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NOT THE ADEM RELATIONS

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0. There are well known connections between the Chern character or Adams operators in complex K-theory and the Cyclic Reduced Powers in cohomology. In particular under suitable hypotheses, the commuting of the Adams operators leads to Adem relations. In this note we derive these Adem relations explicitly for $p = 2$.

1. Let $\mathbb{F}_2\{s_0, s_1, ..., s_i, ...\}$ be the free, graded, associative algebra over $\mathbb{F}_2$ on generators s_i of degree i where s_0 is the unit. We define q_n by $q_0 = s_0$ and, for $n > 0$, $\sum s_i q_{n-i} = 0$ summing from 0 to n. We write $p_{n,N}$ for $\sum \binom{n+i}{i} s_{n+i} q_{N-n-i}$ for integers $n > 0$ and $N \geq n + 1 + \nu_2(n!)$ where the summation is from $i = 0$ to $i = N - n$. Let I be the two sided ideal generated by all such $p_{n,N}$ and $A(2) = \mathbb{F}_2\{s_0, s_1, ..., s_i, ...\}/I$.

There exists a surjective graded ring homomorphism $h : A(2) \rightarrow \mathcal{A}(2)$, with image the mod 2 Steenrod algebra, where $h(s_i) = \mathrm{Sq}^i$. It is not an isomorphism; for example, the lowest degree in which h has a non-trivial kernel is 5 and it is generated by $s_4 s_1 + s_1 s_4 + s_2 s_1 s_2$. However $A(2)$ shares some properties in common with $\mathcal{A}(2)$. As an algebra, $A(2)$ is generated by s_0 and $s_{2^i}, i \geq 0$. A derivation is defined by setting $d(s_i) = s_{i-1}$. Also $A(2)$ has a Hopf algebra structure for which h is a Hopf algebra homomorphism. These matters will be discussed in section 2. We illustrate the use of $A(2)$ by deriving some periodic relations in the Steenrod algebra.

THEOREM (1.1). *Let n and N be positive integers with* $n < 2^{t+1}$ *and* $N \geq 2^{t+2} - t - 2$. *Then*

$$\mathrm{Sq}^n \chi(\mathrm{Sq}^{N-n}) + \mathrm{Sq}^{n+2^{t+1}} \chi(\mathrm{Sq}^{N-n-2^{t+1}}) + \dots$$
$$+ \mathrm{Sq}^{n+i.2^{t+1}} \chi(\mathrm{Sq}^{N-n-i.2^{t+1}}) + \dots = 0$$

COROLLARY (1.2). *Let* $N \geq 2^{t+2} - t - 2$. *Then*

$$\chi(\mathrm{Sq}^N) + \mathrm{Sq}^{2^{t+1}} \chi(\mathrm{Sq}^{N-2^{t+1}}) + \dots + \mathrm{Sq}^{i.2^{t+1}} \chi(\mathrm{Sq}^{N-i.2^{t+1}}) + \dots = 0$$

These results are not new, see [2],[3],[7], although the methods of proof are completely different. For example, if we adopt temporarily the notation of Theorem 2 of [3] by setting $N = 2^n - k$ with $k \leq n$ in Corollary (1.2), we obtain $\chi(\mathrm{Sq}^{2^n-k}) = \mathrm{Sq}^{2^{n-1}} \chi(\mathrm{Sq}^{2^{n-1}-k})$ which by iteration gives the first part of that theorem. In fact, this follows from a weaker result than Corollary (1.2) which leads also to the last part of Theorem 2 of [3]; we will return to this in section 4.

One can set up an iterative procedure from Corollary (1.2) to express $\chi(\mathrm{Sq}^N)$ in terms of products of Sq^i. Let $N_1 = m + 2^{t+1}$ and $N_2 = m + 2^t + 2^{t+1}$

where $m < 2^t$. Then

$$\chi(\mathrm{Sq}^{N_1}) = \mathrm{Sq}^{2^t}\chi(\mathrm{Sq}^{m+2^t}) + \mathrm{Sq}^{2^{t+1}}\chi(\mathrm{Sq}^m)$$

$$\chi(\mathrm{Sq}^{N_2}) = \{\mathrm{Sq}^{2^t}\mathrm{Sq}^{2^t} + \mathrm{Sq}^{2^{t+1}}\}\chi(\mathrm{Sq}^{m+2^t}) + \{\mathrm{Sq}^{2^t}\mathrm{Sq}^{2^{t+1}} + \mathrm{Sq}^{3.2^t}\}\chi(\mathrm{Sq}^m)$$

