Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

FIBREWISE CONFIGURATION SPACES

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Although configuration spaces arise in other branches of mathematics, for topologists they were first considered by Fadell and Neuwirth [7] some thirty years ago. The first mention of *fibrewise* configuration spaces in the literature appears to be in a note by Duvall and Husch [4]; the idea had occurred to others such as Cohen* and Taylor [2] and Harvey [10] about the same time. Our object, in this paper, is to obtain fibrewise versions of some of the original results of Fadell and Neuwirth. This forms part of a general project the aim of which is to develop fibrewise versions of appropriate concepts and results in topology and especially homotopy theory. For the basic theory of fibrewise topology, particularly terminology and notation, we refer to [12]. For the theory of fibrewise manifolds, however, we give an outline here, sufficient for present purposes; a more thorough account will be contained in [5].

1. Introduction

It may be helpful to begin with a brief outline of the relevant definitions and results in the ordinary theory before we start to discuss the fibrewise version. Recall that the nth configuration space $F^n(X)$ of a space X is defined as the subspace of the topological nth power $\prod^n(X)$ consisting of n-tuples of distinct points of X. We can also, when X is Hausdorff, think of $F^n(X)$ as the space $\mathrm{emb}(Q_n, X)$ of embeddings in X of the discrete space $Q_n = \{1, 2, ..., n\} \subset \mathbf{R}$. Thus $F^1(X) = X$, while $F^2(X)$ is the complement of the diagonal in $X \times X$.

The main results of Fadell and Neuwirth [7] concern the case when X is a manifold†. Then $F^n(X)$ is also a manifold. Further, if X is connected then $F^n(X)$ is a fibre bundle (without structural group) over $F^r(X)$ for $r = 1, 2, ..., n-1$. Some conditions are given for the existence of sections.

The special case when $X = \mathbf{R}^k$, the real k-plane, is of particular interest. Clearly $F^2(\mathbf{R}^k)$ can be identified with $\mathbf{R}^k \times (\mathbf{R}^k - \{0\})$ through the transformation

$$(x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2).$$

We note, for later use, that this transformation is $\mathbf{Z}/2$ equivariant, where $\mathbf{Z}/2$ acts on $F^2(\mathbf{R}^k)$ by switching factors, acts on $\mathbf{R}^k$ trivially, and acts on $\mathbf{R}^k - \{0\}$ by the antipodal transformation. Recently Massey [15] has shown that $F^3(\mathbf{R}^k)$ is a fibre bundle over $F^2(\mathbf{R}^k)$ with structural group the orthogonal group $O(k-1)$. Moreover the bundle is trivial if and only if $k = 1, 2, 4$ or 8.

*We are most grateful to Professor Cohen for sending us a copy of this article.

†By a manifold, in this paper, we mean a finite-dimensional, smooth manifold without boundary, of constant dimension, which satisfies the Hausdorff and first countability conditions.

Another case of special interest is when $X = S^{k-1}$, the $(k-1)$-sphere. Then $F^2(S^{k-1})$ can be identified with the tangent bundle $T(S^{k-1}) \subset S^{k-1} \times \mathbf{R}^k$ by projecting x_2 from $-x_1$ onto the tangent plane $\{x_1\} \times \mathbf{R}^k$ at x_1.

The term configuration space is also used in the literature for the space of unordered n-tuples, rather than the space of ordered n-tuples, but here we only use it in the latter sense.

2. Fibrewise configuration spaces

By a *covering space*, in this paper, we simply mean a fibre bundle with discrete fibre. Let B be a space and let E be a covering space of B, in this sense. If $\sigma : B \to E$ is a section of E the complement $E' = E - \sigma B$ is also a covering space of B. We shall mainly be concerned with finite coverings, such as $B \times Q_n$ $(n = 1, 2, ...)$.

Given a fibrewise space X over B the fibrewise mapping-space $\mathrm{map}_B(E, X)$ is defined with the fibrewise compact open topology as in §9 of [12]. In this special case the topology can be described quite simply as follows. The fibre at the point b of B is just the space $\mathrm{map}(E_b, X_b)$ of maps of the discrete set E_b into X_b. Now b admits a neighbourhood W over which E is trivial. So for each point $e \in E_b$ there exists a section $s_e : W \to E_W$ such that $s_e(b) = e$. The restriction $\mathrm{map}_W(E_W, X_W)$ of $\mathrm{map}_B(E, X)$ to W is topologized so that each of the functions

$$s_e^* : \mathrm{map}_W(E_W, X_W) \to X_W$$

is continuous, where $s_e^*\phi = \phi s_e(\beta)$ whenever $\beta \in W$ and $\phi : E_\beta \to X_\beta$. Then $\mathrm{map}_B(E, X)$ itself is topologized so that a subset is open if and only if it meets each of the spaces $\mathrm{map}_W(E_W, X_W)$ in an open set. As shown in §9 of [12] the topology is independent of the choice of sections.

