# SECONDARY CHARACTERISTIC CLASSES AND THE IMMERSION PROBLEM

By José Adem and Samuel Gitler

## Introduction

The Stiefel-Whitney classes of a real vector bundle are determined by the Thom isomorphism and the action of the Steenrod squares on the fundamental class of the bundle. With an analogous procedure, we introduce secondary characteristic classes using stable secondary cohomology operations acting on this fundamental class. These secondary classes satisfy properties which are similar to the well known properties of Stiefel-Whitney classes. They are invariants of the stable fibre homotopy type of the vector bundle, and, under suitable hypotheses, they satisfy a Whitney type product formula. For the tangent bundle of a differentiable manifold $M$, they satisfy a Wu type formula which, in many cases, determines them by means of the action of cohomology operations on a suitable class of $H^*(M \times M)$.

We compute secondary characteristic classes for the tangent and normal bundles of the complex and real projective spaces and use them to obtain some non-immersion results. In particular, we show that the real projective space $RP^n$ cannot be immersed in $R^{2n-9}$ if $n = 2^r + 2^s + 3$ and $r > s \geq 2$. In this case, according to Sanderson in [13], $RP^n$ can be immersed in $R^{2n-8}$. Also, combining this non-immersion result with the results of James ([6]), it follows that $RP^n$ cannot be immersed in $R^{n+[n/2]}$ for all $n \geq 32$.

## 1. Secondary characteristic classes*

Unless otherwise stated, throughout this paper *we will use singular cohomology with coefficients* $Z_2$, the cyclic group of order 2, and in general we will omit the coefficient group; thus $H^q(B)$ will stand for $H^q(B;\, Z_2)$.

Let $\xi = (E, B, \pi)$ be an $n$-vector bundle with base $B$, a paracompact and connected space, projection $\pi\colon E \to B$, and fibre $R^n$, an $n$-dimensional vector space over the reals. Let $E_0 \subset E$ be the subspace of non-zero vectors of $E$. As in [9; p. 34], we define the Thom isomorphism,

$$T\colon H^q(B) \to H^{n+q}(E, E_0),$$

by the composition,

$$H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\smile U} H^{n+q}(E, E_0),$$

where $U \in H^n(E, E_0)$ is the fundamental class of $\xi$. Explicitly, if $x \in H^q(B)$, $T(x) = (\pi^* x) \smile U$.

* Our secondary characteristic classes are different from the ones defined by Peterson and Stein in "Secondary Characteristic Classes," *Ann. of Math.*, **76** (1962), 510–23.

We denote by $A$ the Steenrod algebra over $Z_2$. For each $n$-vector bundle $\xi$, we define a $Z_2$-homomorphism,

$$\xi^*: A \to H^*(B)$$

by $\xi^*(\theta) = T^{-1}\theta(U)$, or equivalently $\theta(U) = [\pi^*\xi^*(\theta)] \smile U$, for every $\theta \in A$. This homomorphism is natural with respect to bundle maps.

In general $\xi^*(\theta)$ is a *primary characteristic class* of $\xi$, and in particular, $\xi^*(\mathrm{Sq}^i) = W_i(\xi)$ is the $i$th Stiefel-Whitney class. If $\mathrm{Sq} = \sum_{i=0}^{\infty} \mathrm{Sq}^i$, $\xi^*(\mathrm{Sq}) = W(\xi)$ is the total Stiefel-Whitney class of $\xi$.

*The characteristic ring* of $\xi$ is the subring $C^*(\xi) \subset H^*(B)$ generated by $W_0(\xi), \cdots, W_n(\xi)$. The Wu formulae imply that $C^*(\xi)$ is an $A$-module and that $\xi^*A \subset C^*(\xi)$. Clearly, the subring of $H^*(B)$ generated by $\xi^*A$ coincides with $C^*(\xi)$.

In our notation, the Whitney product formula is given as follows. Let $\xi$, $\eta$ be two vector bundles over the same base and $\xi \oplus \eta$ their Whitney sum, then

$$(\xi \oplus \eta)^*(\mathrm{Sq}) = \xi^*(\mathrm{Sq}) \smile \eta^*(\mathrm{Sq}).$$

The general considerations we have made on primary characteristic classes, indicate clearly the way to introduce secondary characteristic classes. These will be determined by the action of secondary cohomology operations in the class $U$.

