < \epsilon/2,$$

for all p sufficiently large. Therefore

$$\begin{aligned}
&|(f_N(v_N) - f(v_N)) - (f_N(u_N) - f(u_N))| \leq \\
&\qquad \left|\left(f_N(v_N) - f_{N+p}(v_N)\right) - \left(f_N(u_N) - f_{N+p}(u_N)\right)\right| + \\
&\qquad\qquad \left|\left(f_{N+p}(v_N) - f(v_N)\right) - \left(f_{N+p}(u_N) - f(u_N)\right)\right| < \epsilon,
\end{aligned}$$

contradicting our choice of v_N and u_N. Thus $\beta_{f_n - f} \to 0$.

Furthermore observe that $|\beta_{f_n} - \beta_f| \leq \beta_{f_n - f}$, hence $\beta_f = \lim_{n\to\infty} \beta_{f_n}$.

Finally for any vertex v, note that

$$f(v) = \lim_{n\to\infty} f_n(v) = \lim_{n\to\infty} \frac{1}{s+1} \sum_{w\sim v} f_n(w) = \frac{1}{s+1} \sum_{w\sim v} f(w),$$

since this is a finite sum. Thus the limit function is harmonic. The completeness is thus established. □

The following result is a precise analogue of the classical case of Bloch analytic functions on the open unit disk which shows how they can be characterized in terms of normal families. We recall that a family of functions between metric spaces is called *normal* if every sequence in the family has a subsequence which converges uniformly on compact subsets — which in the tree case means that it converges pointwise — to a function not necessarily in the family. Let us denote by $\mathcal{A}$ the group of all automorphisms of the tree.

THEOREM (3). *Let $v_0 \in T$ be a fixed vertex. A function $f : T \to \mathbb{C}$ is Bloch if and only if the family $\{f \circ S - f(S(v_0)) : S \in \mathcal{A}\}$ is normal.*

Proof. Assume first that f is Bloch, and let $\{S_n\}$ be a sequence of automorphisms of T. Let $g_n = f \circ S_n - f(S_n(v_0))$. We shall define $g : T \to \mathbb{C}$ at a vertex v by induction on $d(v, v_0)$ in such a way that g is the limit of a subsequence of

$\{g_n\}$. Since $g_n(v_0) = 0$ for all $n \in \mathbb{N}$, we set $g(v_0) = 0$. Given a non-negative integer m, assume that we have constructed a subsequence $\{g_{n_k}\}$ such that for all $v \in T$ with $d(v, v_0) \leq m$, the sequence $\{g_{n_k}(v)\}$ converges to the limit $g(v)$. Let w be a vertex with $d(w, v_0) = m+1$. Let v be the neighbor of w such that $d(v, v_0) = m$. Notice that $S_n(v) \sim S_n(w)$ for all $n \in \mathbb{N}$. Since f is Bloch, we have $|g_n(v) - g_n(w)| = |f(S_n(v)) - f(S_n(w))| \leq \beta_f$. From this inequality and the convergence of $\{g_{n_k}(v)\}$ it follows that the sequence $\{g_{n_k}(w)\}$ is bounded, hence some subsequence of it is convergent. Set the corresponding limit equal to $g(w)$. Using a standard diagonalization procedure, we find a subsequence of the original sequence which converges pointwise to g. Thus the family $\{f \circ S - f(S(v_0)) : S \in \mathcal{A}\}$ is normal.

Conversely assume that f is not Bloch. Then there exists a sequence of pairs of neighboring vertices $\{(u_n, v_n)\}$ such that $\lim_{n\to\infty} |f(u_n) - f(v_n)| = \infty$. Let u be any neighbor of v_0. For each n let S_n be any automorphism mapping v_0 to v_n and u to u_n. Let $g_n = f \circ S_n - f(S_n(v_0))$. Since $|g_n(u)| = |f(u_n) - f(v_n)|$, which diverges to infinity, no subsequence of $\{g_n(u)\}$ can converge. Thus the family $\{f \circ S - f(S(v_0)) : S \in \mathcal{A}\}$ is not normal. $\square$

In analogy with the classical case of the unit disk, we now define the little Bloch space on a homogeneous tree.