Setting $m = 0$ in N_1 and reading backwards gives,

$$\mathrm{Sq}^{2^{t+1}} + \mathrm{Sq}^{2^t}\chi(\mathrm{Sq}^{2^t}) + \chi(\mathrm{Sq}^{2^{t+1}}) = 0$$

The existence of $A(2)$ was known in the 1960's but was defined in a different manner which is described in the next section. For a short period at that time, the second author thought that $A(2)$ might be isomorphic to the Steenrod algebra, but lost interest in it when he proved that this was false. The question was discussed with Frank Adams in the 1960's and his thoughts have been influential on what follows; in particular he derived the relations $p_{n,N}$ for n = 1,2, and 3 in a slightly different manner. We are grateful to Wilson Sutherland for reading the manuscript.

2. Let $A = \mathbb{Z}\{S^0, S^1, ..., S^i, ...\}$ be the graded, free, associative ring on generators S^i of degree i where $S^0 = 1 \in A_0$. We define Q^n in A by $Q^0 = 1$ and, for $n > 0$, $\sum S^{n-i}Q^i = 0$, summing from $i = 0$ to n.

A definition of $A(2)$ is implicit in [4], based on properties of Adams operators in complex K-theory. It is shown there that if X is a CW-complex of finite type without homology 2-torsion, then A acts as an algebra of operators on $\bigoplus H^{2i}(X, \mathbb{Z}_{(2)})$ with $S^q : H^{2i}(X, \mathbb{Z}_{(2)}) \to H^{2i+2q}(X, \mathbb{Z}_{(2)})$ a homomorphism. Here $\mathbb{Z}_{(2)}$ denotes the integers localised at 2 and the action is not canonical. If $H^*(X, \mathbb{F}_2)$ is identified with $H^*(X, \mathbb{Z}_{(2)}) \otimes \mathbb{F}_2$, then S^q mod 2 becomes Sq^{2q}. Further, for each $x \in H^{2i}(X, \mathbb{Z}_{(2)})$ and any odd integer k, $(k^nS^n + k^{n-1}S^{n-1}Q^1 + \ldots + Q^n)x$ is divisible by 2^n in the free $\mathbb{Z}_{(2)}$-module $H^{2i+2n}(X, \mathbb{Z}_{(2)})$, that is, $k^nS^n + k^{n-1}S^{n-1}Q^1 + ... + Q^n \equiv 0 \bmod 2^n$ (as an operator acting on $\bigoplus H^{2i}(X, \mathbb{Z}_{(2)})$). We consider all relations mod 2 implied by this last equation as k varies. (For example, when $n = 2$, $k^2S^2 + kS^1Q^1 + Q^2 \equiv 0 \bmod 4$. As $S^2 + S^1Q^1 + Q^2 = 0$, this gives $(k^2 - 1)S^2 + (k - 1)S^1Q^1 \equiv 0 \bmod 4$, and setting $k = 3$ shows that $2S^1Q^1 \equiv 0 \bmod 4$ or $S^1Q^1 \equiv 0 \bmod 2$). These mod 2 relations imply relations among the Sq^{2i} and $\chi(\mathrm{Sq}^{2i})$. (When n = 2, $S^1Q^1 \equiv 0 \bmod 2$ gives $\mathrm{Sq}^2\chi(\mathrm{Sq}^2) = 0$). But any relations which hold among the Sq^{2i} acting on the cohomology of spaces without homology 2-torsion imply relations on the $\mathbb{F}_2$-cohomology of any space by mapping Sq^{2i} to Sq^i, that is, Adem relations.