Let

$$\alpha\beta = \sum_{k=1}^{j} \alpha_k\beta_k = 0 \tag{1.1}$$

be a homogeneous relation of degree $r + 1$ in $A$, with $s_k$ = degree $\alpha_k > 0$ and $t_k$ = degree $\beta_k > 0$. Following Adams ([1]), let $\Phi$ be a stable secondary cohomology operation associated with the relation (1.1). For every space $X$ and every $q > 0$, the operation $\Phi$ is a homomorphism,

$$\Phi: K^q(\Phi; X) \to H^{q+r}(X)/Q^{q+r}(\Phi; X),$$

where

$$K^q(\Phi; X) = \bigcap_{k=1}^{j} [\mathrm{Ker}(\beta_k : H^q(X) \to H^{q+t_k}(X))], \quad \text{and} \tag{1.2}$$

$$Q^{q+r}(\Phi; X) = \sum_{k=1}^{j} \alpha_k H^{q+t_k-1}(X). \tag{1.3}$$

If $\xi = (E, B, \pi)$ is an $n$-vector bundle, suppose that $\xi^*(\beta_k) = 0$, or, equivalently, $\beta_k(U) = 0$, for $k = 1, \cdots, j$. With these hypotheses, the secondary operation $\Phi(U)$ is defined and

$$\Phi(U) \in H^{n+r}(E, E_0)/Q^{n+r}(\Phi; E, E_0).$$

Using the inverse of the Thom isomorphism induced in the factor group, we obtain a *secondary characteristic class* $\xi^*(\Phi)$ defined by

$$\xi^*(\Phi) = T^{-1}\Phi(U), \tag{1.4}$$

where

$$\xi^*(\Phi) \in H^r(B)/Q^r(\Phi; \xi),$$

with $Q^r(\Phi; \xi) = T^{-1}Q^{n+r}(\Phi; E, E_0)$. The direct extension of the cup-product to cosets allows us to express (1.4) in the following equivalent form:

$$\Phi(U) = [\pi^*\xi^*(\Phi)] \smile U. \tag{1.5}$$

The indeterminacy $Q^r(\Phi; \xi)$ can be given explicitly in terms of $H^*(B)$ as follows. Let $\psi\colon A \to A \otimes A$ be the diagonal map, which turns $A$ into a Hopf algebra. The composition

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{1 \otimes \xi^*} A \otimes H^*(B)$$

defines $\varphi = (1 \otimes \xi^*)\psi$. Let $\mu\colon A \otimes H^*(B) \to H^*(B)$ be the homomorphism given by the standard action of $A$ in $H^*(B)$, which makes $H^*(B)$ an $A$-module. Let $\tau\colon H^*(B) \otimes H^*(B) \to H^*(B) \otimes H^*(B)$ be the homomorphism that interchanges the factors, and, finally, let $d^*\colon H^*(B) \otimes H^*(B) \to H^*(B)$ be the homomorphism induced by the diagonal map $d\colon B \to B \times B$. With these homomorphisms define $\lambda\colon A \otimes H^*(B) \to H^*(B)$ to be the composition

$$\lambda = d^*(\mu \otimes 1)(1 \otimes \tau)(\varphi \otimes 1).$$

PROPOSITION 1.6. *The following diagram is commutative:*

$$\begin{array}{ccc} A \otimes H^*(B) & \xrightarrow{\lambda} & H^*(B) \\ {\scriptstyle 1 \otimes T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ A \otimes H^*(E, E_0) & \xrightarrow{\mu} & H^*(E, E_0) \end{array}$$

*Proof.* If $\theta \in A$, suppose $\psi(\theta) = \sum \theta_i \otimes \theta_i'$. With $x \in H^*(B)$ we have:

$$\begin{aligned} T\lambda(\theta \otimes x) &= T\textstyle\sum \theta_i(x) \smile \xi^*(\theta_i') = \sum \theta_i(\pi^*x) \smile (\pi^*\xi^*(\theta_i') \smile U) \\ &= \textstyle\sum \theta_i(\pi^*x) \smile \theta_i'(U) = \theta(\pi^*x \smile U) = \mu(1 \otimes T)\,(\theta \otimes x). \end{aligned}$$