Let X be a fibrewise Hausdorff, fibrewise space over B. For each finite covering space E of B the *fibrewise configuration space* $F_B^E(X)$ is defined as the subspace $\mathrm{emb}_B(E, X)$ of the fibrewise mapping-space $\mathrm{map}_B(E, X)$ consisting of embeddings. Thus the fibre of $F_B^E(X)$ at the point b of B is just the configuration space $\mathrm{emb}(E_b, X_b)$. When $E = B \times Q_n$ we may write $F_B^n(X)$ instead of $F_B^E(X)$; most previous work on fibrewise configuration spaces is limited to this special case.

As we shall soon see, the reduction formula

$$F_B^n(X) = F_{F_B^m(X)}^{n-m}(F_B^{m+1}(X)) \quad (n \geq m \geq 1) \tag{2.1}$$

plays a useful role in the theory. In fact the fibrewise theory can be used to retrieve some of the original results in the ordinary theory.

We have $F_B^1(X) = X$, of course, and $F_B^2(X)$ is just the complement of the diagonal in $X \times_B X$. When X is a k-plane bundle over B we can identify $F_B^2(X)$ with the fibrewise product $X \times_B (X - B)$, where B is embedded as the zero-section, using in each fibre the transformation mentioned in §1. This identification is $\mathbf{Z}/2$ equivariant, the group acting on $F_B^2(X)$ by switching factors, on X by the identity, and on $X - B$ by the antipodal transformation.

In most cases of interest X is a fibre bundle over B. Note that $F_B^E(X)$ is then also a fibre bundle over B. If X is a finite covering space of B so is $F_B^E(X)$.

Suppose that X is a G-bundle over B with fibre A, where G is a topological group and A is a G-space. Then X may be identified with the mixed product $P \times_G A$, where P is the associated principal G-bundle. The configuration space $F^n(A)$ is also a G-space, under the diagonal action, and the fibrewise configuration space $F_B^n(X)$ may be identified with $P \times_G F^n(A)$.

3. Fibrewise manifolds

Manifolds over a base have been considered by Atiyah and Singer [1]; we prefer the term *fibrewise manifold*. In [1] a fibrewise manifold X over a base space B is defined to be a fibre bundle with fibre a compact manifold A and structural group the group $\mathrm{Diff}(A)$ of self-diffeomorphisms of A. Up to a point we could work with this definition but it is unsatisfactory to be restricted to compact fibres. For non-compact fibres the Atiyah-Singer definition is inappropriate although it provides a guide as to how to proceed.

Fibrewise manifolds form a category in which the morphisms are called fibrewise smooth maps. An example of a fibrewise manifold over B is the product $B \times A$ where A is a manifold in the ordinary sense. An example of a fibrewise smooth map is a fibrewise map

$$\theta : B \times A \to B \times A',$$

where A and A' are manifolds, such that the second projection

$$\pi_2\theta : B \times A \to A'$$

defines, for each point b of B, a smooth map $\tau_b : A \to A'$ for which derivatives of all orders exist and vary continuously with b. These special cases are required for the general definitions, as follows.

We say that a fibrewise space X over B is a *fibrewise manifold* if there is given a numerable open covering of B and for each member U of the covering a local trivialization

$$\phi_U : X_U \to U \times A_U,$$

where ϕ_U is fibrewise over U and A_U is a smooth manifold, such that the transition functions are fibrewise smooth. Specifically, if U, V are members of the covering then the map

$$(U \cap V) \times A_U \to (U \cap V) \times A_V$$

determined by $\phi_V \circ \phi_U^{-1}$ is fibrewise smooth. Note that the fibres are manifolds. If $\dim A_U = k$, independently of U, we say that X is k-dimensional.

For example a fibre bundle over B with structural group a Lie group G and fibre a smooth G-manifold is a fibrewise manifold over B.

Returning to the general case let $f : X \to X'$ be a fibrewise map, where X and X' are fibrewise manifolds over B. We describe f as *fibrewise smooth* if when

$$\phi_U : X_U \to U \times A_U,\ \phi'_{U'} : X'_{U'} \to U' \times A'_{U'}$$

are trivializations, as above, the map

$$(U \cap U') \times A_U \to (U \cap U') \times A'_{U'}$$

determined by $\phi'_{U'} \circ f \circ \phi_U^{-1}$ is fibrewise smooth.

The definition of fibrewise manifold we have given does not ensure that open subsets of fibrewise manifolds are also fibrewise manifolds. However the definition of fibrewise smooth map can still be used in the case of such open subsets.

Note that if X and X' are fibrewise manifolds over B then so is the fibrewise topological product $X \times_B X'$. To demonstrate this we combine the numerable covers for X and X' by taking intersections in the usual way.