Definition. Let $\mathcal{B}_0$ be the subspace of $\mathcal{B}$ consisting of those functions f such that the set $M(f, \epsilon) = \{v \in T : |f(v) - f(u)| \geq \epsilon \text{ for some } u \sim v\}$ is finite for every positive number ϵ. The set $\mathcal{B}_0$ is called the *little Bloch space*.

First notice that $f \in \mathcal{B}_0$ if and only if, for any sequence of pairs of neighboring vertices $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ approaching the boundary Ω, we have $\lim_{n\to\infty} |f(v_n) - f(u_n)| = 0$. Observe that this limit condition is the precise analogue of that used to define the little Bloch space of the unit disk.

Next we see that $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$. Given u and v neighboring vertices and f and g Bloch functions, note that $|f(u) - f(v)| \leq \beta_{f-g} + |g(u) - g(v)|$, yielding the inclusion $M(f, \epsilon) \subset M(g, \epsilon - \beta_{f-g})$ for any $\epsilon > 0$. Now if f is in the closure of $\mathcal{B}_0$ and $\epsilon > 0$, there exists $g \in \mathcal{B}_0$ such that $\beta_{f-g} < \epsilon$. Then the set $M(g, \epsilon - \beta_{f-g})$ is finite, hence $M(f, \epsilon)$ is finite. Thus $f \in \mathcal{B}_0$.

We will prove that, just as in the classical case of the Rubel-Shields Theorem for the unit disk (cf. [11]), the double dual of $\mathcal{B}_0$ is isomorphic to $\mathcal{B}$.

We now give an example of a Bloch function which is not in the little Bloch space that will be used in the proof of Theorem (5).

Example (2). Let $p = [\ldots, v_{-1}, v_0, v_1, \ldots]$ be a doubly infinite path. For all $v \in T$ define $f(v) = k \in \mathbb{Z}$, where v_k is the closest vertex of p to v. Observe that for $v \notin p$ the value of f at v and all its neighbors is the same, so f satisfies the mean value property at v. If $k \in \mathbb{Z}$, the value of f at v_k and all its neighbors except v_{k-1} and v_{k+1} is k. At v_{k-1}, v_{k+1} it is $k-1$, $k+1$, respectively, so the mean value is equal to k. Thus f is harmonic. Clearly $\beta_f = 1$, but $M(f, 1) = \{v_k\}_{-\infty}^{\infty}$. Hence $f \in \mathcal{B} \setminus \mathcal{B}_0$.

Notice that the function in Example (2) is unbounded. In fact there exist bounded functions not in the little Bloch space. We give an example of such a function below. The construction is more complicated than that of Example (2) but is in the same spirit.

Example (3). Let $p = [\ldots, v_{-1}, v_0, v_1, \ldots]$ be a doubly infinite path. Given $v \in T$, assume that v_k is the closest vertex of p to v and let $n = d(v_k, v)$. Define $f : T \to \mathbb{C}$ by

$$f(v) = \begin{cases} -\dfrac{2s}{(s-1)^2}\left(1 - s^{-n}\right) & \text{if k is even} \\ 1 + \dfrac{2s}{(s-1)^2}\left(1 - s^{-n}\right) & \text{if k is odd} \end{cases}.$$

Observe that for $v \notin p$, the function f satisfies the mean value property at v because locally it has the form $A + Bs^{-n}$ where n is the distance from some fixed vertex. We need only check harmonicity at each v_k. If k is even so that $f(v_k) = 0$, then $f(v_{k-1}) = f(v_{k+1}) = 1$, whereas the value of f at the other neighbors of v_k is $-\dfrac{2}{s-1}$. Thus the mean value of f at the neighbors of v_k is $0 = f(v_k)$. If k is odd so that $f(v_k) = 1$, then $f(v_{k-1}) = f(v_{k+1}) = 0$, whereas the value of f at the other neighbors of v_k is $\dfrac{s+1}{s-1}$. Thus the mean value of f at the neighbors of v_k is $1 = f(v_k)$. So f is harmonic. Now for all $v \in T$ we have $|f(v)| < 1 + \dfrac{2s}{(s-1)^2}$, so f is bounded, hence Bloch. On the other hand, for all $k \in \mathbb{Z}$ we note that $|f(v_k) - f(v_{k+1})| = 1$, so f is not in the little Bloch space.