It follows from (1.6) that $\theta H^*(E, E_0) = T\lambda(\theta \otimes H^*(B))$. Hence

$$Q^r(\Phi; \xi) = \textstyle\sum_{k=1}^{j} \lambda(\alpha_k \otimes H^{t_k-1}(B)). \tag{1.7}$$

The naturality of $\xi^*$ and that of $\Phi$ imply the naturality of $\xi^*(\Phi)$ with respect to bundle maps. In particular, the secondary characteristic classes of a trivial bundle are zero.

The operation $\Phi$ associated with relation (1.1) in general is not unique. If $\Phi'$ is another operation associated with (1.1), we have ([1; p. 70]):

$$\Phi(U) - \Phi'(U) = \theta(U) \tag{1.8}$$

for some $\theta \in A$. This implies that

$$\xi^*(\Phi) - \xi^*(\Phi') = \xi^*(\theta); \tag{1.9}$$

that is, two secondary characteristic classes determined by the same relation in $A$, differ by a primary characteristic class.

## 2. Invariance under fibre homotopy type

Let $(E_1, B_1, F_1, \pi_1)$ and $(E_2, B_2, F_2, \pi_2)$ be two fibre bundles in the sense of Steenrod, where the bases $B_1$, $B_2$ are finite $CW$-complexes and the fibres $F_1$, $F_2$ are locally compact spaces. A map $f: E_1 \to E_2$ is an $F$-map if $f$ sends fibres into fibres and $f$ restricted to each fibre is a homotopy equivalence. Every $F$-map $f$ induces a map $\bar{f}: B_1 \to B_2$ such that $\pi_2 f = \bar{f}\pi_1$.

Let $\xi = (E_1, B, F, \pi_1)$ and $\eta = (E_2, B, F, \pi_2)$ be two fibre bundles over the same base $B$ and with same fibre $F$. The bundles $\xi$, $\eta$ are of the same fibre homotopy type, in symbols $\xi \sim \eta$, if there exist maps $f: E_1 \to E_2$ and $f': E_2 \to E_1$ which preserve fibres and such that $ff' \simeq$ identity and $f'f \simeq$ identity, under homotopies which also preserve fibres. A theorem of Dold asserts that $\xi \sim \eta$ if and only if there exists an $F$-map $g: E_1 \to E_2$ such that $\bar{g}$ = identity ([5; p. 120]).

Now let $\xi = (E, B, R^n, \pi)$ be a vector bundle over $B$. Choosing a riemannian metric on $E$, we determine $E(1)$ as the subspace of $E$ of vectors of length $\leq 1$, and $\dot{E}(1)$ as the subspace of vectors of length 1. The bundle $\dot{E}(1) \to B$ is essentially independent of the metric and is the orthogonal sphere bundle associated to $\xi$, we denote it by $(\xi)$.

Following Atiyah ([4; p. 292]), we say that two orthogonal sphere bundles $(\xi)$, $(\eta)$ over the same base $B$ are of the same stable fibre homotopy type, in symbols $(\xi) \overset{s}{\sim} (\eta)$, if there exist trivial vector bundles 0, 0$'$, over $B$, such that $(\xi \oplus 0) \sim (\eta \oplus 0')$. The equivalence class of $(\xi)$ under $\overset{s}{\sim}$ is denoted by $J(\xi)$. The set of all classes forms the group $J(B)$, introduced by Atiyah, where the group structure is induced by the Whitney sum for vector bundles.

THEOREM 2.1. *The primary and secondary characteristic classes are $J$-invariants.*

The result for primary classes is well known, and is due originally to Thom ([18; p. 166]). However we will give another proof using only the concepts Milnor introduces in his study of characteristic classes.

Let $\xi = (E, B, R^m, \pi)$, $\eta = (E', B, R^n, \pi')$ be two vector bundles over the same base $B$ and suppose $J(\xi) = J(\eta)$. This is equivalent with $(\xi \oplus 0) \sim (\eta \oplus 0')$, where 0, 0$'$ are trivial vector bundles over $B$. The proof of (2.1) is an immediate consequence of the following two lemmas.