Let $\pi : X \to Y$ be a fibrewise smooth map of fibrewise manifolds over B. We say that X is a *fibrewise smooth fibre bundle* over Y if there exists an open cover of Y by subsets U over which there exists a smooth (over B) trivialization

$$\pi^{-1}U \to U \times A.$$

Here A is a smooth manifold and $U \times A$ is open in the fibrewise manifold $Y \times A$ over B. We prove

PROPOSITION (3.1) *Let $\pi : X \to Y$ be a fibrewise smooth map, where X and Y are fibrewise manifolds over B. Suppose that there exists a numerable open covering of B and for each member U of the covering local trivializations*

$$\phi : X_U \to U \times A_U^X,\ \psi : Y_U \to U \times A_U^Y$$

such that the map

$$\psi \circ \pi \circ \phi^{-1} : U \times A_U^X \to U \times A_U^Y$$

is of the form id $\times\, \pi_U$, where A_U^X is a smooth fibre bundle over A_U^Y with projection π_U. Then π is a numerable fibrewise smooth fibre bundle.

To see this, use a partition of unity for A_U^Y for each U of the numerable family. This determines a partition of unity of $U \times A_U^Y$ and hence of Y_U for each U. In this way we obtain a partition of unity for Y itself from which it follows that X is a fibrewise smooth fibre bundle over Y, as asserted.

The term *fibrewise smooth vector bundle* is defined in a similar fashion. For our purposes the important example is the *fibrewise tangent bundle* T_BX of a fibrewise manifold X, constructed as follows. As a fibrewise set

$$T_BX = \coprod_{b\in B} TX_b,$$

the disjoint union of the tangent bundles to the manifolds X_b. We topologize T_BX using the smooth local trivializations of X. Specifically, if U is a member of the open covering of B, over which X is locally trivial, and

$$\phi : X_U \to U \times A$$

is the corresponding local trivialization, then T_UX_U receives the topology induced by

$$T_\phi : T_UX_U \to U \times TA.$$

Then the open sets of T_BX are the subsets which meet each of the T_UX_U in an open set. The local trivializations make T_BX a fibrewise manifold over B. Moreover the projections

$$U \times TA \to U \times A$$

and the numerable local trivializations of TA over A combine to provide a numerable family of local trivializations of T_BX, as required to show that T_BX is a numerable fibrewise smooth vector bundle over X.

For example, let X be a sphere bundle over B. As in §1 we can identify $F^2_B(X)$ with T_BX and then, using (2.1), identify $F^{n+1}_B(X)$ with $F^n_X(T_BX)$ for all $n \geq 1$.

4. Local trivializations

Let X be a fibrewise manifold over B and let E be a numerable finite covering space of B. The local trivializations $U \times A$ of X and $U \times Q_n$ of E over an open set $U \subset B$ determine a local trivialization $U \times F^n(A)$ of $F^E_B(X)$, which thus becomes a fibrewise manifold. (We combine numerable covers by taking intersections in the usual way.) When E admits a section we prove

PROPOSITION (4.1) *Let E be a numerable finite covering space of B and let X be a fibrewise manifold over B. Suppose that E has a section σ. Then $F^E_B(X)$ is a numerable fibrewise smooth bundle over X with projection σ^* induced by σ.*

Locally σ^* is of the form $id \times \pi : U \times F^n(A) \to U \times A$, where $F^n(A)$ is the smooth fibre bundle over A with projection π given by evaluation at $\sigma(b_0)$ for some point $b_0 \in U$, as in [7]. So the proposition follows at once from (3.1) above. As an illustration of fibrewise techniques we extract the main step in the argument in [7], Theorem 1, as a lemma which could well have other implications.

LEMMA (4.2) *Let B be a space and let X be a fibre bundle over B with a manifold as fibre. If X admits a section s then X (with this section) is locally trivial as a fibrewise pointed space, so that X − sB is a fibre bundle over B. Moreover, if B is a manifold, X is a smooth fibre bundle over B and s is a smooth section, then the local trivializations may be taken to be smooth.*

In other words, under these conditions a bundle of spaces which admits a section is equivalent, as a fibrewise pointed space, to a bundle of pointed spaces.

Clearly it is sufficient to deal with the case when $X = B \times A$, for some manifold A, and the section is given by a map $s : B \to A$. Fix $b_0 \in B$ and choose a coordinate chart $U \subset A$ about $a_0 = s(b_0)$. Restrict attention to the neighbourhood $s^{-1}U$ of b_0. The fibrewise pointed space $s^{-1}U \times A$ over $s^{-1}U$ is equivalent to the pull-back of the fibrewise pointed space $U \times A$ over U, using the diagonal section. It is sufficient to consider this special case.

We shall construct, for each point b of U, a diffeomorphism θ_b of A which is the identity outside U and is such that the self-map

$$(b, a) \mapsto (b, \theta_b(a))$$

of $U \times A$ is a diffeomorphism. Furthermore $\theta_b(a_0) = s(b)$ for all b in a neighbourhood V of b_0, so that the self-map transforms $V \times A$ with the axial section over V into $V \times A$ with the diagonal section. This will establish our result in the special case and hence in general.