For the remainder of this section, we fix an end $\omega_0 \in \Omega$. This allows us to define an ordering between neighboring vertices as follows: if u and v are neighbors, then we write $v \prec u$, $u \succ v$, or $v = u^-$, and say that v is the *predecessor* of u, or u is the *successor* of v, in the case that the path representing ω_0 and beginning with u contains v. Clearly, if $u \sim v$, then either $u \prec v$ or $u \succ v$. In a homogeneous tree of degree $s+1$, every vertex has exactly one predecessor and s successors, which constitute its $s+1$ neighbors.

Let $\mathbb{C}[T]$ be the space of all complex-valued functions on T. We now define two operators on $\mathbb{C}[T]$. If $f \in \mathbb{C}[T]$ define its *derivative* $f' : T \to \mathbb{C}$ by $f'(v) = f(v) - f(v^-)$ and $\varphi f : T \to \mathbb{C}$ by $\varphi f(v) = f(v) - \sum_{u \succ v} f(u)$. Notice that f is harmonic if and only if $\varphi f'$ is identically 0, and a harmonic function f is Bloch if and only f' is bounded.

The first observation follows since

$$\begin{aligned}\varphi f'(v) &= f'(v) - \sum_{u \succ v} f'(u) = (f(v) - f(v^-)) - \sum_{u \succ v}(f(u) - f(v)) \\ &= (s+1)f(v) - \sum_{u \sim v} f(u).\end{aligned}$$

The second observation is an immediate consequence of the definition of Bloch function, noting that $\beta_f = \|f'\|_\infty$.

We can also define the definite integral over a finite path of any $f \in \mathbb{C}[T]$ as follows:

(i) If $u \sim v$ define $\displaystyle\int_u^v f = \begin{cases} f(v) & \text{if } u \prec v \\ -f(u) & \text{if } u \succ v \end{cases}$.

(ii) If $[u = v_0, v_1, \ldots, v_n = v]$ is the path from u to v, then define $\displaystyle\int_u^v f = \sum_{j=1}^{n} \int_{v_{j-1}}^{v_j} f$. (Of course, $\int_v^v f = 0$.)

It follows immediately that the Fundamental Theorem of Calculus holds, and in particular, fixing a vertex $v_0 \in T$, we have

$$f(v) = \int_{v_0}^{v} f' + f(v_0), \quad \text{for all } v \in T.$$

Thus the function $\xi : \mathbb{C}[T] \to \mathbb{C}[T] \oplus \mathbb{C}$ defined by $\xi(f) = (f', f(v_0))$ is a vector space isomorphism. Furthermore, by the above remarks, if we let $\mathcal{H}(T)$ denote the set of all harmonic functions on T, the map ξ induces an isomorphism $\mathcal{H}(T) \to \ker \varphi \oplus \mathbb{C}$. In addition, let $\ell^\infty(T)$ be the set of bounded functions on T and $\mathcal{E} = \ker \varphi \bigcap \ell^\infty(T)$. Then ξ also induces an isomorphism $\mathcal{B} \to \mathcal{E} \oplus \mathbb{C}$. Since for $f \in \mathcal{B}$ the norm of f is given by $\|f\| = \|f'\|_\infty + |f(v_0)|$, if we define $\|(g, \lambda)\| = \|g\|_\infty + |\lambda|$, for $(g, \lambda) \in \ell^\infty(T) \oplus \mathbb{C}$, this latter isomorphism is an isometry of Banach spaces.

Let $\mathcal{E}_0 \subset \mathcal{E}$ be the subset consisting of all functions f which *vanish at infinity*, that is, for which the set $\{v \in T : |f(v)| \geq \epsilon\}$ is finite for all $\epsilon > 0$. Then ξ also induces an isometry $\mathcal{B}_0 \to \mathcal{E}_0 \oplus \mathbb{C}$.