LEMMA 2.2. *Theorem* 2.1 *holds for $\xi$ and $\eta = \xi \oplus 0$, where* 0 *is a trivial vector bundle over $B$.*

LEMMA 2.3. *Theorem* 2.1 *holds for $\xi$ and $\eta$ such that $(\xi) \sim (\eta)$.*

*Proof of 2.2.* The Whitney product formula implies that:

$$(\xi \oplus 0)^*(\mathrm{Sq}) = \xi^*(\mathrm{Sq})0^*(\mathrm{Sq}) = \xi^*(\mathrm{Sq});$$

and from this the result for primary characteristic classes follows readily. Also we obtain that for any stable secondary operation $\Phi$, $(\xi \oplus 0)^*(\Phi)$ is defined if and only if $\xi^*(\Phi)$ is defined.

Suppose that 0 has fibre $R^k$. Clearly we may suppose that 0 is a product bundle, and if $E$ is the total space of $\xi$, we may consider $E' = E \times R^k$ as the total space of $\xi \oplus 0$. It then follows that $(E', E_0') = (E, E_0) \times (R^k, R^k - 0)$. If $U' \in H^{n+k}(E', E_0')$, $U \in H^n(E, E_0)$, and $w \in H^k(R^k, R^k - 0)$ are the fundamental classes, we have $U' = U \times w$, and this in turn implies that, for all $x \in H^q(B)$, $T'(x) = T(x) \times w$, where $T'$, $T$ are the Thom isomorphisms for $\xi \oplus 0$, and $\xi$ respectively. This, together with the Cartan product formula, implies $Q(\Phi; \xi) = Q(\Phi; \xi \oplus 0)$. Finally, if $\xi^*(\Phi)$ is defined, the stability of $\Phi$ implies that $\Phi(U \times w) = \Phi(U) \times w$, and then it follows that $\xi^*(\Phi) = (\xi \oplus 0)^*(\Phi)$.

*Proof of 2.3.* For every vector bundle, $\xi = (E, B, R^n, \pi)$ we have a natural isomorphism

$$H^*(E, E_0) \xrightarrow{\approx} H^*(E(1), \dot{E}(1))$$

induced by the composition of the inclusions

$$(E(1), \dot{E}(1)) \xrightarrow{i} (E(1), E_0(1)) \xrightarrow{j} (E, E_0),$$

where $E_0(1) = E_0 \frown E(1)$. Since $\dot{E}(1)$ is a deformation retract of $E_0(1)$, $i$ induces an isomorphism. Also $j$ induces an isomorphism, since it is an excision. Then, in the definition of characteristic classes we may replace the pair $(E, E_0)$ by the pair $(E(1), \dot{E}(1))$ and the class $U \in H^n(E, E_0)$ by its image $U_1 \in H^n(E(1), \dot{E}(1))$. If $F \simeq R^n$ is the fibre over the point $b$, $F(1) = F \frown E(1)$ and $\dot{F}(1) = F \frown \dot{E}(1)$ are the corresponding fibres over $b$ of the associated bundles $E(1) \to B$, $\dot{E}(1) \to B$. The inclusion $i$: $(F(1), \dot{F}(1)) \to (E(1), \dot{E}(1))$ induces an isomorphism in dimension $n$,

$$i^*: H^n(E(1), \dot{E}(1)) \to H^n(F(1), \dot{F}(1)),$$

such that $i^* U_1$ is the generator of $H^n(F(1), \dot{F}(1))$. This property, for all $b$, characterizes $U_1$([9; p. 34]).