We formalise the above as follows. Let $B = \mathbb{Q}\{S^0, S^1, ..., S^i, ...\}$ be the graded, free, associative algebra over the rationals $\mathbb{Q}$ on the S^i used to define A. Then A is a subring of B. In B, let

$$r_{n,k} = 2^{-n}\{k^n S^n + k^{n-1}S^{n-1}Q^1 + \dots + Q^n\}$$

for each integer $n > 0$ and odd integer k. Set $r_{o,k} = 1$ and $r_{n,k} = 0$ for $n < 0$. We define C to be the smallest subring of B which contains A and all the elements $r_{n,k}$, or equivalently, C is the subring of B generated by $S^i, i \geq 0$, and $r_{n,k}, n \geq 1, 1 < k < 2^n$. There is an embedding of graded rings $A \to C$ and, in each degree, both are free, finitely generated Abelian groups. So there is an induced homomorphism $i : A \otimes \mathbb{F}_2 \to C \otimes \mathbb{F}_2$. We define $A(2)$ to be the image of i.

Let the images of S^n and Q^n under the composition $A \to A \otimes \mathbb{F}_2 \to C \otimes \mathbb{F}_2$ be s_n and q_n. To prove that this definition of $A(2)$ coincides with that given in section 1, we must show that the mod 2 relations implied by $k^n S^n + k^{n-1}S^{n-1}Q^1 + \dots + Q^n \equiv 0 \bmod 2^n$, as k varies over odd integers, are precisely the relations $p_{n,N} = 0$ for $n \geq 1, N \geq n + 1 + \nu_2(n!)$. This we establish in section 3 and from it follows immediately that $h : A(2) \to \mathcal{A}(2)$ of section 1 is a surjective homomorphism of graded rings. It will be proved in section 4 that $A(2)$ is generated by s_0 and $s_{2^i}, i \geq 0$.

A connected, associative, coassociative, strictly co-commutative Hopf algebra structure is determined on B by requiring that its comultiplication ψ satisfies $\psi(S^n) = \sum S^{n-i} \otimes S^i$. One checks that this implies that $\psi(r_{n,k}) = \sum r_{i,k} \otimes r_{n-i,k}$. Mapping S^i to Q^i gives an anti-automorphism of this Hopf algebra. In addition, B has a derivation determined by $d(S^i) = S^{i-1}$ (and therefore $d(Q^i) = -Q^{i-1}$). One must check that $d(r_{n,k}) = \frac{(k-1)}{2} r_{n-1,k}$.

The corresponding formulae for $\psi(S^n)$ and $d(S^n)$ define corresponding structures on A, and therefore, on $A(2)$. Thus $A(2)$ is a connected, graded, associative, coassociative, cocommutative Hopf algebra over $\mathbb{F}_2$ with an anti-automorphism χ defined by $\chi(s_i) = q_i$ and a derivation d defined by $d(s_i) = s_{i-1}$.

Finally in this section, we comment on the missing Adem relations in $A(2)$. It is shown in [8] that to define $\mathcal{A}(2)$ from $\mathbb{F}_2\{\mathrm{Sq}^0, \mathrm{Sq}^1, \dots, \mathrm{Sq}^i, \dots\}$ it is sufficient to express each Sq^n for $n \neq 2^k$ in terms of the Sq^{2^i} and to be able to write $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}$ and $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^j} + \mathrm{Sq}^{2^j}\mathrm{Sq}^{2^i}$ for $j \geq i + 2$ in terms of Sq^{2^k} for $k < i$. It is the third set of relations which is absent from $A(2)$.

3. We need to examine the system of equations,

$$k^N S^N + k^{N-1}S^{N-1}Q^1 + \dots + Q^N \equiv 0 \bmod 2^N$$

where k is an odd integer and $1 < k < 2^N$, or equivalently,

$$(1) \quad S^1 Q^{N-1}(k-1) + S^2 Q^{N-2}(k^2 - 1) + \dots + S^N(k^N - 1) \equiv 0 \bmod 2^N,$$

as explained in the section above.

First we recall Newton's interpolation formula.