We can now prove the Rubel-Shields Theorem on trees:

THEOREM (4). *The double dual $\mathcal{B}_0^{**}$ of $\mathcal{B}_0$ is $\mathcal{B}$.*

Proof. Using the isometries induced by ξ, it is sufficient to prove that $\mathcal{E}_0^{**}$ is isomorphic to $\mathcal{E}$. Let $c_0(T)$ be the subspace of $\mathbb{C}[T]$ of functions vanishing at infinity. Then $\mathcal{E}_0$ is a closed subspace of $c_0(T)$ and $\mathcal{E}$ is a closed subspace of $\ell^\infty(T)$, in fact in each case they are the intersections of the larger spaces with $\ker \varphi$. There is a natural isometry between $(c_0(T))^{**}$ and $\ell^\infty(T)$, because $(c_0(T))^*$ is isometrically equivalent to the space $\ell^1(T)$ of absolutely summable functions on T, and $(\ell^1(T))^*$ is isometrically equivalent to $\ell^\infty(T)$ (cf. [9, pp. 55-57]).

For convenience, we shall use φ_0 to represent φ considered as a map from $c_0(T)$ to itself, and φ_∞ to represent φ considered as a map from $\ell^\infty(T)$ to itself. Thus $\mathcal{E}_0 = \ker \varphi_0$ and $\mathcal{E} = \ker \varphi_\infty$. We shall prove the result in two steps. First we shall use the following lemma to show that $\mathcal{E}_0^{**} = \ker(\varphi_0^{**})$, and then we shall verify directly that $\varphi_0^{**} = \varphi_\infty$.

LEMMA (6). *If $f : A \to B$ is a continuous linear operator of Banach spaces, then* $(\ker f)^{**} = \ker(f^{**})$.

Proof. An easy application of the Closed Graph Theorem says that f factors as

$$A \xrightarrow{p} A/\ker f \xrightarrow{\psi} \mathrm{Im} f \xrightarrow{i} B,$$

where p is the projection, ψ is an isomorphism of Banach spaces, and i is the inclusion map. Applying the transpose operator, we obtain that f^* factors as

$$B^* \xrightarrow{i^*} (\mathrm{Im} f)^* \xrightarrow{\psi^*} (A/\ker f)^* \xrightarrow{p^*} A^*.$$

Note that p^* is an injection, and by the Hahn-Banach Theorem, i^* is a surjection. Thus the image of f^* is exactly $(A/\ker f)^*$, which is equal to $(\ker f)^{\perp}$ by Theorem (4.9(b)) of [12]. So $(\ker f)^{\perp\perp} = (\mathrm{Im}(f^*))^{\perp} = \ker(f^{**})$, where the latter equality follows from [12], Theorem (4.12).

Finally we show that $(\ker f)^{\perp\perp} = (\ker f)^{**}$. Now $\alpha \in (\ker f)^{\perp\perp}$ if and only if $\alpha \in A^{**}$ and $\alpha((\ker f)^{\perp}) = 0$. But this is equivalent to saying that there exists a linear functional $\beta : \ker f^* = A^*/(\ker f)^{\perp} \to \mathbb{C}$ such that $i^{**}(\beta) = \beta \circ i^* = \alpha$, i.e. $\alpha \in i^{**}(\ker f)^{**}$. Since i^{**} is just the inclusion of $(\ker f)^{**}$ in A^{**}, this proves $(\ker f)^{\perp\perp} = (\ker f)^{**}$. □

We are now ready to show that φ_0^{**} is exactly φ_∞.

For any vertex v, let $\delta_v : T \to \mathbb{C}$ be the evaluation at v. Given any $f \in c_0(T)$, we have $\varphi_0 f(v) = f(v) - \sum_{u \succ v} f(u)$. Now let $g \in c_0(T)^* = \ell^1(T)$. By this identification, $(\varphi_0^* g)(v)$ is the same as $\varphi_0^* g$ applied to the evaluation δ_v. But $\varphi_0^* g(\delta_v) = g(\varphi_0 \delta_v) = \sum_w g(w)(\delta_v(w) - \sum_{u \succ w} \delta_v(u)) = g(v) - g(v^-)$. Now let $h \in \ell^\infty(T)$. Then

$$\begin{aligned}(\varphi_0^{**} h) g = h(\varphi_0^* g) &= \sum_w h(w)(g(w) - g(w^-)) = \sum_w h(w) g(w) - \sum_w h(w) g(w^-)\\ &= \sum_w h(w) g(w) - \sum_w \sum_{u \succ w} h(u) g(w) = \sum_w [h(w) - \sum_{u \succ w} h(u)] g(w)\\ &= \sum_w \varphi_\infty h(w) g(w) = (\varphi_\infty h) g.\end{aligned}$$