Without real loss of generality we may replace (U, b_0) by the pair $(\mathbf{R}^k, \{0\})$, where $k = \dim A$. Choose a C^∞-bump function $\alpha : [0, \infty) \to \mathbf{R}$ such that $0 \leq \alpha(t) \leq 1$ for all t, while $\alpha(t) = 1$ for $t \leq 1$ and $\alpha(t) = 0$ for $t \geq 4$. For $b \in \mathbf{R}^k$ consider the vector field v_b given by

$$v_b(x) = \alpha(||x||^2)b.$$

This has compact support, since $v_b(x) = 0$ for $||x|| \geq 2$ and $v_b(x) = b$ for $||x|| \leq 1$. Take $\theta_b \mid U$ to be $\Theta_b(1)$, where Θ_b is the flow determined by the vector field. This gives the required family of diffeomorphisms, with V the unit disc in $\mathbf{R}^k$, thus proving the first assertion of the lemma. The remaining statements are easy observations.

The deduction that $F^n(A)$ is a smooth fibre bundle over A runs as follows. We have already noted that $F^2(A)$ is the complement of the diagonal section in $A \times A$, regarded as a fibrewise space over A using the first projection. By (4.2), therefore, $F^2(A)$ is a smooth fibre bundle over A. Now we can use the reduction formula

$$F^n(A) = F_A^{n-1}(F^2(A))$$

to see at once that $F^n(A)$ is a fibre bundle over A.

Similar arguments show more generally that $F^n(A)$ is a smooth fibre bundle over $F^r(A)$ for $1 \leq r < n$. Indeed $F^{r+1}(A)$ as a fibrewise space over $F^r(A)$ can be regarded as the complement of the union of the r canonical sections of the trivial bundle $F^r(A) \times A$ over $F^r(A)$. Removing the sections one by one we see from (4.2) that $F^{n+1}(A)$ is a fibre bundle over $F^r(A)$. Then

$$F^n(A) = F^{n-r}_{F^r(A)}(F^{r+1}(A))$$

is clearly a smooth fibre bundle over $F^r(A)$. By applying (3.1) again we can generalize (4.1) to

PROPOSITION (4.3) *Let E_1 and E_2 be numerable finite covering spaces of B, and let X be a fibrewise manifold over B. Then $F_B^{E_1 \sqcup E_2}(X)$ is a numerable fibrewise smooth fibre bundle over $F_B^{E_1}(X)$.*

5. Sections

Under certain conditions the fibrewise fibrations considered in §4 admit sections. In special cases ad hoc constructions can be used but for a general result we need to rely on the theory of fibrewise manifolds, particularly the following result.

PROPOSITION (5.1) *Let X be a fibrewise manifold over B. Then there exists a fibrewise smooth map*

$$e : T_B X \to X \times_B X$$

over X which sends the zero-section of $T_B X$ into the diagonal of $X \times_B X$ and is injective on each fibre over X.

Here we regard $X \times_B X$ as a fibrewise manifold over X using the second projection. When B is a point, so that X is just a manifold, the above result is standard and can be found in [14], for example. The map e is constructed as an appropriately scaled exponential map. An outline of the proof in the general case is given in an appendix to this paper. We use (5.1) to prove

PROPOSITION (5.2) *Let E be a numerable finite covering space of B with section σ and let X be a fibrewise manifold over B with projection p. Suppose that the fibrewise configuration space $F_X^{p^*E}(T_B X)$ admits a section over X. Then $F_B^E(X)$, regarded as a fibrewise space over X with projection σ^*, admits a section.*

For a section of $F_X^{p^*E}(T_B X)$ over X determines a fibrewise embedding of p^*E in $T_B X$ over X. Without loss of generality we may assume that the pull-back of σ corresponds to the zero-section of $T_B X$. Composition with the map e in (5.1) gives the required section of $F_B^E(X)$ over X. To be precise, for $x \in X_b$ let us denote by $i_x : E_b \to (T_B X)_x$ the embedding given by the section: thus $i_x(\sigma(b)) = 0$. Write $e_x : (T_B X)_x \to X_b$ for the restriction of e to fibres over

x. Then $e_x \circ i_x : E_b \to X_b$ is the required embedding in $F_B^E(X)_b$. When E is trivial this result simplifies to

PROPOSITION (5.3) *Let X be a fibrewise manifold over B. Suppose that the fibrewise tangent bundle $T_B X$ admits a nowhere-zero section over X. Then the fibrewise configuration space $F_B^n(X)$ admits a section over X for all $n \geq 1$.*