LEMMA (3.3). *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$. *Then*

$$n!a_n = f(n) - \binom{n}{1}f(n-1) + \dots + (-1)^i\binom{n}{i}f(n-i) + \dots + (-1)^n f(0)$$

We apply Lemma (3.3) to $f_m(x) = (2x+1)^m - 1$. Let

$$c_{n,m} = f_m(n) - \binom{n}{1}f_m(n-1) + \dots + (-1)^n f_m(0)$$

Then $c_{n,m} = 0$ if $n > m$ and $c_{n,n} = n!2^n$, setting $n = m$. Let ρ_n denote equation (1) with $k = 2n+1$. So

$$\rho_n - \binom{n}{1}\rho_{n-1} + \dots + (-1)^i\binom{n}{i}\rho_{n-i} + \dots + (-1)^{n-1}\binom{n}{n-1}\rho_1$$

becomes

$$(2) \qquad \begin{aligned} &c_{n,n}S^nQ^{N-n} + c_{n,n+1}S^{n+1}Q^{N-n-1} + \dots \\ &\quad + c_{n,n+i}S^{n+i}Q^{N-n-i} + \dots + c_{n,N}S^N \equiv 0 \bmod 2^N \end{aligned}$$

for $1 \le n < 2^{N-1}$, and the systems of equations (1) and (2) are equivalent.

We require a more tractable formula for the coefficients $c_{n,m}$.

LEMMA (3.4). *As formal power series in* t,

$$\frac{c_{n,n}}{(1-t)(1-3t)\dots(1-(2n+1)t)} = \sum c_{n,n+i}t^i$$

The proof of Lemma (3.4) is a routine exercise in expressing the left hand side as a sum of partial fractions.

COROLLARY (3.5). $c_{n,n}$ *divides* $c_{n,m}$.

It follows from Corollary (3.5) that the equations (2) are identically zero for $N < 1 + n + \nu_2(n!)$ as $\nu_2(c_{n,n}) = n + \nu_2(n!)$. Let $d_{n,m} = c_{n,m}/c_{n,n}$. The non-trivial relations in (2) are equivalent to

$$S^nQ^{N-n} + d_{n,n+1}S^{n+1}Q^{N-n-1} + \dots + d_{n,N}S^N \equiv 0 \bmod 2^{N-n-\nu_2(n!)}$$

Thus the mod 2 relations following from (1) are

$$S^nQ^{N-n} + d_{n,n+1}S^{n+1}Q^{N-n-1} + \dots + d_{n,N}S^N \equiv 0 \bmod 2$$

for $N \ge 1 + n + \nu_2(n!)$.

From Lemma (3.4), it follows that $d_{n,n+i}$ mod 2 equals the coefficient mod 2 of t^i in $(1-t)^{-(n+1)}$ which is $\binom{n+i}{i}$. Thus we have established that in $A(2)$ of section 2, the relations among the s_i and q_i are

$$p_{n,N} = \sum \binom{n+i}{i} s_{n+i}q_{N-n-i} = 0$$

for $N \geq 1 + n + \upsilon_2(n!)$ which gives the definition of the introduction.

4. We consider the system of equations

$$s_n q_{N-n} + \binom{n+1}{1} s_{n+1} q_{N-n-1} + \dots + \binom{n+i}{i} s_{n+i} q_{N-n-i} + \dots = 0,$$

where $N \geq 1 + n + \nu_2(n!)$, but, except when we are describing phenomena exclusively relating to $A(2)$,we will write Sq^i for s_i and $\chi(\mathrm{Sq}^i)$ for q_i.

An array of coefficients for small values of n and N is given in Table 1. For example, row seven means that,

$$\mathrm{Sq}^7\chi(\mathrm{Sq}^{N-7}) + \mathrm{Sq}^{15}\chi(\mathrm{Sq}^{N-15}) + \dots + \mathrm{Sq}^{7+8i}\chi(\mathrm{Sq}^{N-7-8i}) + \dots = 0$$

for $N \geq 12$.

In computations which follow, we will use without further comment the well known formulae :-

Let the binary expansions of n and r be $\sum a_i 2^i$ and $\sum b_i 2^i$ respectively. Then

$$\text{(a)} \;\; \nu_2(n!) = n - \sum a_i, \qquad\qquad \text{(b)} \;\; \binom{n}{r} \equiv \prod \binom{a_i}{b_i} \bmod 2$$

We return to considering the theorem of Don Davis mentioned in the introduction. We consider row $2^{n-1} - 1$ with $N \geq 2^n - n$. Then

$$\mathrm{Sq}^{2^{n-1}-1}\chi(\mathrm{Sq}^{2^{n-1}-k}) = 0 \tag{3}$$

for $1 \leq k \leq n-1$ and

$$\mathrm{Sq}^{2^{n-1}-1}\chi(\mathrm{Sq}^{2^{n-1}}) = \mathrm{Sq}^{2^n-1} \tag{4}$$

These follow as $\binom{2^n-1+s}{s} \equiv 0 \bmod 2$ for $1 \leq s \leq 2^{n-1} - 1$ and $\binom{2^n-1}{2^n-1} \equiv 1 \bmod 2$.