This shows that $\varphi_0^{**} h = \varphi_\infty h$, as promised. □

In the classical case of the open unit disk, every analytic function f on Δ can be expressed as the uniform limit on compact subsets of bounded functions in $\mathcal{B}_0(\Delta)$. The construction is quite simple: for $0 \le r < 1$, define $f_r(z) = f(rz)$. Since each f_r can be extended to a neighborhood of the disk, it is analytic at the boundary and is thus in $\mathcal{B}_0(\Delta) \cap L^\infty(\Delta)$. This construction is very useful for proving results in complex analysis. We observe here that a similar construction exists for harmonic functions on a homogeneous tree.

PROPOSITION (1). *Fix $v_0 \in T$. There is a sequence of transformations $\chi_n : \mathcal{H}(T) \to \mathcal{B}_0 \cap \ell^\infty(T)$ such that for all $v \in T$ and $f \in \mathcal{H}(T)$,*

$$\chi_n f(v) = f(v) \text{ for each } n \in \mathbb{N}, \text{ with } n \geq d(v_0, v).$$

In particular, the sequence $\{\chi_n f\}$ converges to f pointwise.

Proof. Let $n \in \mathbb{N}$. Given $f \in \mathcal{H}(T)$ and $v \in T$, define

$$\chi_n f(v) = \begin{cases} f(v) & \text{if } d(v_0, v) \leq n \\ f(v_1) + (f(v_2) - f(v_1)) \dfrac{s - s^{-p}}{s - 1} & \text{if } d(v_0, v) = n + p, \ p \geq 1 \end{cases},$$

where v_1, v_2 are the vertices on the path from v_0 to v such that $d(v_0, v_1) = n-1$ and $d(v_0, v_2) = n$.

Notice that the second formula for $\chi_n f(v)$ agrees with the first also for $p = -1$, where $v = v_1$, and for $p = 0$, where $v = v_2$.

By the construction $\chi_n f$ satisfies the mean value property at v if $d(v_0, v) \leq n - 1$. Now for $d(v_0, v) = n + p$, $p \geq 0$, we note that v has s neighbors at distance $n+p+1$ from v_0 and one neighbor at distance $n+p-1$ from v_0. Thus to complete the proof that $\chi_n f$ is harmonic we need only observe that

$$\begin{aligned} &(s+1)\left[f(v_1) + (f(v_2) - f(v_1)) \frac{s - s^{-p}}{s-1}\right] \\ &= s\left[f(v_1) + (f(v_2) - f(v_1)) \frac{s - s^{-(p+1)}}{s-1}\right] + \\ &\quad \left[f(v_1) + (f(v_2) - f(v_1)) \frac{s - s^{-(p-1)}}{s-1}\right], \end{aligned}$$

or equivalently, that $(s+1)s^{-p} = s \cdot s^{-(p+1)} + s^{-(p-1)}$, which is obvious. So $\chi_n f \in \mathcal{H}(T)$. For any neighbors u and v such that $d(v_0, v) = n + p$ and $d(v_0, u) = n + p + 1$, we see that $\chi_n f(v) - \chi_n f(u) = \dfrac{f(v_1) - f(v_2)}{s^{p+1}}$ so that $\chi_n f \in \mathcal{B}_0$.

Finally notice that for each fixed n, there are only finitely many choices for v_1 and v_2. Thus it is easy to see from the definition that $|\chi_n f(v)|$ is bounded. Therefore $\chi_n f \in \ell^\infty(T)$. □

We can construct a rich family of elements of $\mathcal{B}_0$ as follows. Fixing a vertex v_0 and a positive integer n, we let $A_n = \{v \in T | d(v, v_0) = n\}$. Let $g : A_n \to \mathbb{C}$ be any function. By solving the Dirichlet problem (cf. [2], Lemma (4.3)) we extend g to a function h harmonic on $B_n = \{v \in T | d(v, v_0) \leq n\}$. Observing that the definition of $\chi_n f$ depends only on $f | B_n$, we can extend h to $\chi_n h$.