Indeed let $v : X \to T_B X$ be the nowhere-zero vector field. Then a section of $F_X^n(T_B X)$ over X is defined by sending each point x into the n-tuple $(u_1, ..., u_n)$, where $u_i = (i-1)v(x) \in (T_B X)_x$ $(i = 1, ..., n)$. We deduce

COROLLARY (5.4) *Let X be a fibrewise manifold over B. Suppose that the pull-back $T_B X \times_B F_B^r(X)$ of $T_B X$ to $F_B^r(X)$ admits a nowhere-zero section for some $r \geq 1$. Then $F_B^n(X)$ admits a section over $F_B^m(X)$ for $n \geq m \geq r$.*

For the hypothesis implies that $T_{F_B^m(X)} F_B^{m+1}(X)$ admits a nowhere-zero section over $F_B^m(X)$. Then (5.3), with B replaced by $F_B^m(X)$ and X replaced by $F_B^{m+1}(X)$, shows that

$$F_{F_B^m(X)}^{n-m}(F_B^{m+1}(X)) = F_B^n(X)$$

has a section over $F_B^m(X)$, as asserted.

When B is a point we retrieve from (5.4) various results of Fadell and Neuwirth. For example we see that sections exist when X is an open manifold, also when X is a compact connected manifold with Euler characteristic zero.

With general B the conclusion of (5.4) holds for affine bundles with $r = 2$, also for sphere-bundles with $r = 3$, using the identifications made at the end of §3. Of course direct geometric constructions can also be used.

For example, let X be a sphere bundle over B, and let

$$\rho : F_B^{n+1}(X) \to F_B^n(X)$$

be defined by dropping the last point from each n-tuple, where $n \geq 3$. Given three distinct points p_1, p_2, p_3 in the same fibre X_b of X consider the line segment L joining p_1 to p_2 in the associated affine bundle. If x is any point of L between p_1 and p_2 we can project x into X_b from p_3, thus obtaining a point x' of X_b. If x is chosen sufficiently close to p_1 the projection x' will be distinct from any previously given set $p_1, ..., p_n$ of distinct points of X_b. Thus a section of ρ is given by the transformation

$$(p_1, p_2, ..., p_n) \mapsto (p_1, p_2, ..., p_n, x').$$

Full details are given by Fadell [6] in the case where B reduces to a point.

6. Trivializations

Again we suppose that E is a finite covering space of B with section σ. We also suppose that X is a fibre bundle over B with section s. The projection

$$\sigma^* : F_B^E(X) \to X$$

is defined as before, and the fibrewise fibre $\rho^{-1}(sB)$ is equivalent to the fibrewise configuration space $F_B^{E'}(X')$, where $E' = E - \sigma B$ and $X' = X - sB$.

When X is a vector bundle over B and s is the zero section it is easy to see that $F_B^E(X)$ is equivalent, as a fibrewise pointed space, to the fibrewise topological product $X \times_B F_B^{E'}(X')$. In fact a trivialization

$$\xi : X \times_B F_B^{E'}(X') \to F_B^E(X)$$

is given in each fibre by the formula $\xi(x, u') = u$, where $x \in X_b$ $(b \in B)$, $u' : E'_b \to X'_b$ and $u : E_b \to X_b$ are related by

$$u(s(b)) = x - s(b), \; u(e') = x - u'(e').$$

The same conclusion may be reached in other cases. What follows is suggested by the "elementary considerations" in the work of Cook and Crabb [3] on fibrewise Hopf structures, and by the discussion in Fadell and Neuwirth [7] of "suitability".

By a *Hopf structure* on a pointed space A we mean a pointed map $m : A \times A \to A$, called the *multiplication*, which coincides with the folding map on the wedge product $A \vee A$. We describe the Hopf structure as *special* if each of the left translations is a homeomorphism. For example the classical Hopf structures on S^q, for $q = 1, 3$ or 7, are special in this sense.

Now let A be a pointed G-space, where G is a topological group. Then $A \times A$ is a pointed G-space, with the diagonal action, and G-equivariant Hopf structures can be considered, as in [3].

For example, take the classical Hopf structure on S^q, for $q = 1, 3$ or 7. In the case of S^1 this is given by complex multiplication, which is $O(1)$ equivariant, in the case of S^3 by quaternionic multiplication, which is $SO(3)$-equivariant, and in the case of S^7 by Cayley multiplication, which is G_2-equivariant.

Equivariant Hopf structures lead to fibrewise Hopf structures, as follows. Let B be a space and let X be a sectioned G-bundle over B with fibre the pointed G-space A. Suppose that A admits a G-equivariant Hopf structure m. Then m defines a fibrewise Hopf structure $m' : X \times_B X \to X$ on X. Moreover if m is special then m' is special in the sense that fibrewise left translation is a homeomorphism. This implies that the fibrewise configuration space $F_B^E(X)$ is trivial as a fibrewise fibre bundle over X with fibrewise fibre $F_B^{E'}(X')$. Specifically a trivialization

$$\xi : X \times_B F_B^{E'}(X') \to F_B^E(X)$$

is given in each fibre by the formula $\xi(x, u') = u$, where $x \in X_b$ $(b \in B)$, $u' : E'_b \to X'_b$ and $u : E_b \to X_b$ are related by

$$u(s(b)) = m'(x, s(b)),\ u(e') = m'(x, u'(e')).$$

The conclusion holds in particular, for the fibrewise suspension of every (orthogonal) 0-sphere bundle, of every orientable 2-sphere bundle, and of every 6-sphere-bundle admitting a G_2-structure.