Now with $\xi$ and $\eta$ consider the associated bundles $\pi: E(1) \to B$ and $\pi': E'(1) \to B$. The hypothesis $(\xi) \sim (\eta)$ implies the existence of a fibre homotopy equivalence $f_1 : \dot{E}(1) \to \dot{E}'(1)$. Let $f$: $(E(1), \dot{E}(1)) \to (E'(1), \dot{E}'(1))$ be a radial extension of $f_1$. The commutative diagram

$$\begin{array}{ccc} H^n(E'(1), \dot{E}'(1)) & \xrightarrow{f^*} & H^n(E(1), \dot{E}(1)) \\ \Big\downarrow{\scriptstyle i'^*} & & \Big\downarrow{\scriptstyle i^*} \\ H^n(F'(1), \dot{F}'(1)) & \xrightarrow{f_b^*} & H^n(F(1), \dot{F}(1)), \end{array}$$

and the fact that $f_b = f \mid \dot{F}(1)$ is a homotopy equivalence, implies $f^*U_1' = U_1$. For every $\theta \in A$, we have,

$$\pi^*\xi^*(\theta) \smile U_1 = \theta(U_1) = f^*\theta(U_1') = f^*(\pi'^*\eta^*(\theta) \smile U_1') = \pi^*\eta^*(\theta) \smile U_1$$

and hence $\xi^*(\theta) = \eta^*(\theta)$. Moreover $\xi^*(\Phi)$ is defined if and only if $\eta^*(\Phi)$ is defined, and, by (1.4), $Q^r(\Phi; \xi) = Q^r(\Phi; \eta)$. Finally,

$$\pi^*\xi^*(\Phi) \smile U_1 = \Phi(U_1) = f^*\Phi(U_1') = \pi^*\eta^*(\Phi) \smile U_1;$$

therefore, $\xi^*(\Phi) = \eta^*(\Phi)$. This ends the proof of (2.3), and hence that of (2.1).

COROLLARY 2.4. *The primary and secondary characteristic classes are natural with respect to F-maps.*

*Proof.* It follows from (2.1) and the fact that every $F$-map factorizes into a fibre homotopy equivalence followed by a bundle map (see [5; p. 120]).

COROLLARY 2.5. *Let $M$, $N$ be two compact differentiable manifolds and $f: M \to N$, a homotopy equivalence. Then the primary and secondary tangent and normal classes of $M$ and $N$ correspond under $f^*$, the induced homomorphism in cohomology.*

*Proof.* Let $\xi$, $\eta$ be the tangent bundles to $M$, $N$ respectively; then a result of Atiyah ([4; Th. 3.6]) asserts that $J(f^*\xi) = J(\eta)$. By (2.1) and naturality the result for the tangent bundles follows. Now if $\nu$, $\rho$ are the normal bundles to a given immersion of $M$ and $N$ respectively, we clearly have $J(f^*\rho) = J(\nu)$, and by the same argument the result also follows for the normal characteristic classes.

In particular (2.5) implies that the primary and secondary classes of the tangent bundle are independent of the differentiable structure, and for the normal bundle that they are also independent of the particular immersion.

## 3. Bundles with trivial characteristic ring

Let $E(B)$ be the set of vector bundles with base $B$. $E(B)$ is a commutative monoid under the Whitney sum. Let $E_0(B) \subset E(B)$ be the subset of vector bundles with trivial characteristic ring. The Whitney product formula implies that $E_0(B)$ is a submonoid of $E(B)$.

PROPOSITION 3.1. *If $\Phi$ is a stable secondary operation associated with a relation of type* (1.1), *then for every $\xi \in E_0(B)$, the secondary characteristic class $\xi^*(\Phi)$ is defined. In this case the relation determines the class uniquely and*

$$Q^r(\Phi; \xi) = \sum_{k=1}^{j} \alpha_k H^{t_k-1}(B).$$

*Proof.* Since $\xi \in E_0(B)$, $C^*(\xi) = 1$; but this is equivalent to $A(U) = U$, so $\Phi(U)$ is defined and, by (1.8), it is unique. Now, the homomorphism $\lambda$ of (1.7) coincides with the standard action of $A$ in $H^*(B)$. Indeed, for $x \in H^*(B)$, and $\theta \in A$ with $\psi(\theta) = \sum \theta_i \otimes \theta_i'$ we have:

$$\lambda(\theta \otimes x) = \sum \theta_i(x) \smile \xi^*(\theta_i') = \theta(x) \smile 1 = \theta(x)$$

and this ends the proof of (3.1).