As an application of this construction, we obtain the following result which is an analogue of Theorem (2.1) of [1]:

THEOREM (5). *For $f \in \mathcal{B}$ we have $f \in \mathcal{B}_0$ if and only if $\chi_n f \to f$ in $\mathcal{B}$. Furthermore $\mathcal{B}_0$ is a separable nowhere dense subspace of $\mathcal{B}$.*

Proof. Fix $v_0 \in T$ and let $f \in \mathcal{B}$. Since $\mathcal{B}_0$ is closed in $\mathcal{B}$, if $\chi_n f \to f$ in $\mathcal{B}$, then $f \in \mathcal{B}_0$.

Conversely, assume that $f \in \mathcal{B}_0$. Given $\epsilon > 0$ choose $N \in \mathbb{N}$ so that $v \notin M(f, \epsilon/2)$ whenever $d(v_0, v) \geq N - 1$. Let $n \in \mathbb{N}$, $n \geq N$. By definition of χ_n, the functions $\chi_n f$ and f agree at all vertices v within distance n of v_0. In particular, setting $f_n = \chi_n f - f$ we have $\|f_n\| = \beta_{f_n}$. So to calculate the Bloch norm of $\chi_n f - f$ it is sufficient to estimate the difference of the values of f_n at neighboring vertices whose distance from v_0 is at least n. Let v and u be neighboring vertices such that $d(v_0, v) = n + p$, $d(v_0, u) = n + p + 1$, with p a non-negative integer. Let v_1, v_2 be as in the proof of Proposition (1). Then

$$|\chi_n f(v) - \chi_n f(u)| = \left| \frac{f(v_1) - f(v_2)}{s^{p+1}} \right|.$$

Since v_1 and v are not in $M(f, \epsilon/2)$, we have $|f(v_1) - f(v_2)| < \epsilon/2$ and $|f(v) - f(u)| < \epsilon/2$. So

$$|f_n(v) - f_n(u)| \leq |\chi_n f(v) - \chi_n f(u)| + |f(v) - f(u)| < \frac{\epsilon}{2s^{p+1}} + \frac{\epsilon}{2} < \epsilon.$$

Thus $\|\chi_n f - f\| < \epsilon$, for $n \geq N$. Hence $\chi_n f \to f$ in $\mathcal{B}$.

For each $n \in \mathbb{N}$ the set $S_{n,\mathbb{Q}} = \{\chi_n f | f \in \mathcal{H}(T),\ f(v) \in \mathbb{Q} + i\mathbb{Q} \text{ for } v \in T\}$ is countable since $\chi_n f$ is determined by its values on the finite set $B_n = \{v \in T | d(v_0, v) \leq n\}$. Furthermore $S_{n,\mathbb{Q}}$ is dense in $\{\chi_n f | f \in \mathcal{H}(T)\}$: let $g = \chi_n f$, with $f \in \mathcal{H}(T)$. For all $v \in B_{n-1}$, let u_v be a neighbor of v further away from v_0,and let $B'_n = \{u_v | v \in B_{n-1}\}$, a subset of B_n. Define $h : B_n \to \mathbb{C}$ as follows. Given $\epsilon > 0$, let h be an arbitrary ϵ-approximation to g on the set $B_n - B'_n$ with values in $\mathbb{Q} + i\mathbb{Q}$. There is a unique definition of h on B'_n which extends $h | B_n - B'_n$ harmonically, so that $h \in S_{n,\mathbb{Q}}$. A simple combinatorial argument shows that h is an $as\epsilon$-approximation to g, where a is the $(n+1)$st Fibonacci number.

Thus $\bigcup_{n \in \mathbb{N}} S_{n,\mathbb{Q}}$ is a countable dense subset of $\{\chi_n f | f \in \mathcal{H}(T),\ n \in \mathbb{N}\}$, which is dense in $\mathcal{B}_0$ by the first part. Hence $\mathcal{B}_0$ is separable.