7. Sequences of fibrewise fibrations

As Fadell and Neuwirth show in [7], the homotopy theory of ordinary configuration spaces can be investigated through a sequence of fibrations. Specifically, if A is a connected manifold the fibrations are those associated with the successive configuration spaces $F^n(A)$, $F^{n-1}(A - Q_1), \ldots,$ and $F^1(A - Q_{n-1})$, where $q_1, ..., q_{n-1}$ are distinct points of A and $Q_r = \{q_1, ..., q_r\}$. By homogeneity, the spaces obtained by this procedure are independent of the choice of points to be deleted, up to diffeomorphism.

When we turn to the fibrewise theory it is sections which are to be deleted and even in the simplest cases the result may depend on the choice of section. For example take $B = S^n$ and $X = S^n \times S^n$, regarded as a fibrewise space using the first projection. The complement of the diagonal section is the tangent bundle to S^n, while the complement of the second (axial) insertion is the trivial bundle. Except when S^n is parallelizable, the complements of these sections are not equivalent in the sense of fibrewise homeomorphism. This can be seen as follows. If the tangent bundle is fibrewise homeomorphic to the trivial bundle then its fibrewise Alexandroff compactification is trivial. From such a trivialization we readily obtain a Hopf structure on S^n; hence $n = 1, 3$ or 7.

So let X be a fibrewise manifold over B, with connected fibres. Let $s_1, ...,$ s_{n-1} be mutually non-intersecting sections of X, and write $Q_rB = s_1B \cup \cdots \cup s_rB$, for $r = 1, ..., n-1$. To investigate the fibrewise homotopy of $F^n_B(X)$ we can proceed by inductive arguments through the sequence of fibrewise fibrations associated with the successive fibrewise configuration spaces $F^n_B(X)$, $F^{n-1}_B(X - Q_1B), ..., F^1_B(X - Q_{n-1}B)$. As we have seen in §5, these fibrations may admit sections under certain conditions.

Alternatively we can proceed through the sequence of fibrewise fibrations

$$F^n_B(X) \to F^{n-1}_B(X) \to \ldots \to F^1_B(X) = X$$

where the successive fibres are $X - Q_1B, X - Q_2B, ..., X - Q_{n-1}B$.

For example let X be a euclidean bundle of rank k, with associated sphere-bundle $S(X)$. We suppose that $S(X)$ admits a section s from which we construct the family of mutually non-intersecting sections $s, 2s, 3s, ..., ns$ of X. Then X is fibrewise contractible, $X - Q_1B$ has the same fibrewise homotopy type as $S(X)$, $X - Q_2B$ has the same fibrewise homotopy type as $S(X) \vee_B S(X)$, and so on. In such cases, therefore, calculation of the fibrewise homotopy groups of fibrewise configuration spaces is an accessible problem.

8. Fibrewise embeddings

For any (Hausdorff) space A we can think of the configuration space $F^n(A)$ as the space $\mathrm{emb}(Q_n, A)$ of embeddings $u : Q_n \to A$, where

$$Q_n = \{1, ..., n\} \subset \mathbf{R}$$

is discrete. Such an embedding u determines a $\mathbf{Z}/2$-map

$$F^2(u) : F^2(Q_n) \to F^2(A),$$

where $\mathbf{Z}/2$ acts by switching factors. Thus a map

$$h : \mathrm{emb}(Q_n, A) \to \mathrm{map}^{\mathbf{Z}/2}(F^2(Q_n), F^2(A))$$

is defined. Taking $A = \mathbf{R}^k$ we prove

PROPOSITION (8.1) *For $n \geq 1$, the map*

$$h : \mathrm{emb}(Q_n, \mathbf{R}^k) \to \mathrm{map}^{\mathbf{Z}/2}(F^2(Q_n), F^2(\mathbf{R}^k))$$

is $(2k-3)$-connected.

If we regard Q_n as a 0-dimensional manifold this result may be regarded as a special case of the embedding theorem of Haefliger [8], but it is easily proved by induction on n, as follows.

Observe that the codomain of h corresponds precisely to the space of maps of T^n into $F^2(\mathbf{R}^k)$, where $T^n \subset F^2(Q_n)$ is the set of pairs of integers (i,j) such that $1 \leq i < j \leq n$. In fact a $\mathbf{Z}/2$-map of $F^2(Q_n)$ determines, by restriction, a map of T^n, and conversely. When $n = 2$, in particular, the codomain corresponds to $F^2(\mathbf{R}^k)$, since T^n is just the pair (1,2), and h is a homeomorphism.