It follows that $\mathrm{Sq}^{2^{n-1}-k}\chi(\mathrm{Sq}^{2^{n-1}}) = \mathrm{Sq}^{2^n-k}$, or equivalently $\chi(\mathrm{Sq}^{2^n-k}) = \mathrm{Sq}^{2^{n-1}}\chi(\mathrm{Sq}^{2^{n-1}-k})$, for $1 \leq k \leq n$. We argue by induction; the case $k = 1$ is included in (4). Applying the derivation d to $\mathrm{Sq}^{2^{n-1}-k}\chi(\mathrm{Sq}^{2^{n-1}}) = \mathrm{Sq}^{2^n-k}$ for $k < n$ gives

$$\mathrm{Sq}^{2^{n-1}-k-1}\chi(\mathrm{Sq}^{2^{n-1}}) + \mathrm{Sq}^{2^{n-1}-k}\chi(\mathrm{Sq}^{2^{n-1}-1}) = \mathrm{Sq}^{2^n-k-1}.$$

But

$$\mathrm{Sq}^{2^{n-1}-k}\chi(\mathrm{Sq}^{2^{n-1}-1}) = \chi(\mathrm{Sq}^{2^{n-1}-1}\chi(\mathrm{Sq}^{2^{n-1}-k})) = 0$$

by (3), and so we have the required result.

If we differentiate $\chi(\mathrm{Sq}^{2^n-n}) = \mathrm{Sq}^{2^{n-1}}\chi(\mathrm{Sq}^{2^{n-1}-n})$, we obtain

$$\chi(\mathrm{Sq}^{2^n-n-1}) = \mathrm{Sq}^{2^{n-1}-1}\chi(\mathrm{Sq}^{2^{n-1}-n}) + \mathrm{Sq}^{2^{n-1}}\chi(\mathrm{Sq}^{2^{n-1}-n-1})$$

and replacing n by $n-1$ and substituting for $\chi(\mathrm{Sq}^{2^{n-1}-n})$ this gives,

$$\begin{aligned}\chi(\mathrm{Sq}^{2^n-n-1}) = {} & \mathrm{Sq}^{2^{n-1}-1}\mathrm{Sq}^{2^{n-2}-1}\chi(\mathrm{Sq}^{2^{n-2}-n+1}) \\ & +\mathrm{Sq}^{2^{n-1}-1}\mathrm{Sq}^{2^{n-2}}\chi(\mathrm{Sq}^{2^{n-2}-n}) + \mathrm{Sq}^{2^{n-1}}\chi(\mathrm{Sq}^{2^{n-1}-n-1})\end{aligned}$$

But in $\mathcal{A}(2)$, $\mathrm{Sq}^{2k-1}\mathrm{Sq}^k = 0$, $k > 0$, and so the second term, on the right hand side vanishes. Iteration leads to the formula,

$$\chi(\mathrm{Sq}^{2^n-n-1}) = \mathrm{Sq}^{2^{n-1}-1}\mathrm{Sq}^{2^{n-2}-1}...\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^{2^{n-1}}\chi(\mathrm{Sq}^{2^{n-1}-n-1})$$

This is the final part of theorem 2 of [3].

However in $A(2)$, $s_{2k-1}s_k \neq 0$ in general. (For otherwise $A(2)$ would be isomorphic to $\mathcal{A}(2)$.) For example, $s_1s_1 = 0$, $s_3s_2 = 0$, but $s_5s_3 \neq 0$. We do not know if $s_{2^t-1}s_{2^t-1} = 0$ for all $t > 0$.