Next let f be the function in Example (2) and let $g \in \mathcal{B}_0$ be arbitrary. Then for any $\epsilon > 0$ we have $g_\epsilon = g + \epsilon f \notin \mathcal{B}_0$ but $\|g - g_\epsilon\| = \epsilon$. Thus $\mathcal{B}_0$ is nowhere dense in $\mathcal{B}$. This completes the proof. □

We wish to observe that $\mathcal{B}_0$ is not contained in $\ell^\infty(T)$, so the intersection of these spaces in not all of $\mathcal{B}_0$. We show this as follows:

Example (4). Fix an infinite path $[v_1, v_2, \ldots]$. If $v \in T$ is such that the closest vertex on this path to v is v_n and $d(v, v_n) = p \geq 0$, we let

$$f(v) = \begin{cases} \log n + \dfrac{s(1 - s^{-p})}{(s-1)^2} \log \dfrac{n^2}{n^2 - 1} & \text{for } n \geq 2 \\ -\dfrac{1 - s^{-p}}{s - 1} \log 2 & \text{for } n = 1 \end{cases}.$$

By the similarity with the construction of Proposition (1) it is easy to see that f is harmonic.

Now if u and v are neighbors and the nearest vertex to v on the path is v_n with $n \geq 2$ and $p = d(v, v_n)$, then

$$|f(u) - f(v)| \leq \left(\log \frac{n^2}{n^2 - 1}\right) \frac{1}{s^p(s-1)}.$$

For $n = 1$ the upper bound is $s^{-p} \log 2$. Since these upper bounds approach 0 as either n or p approaches infinity, $f \in \mathcal{B}_0$. Since $f(v_n) = \log n$, we obtain that $f \notin \ell^\infty(T)$.

4. The non-homogeneous case

Many of the results of this paper can be extended to non-homogeneous trees with a few modifications. Let T be an arbitrary tree with only the condition that each vertex has at least two neighbors. In addition, let $P : T \times T \to [0, 1]$ be a *nearest neighbor Markov operator* (or *stochastic* operator); that is, for all $v \in T$, the set $\{u \in T : P(v, u) \neq 0\}$ coincides with the set of neighbors of v, and $\sum_u P(v, u) = 1$. For example, if T is homogeneous of degree $s + 1$, then

$$P(v, u) = \begin{cases} \dfrac{1}{s+1} & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

defines a Markov operator on T.

Notice that a Markov operator P induces a convolution operator on $\mathbb{C}[T]$ as follows: If $f : T \to \mathbb{C}$ is any function, then define $P * f : T \to \mathbb{C}$ by $P * f(v) = \sum_u P(v, u) f(u)$. We say that a function f on T is *harmonic* if $P * f = f$, that is, the value of f at any vertex is the *weighted* mean of its values at the neighboring vertices. Notice that in the homogeneous case, using the above Markov operator, this definition agrees with the earlier one.

In this more general context, it is also necessary to generalize the definition of automorphism of a tree given in §1. An *automorphism* of T is a bijection $S : T \to T$ such that $P(Su, Sv) = P(u, v)$ for all $u, v \in T$. Automorphisms of T necessarily carry edges to edges.

Theorem (2) still holds in this generality. Let $\mathcal{B}'$ be the space of all functions, not necessarily harmonic, satisfying the Bloch condition. Observe that the proof of Theorem (2) first shows that $\mathcal{B}'$ is a Banach space, and then that the subspace of harmonic functions in $\mathcal{B}'$ is closed. In the general case, harmonicity is also preserved under limits, since it is again a local condition regarding a finite sum at each vertex.

In regard to Theorem (3), it is true also in the non-homogeneous case that f Bloch implies that the family $\{f \circ S - f(S(v_0)) : S \in \mathcal{A}\}$ is normal, where $\mathcal{A}$ is the group of automorphisms of T. But the converse is not true in general, because the group $\mathcal{A}$ may be too small to give any information. For example, it would not be hard to construct a tree with $\mathcal{A}$ trivial. It is easy to see, however, that the following weaker version of homogeneity is sufficient to prove the converse.

Definition. A tree T is *semi-homogeneous* if there exists a finite set of pairs of neighbors $\{(u_j, v_j)\}_{j=1}^k$ such that given any pair of neighboring vertices (u, v) there exist an integer n and an automorphism S of T such that $S(u) = u_n,\ S(v) = v_n$.

Now Theorem (3) holds if T is any semi-homogeneous tree.