Now let $n > 2$ and suppose the result is true with $n-1$ in place of n. Consider the commutative diagram shown below.

$$\begin{array}{ccc} \mathrm{emb}(Q_n, \mathbf{R}^k) & \xrightarrow{h} & \mathrm{map}^{\mathbf{Z}/2}(F^2(Q_n), F^2(\mathbf{R}^k)) \\ \downarrow & & \downarrow \\ \mathrm{emb}(Q_{n-1}, \mathbf{R}^k) & \xrightarrow[h]{} & \mathrm{map}^{\mathbf{Z}/2}(F^2(Q_{n-1}), F^2(\mathbf{R}^k)) \end{array}$$

Here the left-hand vertical is the fibration of Fadell and Neuwirth, i.e. a special case of our (4.3), while the right-hand vertical is an example of the equivariant form of the Borsuk fibration, easily proved in this case. Consider the fibre on the left over some $(n-1)$-tuple $e_1, ..., e_{n-1}$ of distinct points of $\mathbf{R}^k$ and the corresponding fibre on the right. The former is equivalent, up to homotopy type, to the wedge product of $n-1$ copies of S^{k-1} while the latter is equivalent to the topological product. Moreover h is similarly equivalent to the inclusion of the wedge product in the topological product, and so induces an isomorphism of homotopy groups up to dimension $2k-3$. Since the lower h, in the diagram, is $(2k-3)$-connected by the inductive hypothesis, so therefore is the upper h by the five lemma. Hence, by induction, we obtain (8.1).

Returning to the fibrewise situation let E be a fibrewise space over B. By using the map h in each fibre we construct a fibrewise map

$$h_B : \mathrm{emb}_B(E,X) \to \mathrm{map}_B^{\mathbf{Z}/2}(F_B^2(E), F_B^2(X)).$$

Sections of the domain correspond precisely to fibrewise embeddings of E in X while sections of the codomain correspond precisely to fibrewise $\mathbf{Z}/2$-maps of $F_B^2(E)$ into $F_B^2(X)$. Moreover if $u : E \to X$ is the fibrewise embedding corresponding to a section s of the domain then $F_B^2(u) : F_B^2(E) \to F_B^2(X)$ is the fibrewise $\mathbf{Z}/2$-map corresponding to the section $h_B \circ s$ of the codomain.

In particular when X is an affine bundle we can apply (3.2) of [11] and then using (8.1) obtain

PROPOSITION (8.2) *Let E be an n-fold covering of a finite complex B, where $n > 1$, and let X be an affine bundle of rank k over B. Then the correspondence given by F_B^2 between classes of fibrewise embeddings of E in X and fibrewise $\mathbf{Z}/2$-maps of $F_B^2(E)$ in $F_B^2(X)$ is surjective when* $\dim B < 2k-3$, *injective when* $\dim B < 2k-4$.

Here, of course, we classify fibrewise embeddings by fibrewise isotopy and fibrewise $\mathbf{Z}/2$-maps by fibrewise $\mathbf{Z}/2$-homotopy.

Note that $F_B^2(X)$ has the same fibrewise $\mathbf{Z}/2$-homotopy type as the sphere-bundle $S(V)$, where V is the vector bundle of translations associated with X and $\mathbf{Z}/2$ acts antipodally on each fibre. Thus (8.2) shows that when $\dim B < 2k-3$ there exists a fibrewise embedding of E in X if there exists a fibrewise $\mathbf{Z}/2$-map of $F_B^2(E)$ into $S(V)$.

There is a reformulation of the condition in the theorem which may be of interest. Write D for the orbit space of $F_B^2(E)$ with respect to the action of $\mathbf{Z}/2$ and write L for the line bundle over D associated with this double cover. Then fibrewise $\mathbf{Z}/2$-maps of $F_B^2(E)$ into $S(V)$ correspond precisely to sections of $S(L \otimes V)$ over D. (In particular if E is a double cover of B then D reduces to B and it is easy to see directly, without any restriction on $\dim B$, that there exists a fibrewise embedding of E in X if $S(L \otimes V)$ admits a section.)

As an application we deduce the result of Hansen [9] that if $\dim B < k$ there exists a fibrewise embedding of E in X. For dimensional reasons there is a section of $S(L \otimes V)$ over D and hence by (8.2), if $k > 2$, a section of $F_B^2(X)$.

The special case $k = 2$ is elementary; indeed $F_B^2(X)$ then has path-connected fibres and so admits a section if $\dim B \leq 1$.

Fibrewise embeddings of covering spaces have been studied by Duvall, Hansen, Husch, Møller and Petersen; we cite only [4] and [9] in our references since the overlap between our results and those in the previous literature seems rather minor.