It is a routine matter to check that $A(2)$ is generated as an algebra by s_0 and $s_{2^i}, i \geq 0$. For, let k have binary expansion $2^u + a_{u+1}2^{u+1} + ... + 2^v$ with $u < v$. Then from row 2^u of Table 1, one has

$$s_{2^u}q_{k-2^u} + ... + s_k = 0,$$

as $\binom{k}{k-2^u} \equiv 1 \bmod 2$. The result then follows by induction on k.

There exists a surjective, degree-halving, homomorphism of algebras $A(2)/(s_1) \to A(2)$ defined by mapping s_{2i} to s_i and s_{2i+1} to zero. But unlike the situation for $\mathcal{A}(2)$, it is not an isomorphism; it is surjective but not injective. The key reason why it is not an isomorphism is that $2M \geq 2n + 1 + \nu_2(2n!)$ implies that $M \geq n + 1 + \nu_2(n!)$ but is not equivalent to it.

We now prove Theorem (1.1) by applying row operations to the general case of Table 1. We consider the $2^r - 1$ rows from row 2^r to $2^{r+1} - 1$ where $N \geq 2^{r+2} - r - 2$. Working from column zero, the entries in each row $2^r + s$ where $0 \leq s \leq 2^{r-1}$ are periodic with period 2^{r+1} as

$$\binom{2^r+s+t}{t} \equiv \binom{2^r+s+t+k2^{r+1}}{t+k2^{r+1}} \bmod 2, -(2^r+s) \leq t < 2^r - s$$

($\binom{n}{t}$ with $t < 0$ is zero, but the formula remains true.)

The first 2^r entries of each such row are zero and it follows that we can perform row reductions, so that each row becomes

$$\mathrm{Sq}^{2^r+s}\chi(\mathrm{Sq}^{N-2^r-s}) + ... + \mathrm{Sq}^{(2i+1)2^r+s}\chi(\mathrm{Sq}^{N-(2i+1)2^r-s}) + ... = 0$$

The resulting array of coefficients is given in Table 2.

Now let $N \geq 2^{t+2} - t - 2$ and consider row $2^r + s$ where $r < t$ and $0 \leq s \leq 2^{r-1}$. The second non-zero element in this row occurs in column $2^{r+1} + 2^r + s$. So if we add row $2^{r+1} + 2^r + s$, with period 2^{r+2}, to row $2^r + s$, with period 2^{r+1}, the new row has period 2^{r+2}. Indeed, letting ρ_i denote the i-th row,

$\rho_{2^r+s} + \rho_{2^{r+1}+2^r+s} + \rho_{2^{r+2}+2^r+s} + \ldots + \rho_{2^t+2^r+s}$ has period 2^{t+1}. Thus by row reduction, for $N \geq 2^{t+2}-t-2$, we can form an equivalent system of equations for $n < 2^{t+1}$ of period 2^{t+1}. This proves Theorem (1.1).

Corollary (1.2) follows immediately by differentiating

$$\begin{aligned} \mathrm{Sq}^1\chi(\mathrm{Sq}^N) + \mathrm{Sq}^{1+2^{t+1}}\chi(\mathrm{Sq}^{N-2^{t+1}}) + \ldots \\ + \mathrm{Sq}^{1+i.2^{t+1}}\chi(\mathrm{Sq}^{N-i.2^{t+1}}) + \ldots = 0 \end{aligned}$$

and again applying Theorem (1.1), with $N+1$ replaced by N

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	$N \geq 2$
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	$N \geq 4$
3	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	$N \geq 5$
4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	$N \geq 8$
5	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	$N \geq 9$
6	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	$N \geq 11$
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	$N \geq 12$
8	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	$N \geq 16$
9	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	$N \geq 17$
10	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	$N \geq 19$
11	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	$N \geq 20$
12	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	$N \geq 23$
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	$N \geq 24$
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$N \geq 26$
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	$N \geq 27$
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$N \geq 32$

Table 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	$N \geq 2$
2	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	$N \geq 5$
3	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	$N \geq 5$
4	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	$N \geq 12$
5	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	$N \geq 12$
6	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	$N \geq 12$
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	$N \geq 12$
8	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	$N \geq 27$
9	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	$N \geq 27$
10	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	$N \geq 27$
11	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$N \geq 27$
12	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	$N \geq 27$
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	$N \geq 27$
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$N \geq 27$
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	$N \geq 27$
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N \geq 58$

Table 2