In order to extend Theorem (4) to the non-homogeneous case, we need to modify some more definitions. First define a function induced by the Markov operator: For $v \in T$ let $P'(v) = P(v^-, v)$. The modifications are now gotten by taking P' into account in several definitions. In this context we define the function $\varphi : \mathbb{C}[T] \to \mathbb{C}[T]$ by $\varphi f(v) = f(v) - \sum_{u \succ v}(f(u) + f(v))$. Notice that if we apply this formula to the homogeneous case, we get the earlier value of φ divided by the degree of homogeneity. Further we let $\mathcal{E}$ be the space of functions $g \in \mathbb{C}[T]$ such that Pg is bounded and $\varphi g = 0$. A function $f \in \mathbb{C}[T]$ is harmonic if and only if $\varphi f' = 0$, and we let $\beta_f = \|P'f'\|_\infty$. This slightly changes the Bloch condition for harmonic functions to

$$\sup_v P'(v)|f'(v)| < \infty.$$

The proof of Theorem (4) proceeds in the non-homogeneous case exactly as before.

Finally, the construction in Proposition (1) can be carried out easily, but it would be very complicated to write down an explicit general formula for the sequence $\{\chi_n\}$. Theorem (5) ought to hold in this generality.

We now turn to possible generalizations of Theorem (1). First note that if T is homogeneous of degree $s + 1$, then our new definition of β_f yields the old definition divided by $s + 1$. Except for this scaling factor there are no further changes in Theorem (1) for the homogeneous case. Even in the non-homogeneous case a bounded harmonic function is always Bloch (clearly, if $f : T \to \Delta$, then $\beta_f \le 2$). The first interesting question, therefore, is how to calculate from tree data the uniform upper bound on the Bloch norms of functions with image in Δ. This will be a very difficult calculation in the most general case. Another question is: can this least upper bound be attained by some bounded harmonic function? We suspect that the answer to this question is yes, but it will be very difficult to classify these extremal functions. Finally, assuming that extremal functions exist, do there necessarily exist extremal functions which realize this maximum value inside the tree? That is, given $\beta = \sup \beta_f$ taken over all harmonic functions $f : T \to \Delta$, does there exist a harmonic function $F : T \to \Delta$, and neighboring vertices v_0, v_1 in T such that $\beta = P(v_0, v_1)|F(v_0) - F(v_1)|$? We leave these as open questions.

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References

[1] J. M. Anderson, J. Clunie and C. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12-37.

[2] C. A. Berenstein, E. Casadio Tarabusi, J. M. Cohen and M. A. Picardello, *Integral geometry on trees*, Amer. J. Math. **113** (1991), 441-470.

[3] J. A. Cima, *The basic properties of Bloch functions*, Internat. J. Math.& Math. Sci. **2** (3) (1979), 369-413.

[4] J. M. Cohen and F. Colonna, *Bounded holomorphic functions on bounded symmetric domains*, Trans. Amer. Math. Soc. to appear.

[5] ———, ———, *Embeddings of trees in the hyperbolic disk*, Complex Variables, to appear.

[6] J. M. Cohen and M. A. Picardello, *The 2-circle and 2-disk problems on trees*, Israel J. Math. **64** (1988), 73-86.

[7] F. Colonna, *The Bloch constant of bounded analytic functions*, J. London Math. Soc. (2) **36** (1987), 95-101.

[8] ———, *The Bloch constant of bounded harmonic mappings*, Indiana Math. J. (4) **38** (1989), 829-840.

[9] J. Horváth, Topological vector spaces and distributions, Vol. I, Addison-Wesley, Reading-Palo Alto-London-Don Mills (1966).

[10] Ch. Pommerenke, *On Bloch functions*, J. London Math. Soc.**2** (1970), 689-695.

[11] L. A. Rubel and A. L. Shields, *The second duals of certain spaces of analytic functions*, J. Austral. Math. Soc. **11** (1970), 276-280.

[12] W. Rudin, Functional Analysis, 2nd Ed. McGraw-Hill, New York (1991).

[13] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and p-valence*, Trans. Amer. Math. Soc. **52** (1942), 128-216.

[14] A. L. Shields and D. L. Williams, *Bounded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. **162** (1971), 287-302.