9. Appendix

Let X be a fibrewise manifold over B. By a *fibrewise smooth metric* on X we mean, roughly, a family of smooth Riemannian metrics on the fibres X_b depending continuously on b (or, formally, a fibrewise smooth section of $T_B^*X \otimes T_B^*X$ over X defining a metric on each fibre). Fibrewise smooth metrics can be constructed by the following procedure. Let $\{U\}$ be the numerable open covering of B and for each member U let

$$\phi_U : X_U \to U \times A_U$$

be the corresponding local trivialization defining the fibrewise smooth structure of X. The smooth manifold A_U admits a smooth Riemannian metric g_U, say, and this determines a fibrewise smooth metric $\phi_U^* g_U$ on X_U. Choose a partition of unity $\{\alpha_U\}$ on B subordinated to $\{U\}$. By composing with the projection we obtain a partition of unity $\{\beta_U\}$ on X subordinated to $\{X_U\}$. Then

$$g = \sum_U \beta_U \phi_U^* g_U$$

is a fibrewise smooth metric on X.

PROPOSITION (9.1) *Let X be a fibrewise manifold over B with fibrewise smooth metric. Then there exists a fibrewise smooth map* $\delta : X \to (0, \infty)$ *such that the exponential map* exp *is defined and injective on the open disc*

$$\{\|v\|_g < \delta(x)\}$$

in the tangent space $T_x X_b$ for each $x \in X_b \subset X$.

Here $\| \, \|_g$ denotes the norm on $T_x X_b$ defined by the fibrewise metric g. We regard $X \times_B X$ as a fibrewise manifold over X using the second projection. By scaling we obtain (5.1).

To prove (9.1) it is sufficient to find such a map δ_U on each $U \times A_U$. For then we can take

$$\delta = \Sigma \beta_U \phi_U^* \delta_U,$$

so that $\delta(x) \leq \max \delta_U(\phi_U(x))$ for all x. Without real loss of generality, therefore, we can assume X is of the form $B \times A$, for some smooth manifold A. Using a partition of unity on A, subordinate to a local covering by coordinate charts, we can reduce this to the case where A is an open disc in $\mathbf{R}^k$ in a similar manner. To be precise: we may assume g given on $B \times \mathbf{R}^k$ and seek $\delta' : B \to (0, \infty)$ so that the exponential map is defined and injective on the open disc

$$\{||v||_{g(b,x)} < \delta'(b)\}$$

for $(b, x) \in B \times D^k$, our partition of unity having support in D^k, the closure of the open disc.

It is convenient to describe the situation in terms of the euclidean norm $|| \ ||$ on $\mathbf{R}^k$ rather than the norm on the tangent space given by g. Set

$$\lambda(b) = \min\{||v||_{g(b,x)} : x \in D^k,\ v \in \mathbf{R}^k,\ ||v|| = 1\}.$$

Since g is continuous and D^k is compact the function $\lambda : B \to (0, \infty)$ is continuous. We assert that there exists a map $\delta'' : B \to (0, \infty)$ such that exp is defined and injective on the open disc

$$\{||v|| < \delta''(v)\}$$

for $(b, x) \in B \times D^k$. Then we take $\delta'(b) = \lambda(b).\delta''(b)$ and the rest follows.

We now have a purely local problem. Let Γ denote the connection form of a metric g on $\mathbf{R}^k$. For each $y \in \mathbf{R}^k$, Γ_y is a bilinear map $\mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$. The equation of a geodesic γ is

$$\ddot{\gamma}(t) + \Gamma_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 0,$$

$$\gamma(0) = x,\ \dot{\gamma}(0) = v,$$

and then $\gamma(1) = \exp_x(v)$. We claim that a constant $\delta'' > 0$ can be manufactured (continuously) from the coefficients of Γ and $D\Gamma$ on the closed disc $\{y \in \mathbf{R}^k : ||y|| \leq 2\}$ such that

(i) for $x \in D^k$, $v \in \mathbf{R}^k$, $||v|| < \delta''$, $\exp_x(v)$ is defined and $||\exp_x(v)|| < 2$;

(ii) for $x \in D^k$, $v, w \in \mathbf{R}^k$, $||v||, ||w|| < \delta''$, we have

$$||\exp_x(v) - \exp_x(w) - (v - w)|| \leq \frac{1}{2}||v - w||,$$

and so $\exp_x(v) = \exp_x(w)$ if and only if $v = w$.

For example, we can take

$$\delta'' = \frac{1}{16}\left\{1 + \sup_{||y|| \leq 2} ||\Gamma_y|| + \sup_{||y|| \leq 2} ||D\Gamma_y||\right\}^{-1}.$$

Since δ'' is given by sup norms of coefficients of Γ and $D\Gamma$ on a compact disc we obtain the map δ'' as asserted. This completes the proof of (9.1) and hence (5.1).

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