The Whitney product formula for secondary characteristic classes is given by the following:

THEOREM 3.2. *Let $\Phi$ be any stable cohomology operation of degree r. Then for every $\xi, \eta \in E_0(B)$ we have*

$$(\xi \oplus \eta)^*(\Phi) = \xi^*(\Phi) + \eta^*(\Phi)$$

*in $H^r(B)/Q^r(\Phi; B)$.*

*Proof.* It follows by the argument given in [9; p. 36] for the Stiefel-Whitney classes, except that instead of using squares and the Cartan formula, one uses secondary operations and the product formula of [3, (8.6)].

As a special case of (3.2) we obtain the following formulation of the Whitney duality for secondary characteristic classes:

COROLLARY 3.3. *Let $M$ be a differentiable n-manifold, such that all the positive dimensional Stiefel-Whitney classes of $M$ vanish. If $\tau$ is the tangent bundle to $M$, and $\nu$ is the normal bundle to an immersion of $M$, then $\tau, \nu \in E_0(M)$, and*

$$\tau^*(\Phi) = \nu^*(\Phi)$$

*in $H^r(M)/Q^r(\Phi; M)$, for every stable secondary operation $\Phi$ of degree r.*

## 4. Secondary characteristic classes of the tangent bundle

Let $\tau = (E, M, \pi)$ be the tangent bundle of a differentiable $n$-manifold. The classes $\tau^*(\Phi)$ are the secondary characteristic classes of $M$. In analogy with the Wu formulae for the Stiefel-Whitney classes of $M$, we will give a criterion to compute $\tau^*(\Phi)$ using the action of $\Phi$ in $H^*(M \times M)$. This is our Theorem (4.10). For this purpose, we will use the notation, definitions, and results given by Milnor in [9; p. 45]. Let

$$\psi: H^*(E, E_0) \to H^*(M \times M, M \times M - \Delta)$$

be the natural isomorphism, where $\Delta \subset M \times M$ is the diagonal. We denote by $\bar{U} \in H^n(M \times M)$ the class defined by $\bar{U} = i^*\psi(U)$, where $U \in H^n(E, E_0)$ is the fundamental class of $\tau$ and

$$i^*: H^*(M \times M, M \times M - \Delta) \to H^*(M \times M)$$

is the homomorphism induced by the inclusion.

THEOREM 4.1. *Let $\theta \in A$ be an element of the Steenrod algebra; then*
(i) *$\theta(U) \neq 0$ if and only if $\theta(\bar{U}) \neq 0$*
(ii) *$\theta(\bar{U}) = (\tau^*(\theta) \times 1) \smile \bar{U}$.*

*Proof.* If $x \in H^*(M)$, the definition of $\psi$ implies that $i^*\psi(\pi^*x \smile U) = (x \times 1) \smile \bar{U}$. On the other hand, the computation, by Milnor, of the class $\bar{U}$ in terms of the cohomology ring of $M$ implies that $(x \times 1) \smile \bar{U} \neq 0$ if and only if $x \neq 0$. Now $\theta(U) = (\pi^*\tau^*(\theta)) \smile U \neq 0$ if and only if $\tau^*(\theta) \neq 0$, and since

$$\theta(\bar{U}) = i^*\psi\theta(U) = i^*\psi(\pi^*\tau^*(\theta) \smile U) = (\tau^*(\theta) \times 1) \smile \bar{U},$$

(i) and (ii) of (4.1) follow immediately.

In order to establish the analogue of (4.1) for secondary characteristic classes we need to study the homomorphism

$$i^*\colon H^{n+r}(M^2, M^2 - \Delta)/Q^{n+r}(\Phi; M^2, M^2 - \Delta) \to H^{n+r}(M^2)/Q^{n+r}(\Phi; M^2),$$

where $M^2 = M \times M$, and $\Phi$ is a stable secondary operation of degree $r$.

The cohomology exact sequence of the pair $(M^2, M^2 - \Delta)$ breaks up into short exact sequences,

$$0 \to H^q(M^2, M^2 - \Delta) \xrightarrow{i^*} H^q(M^2) \xrightarrow{j^*} H^q(M^2 - \Delta) \to 0.$$

This follows from the fact that $i^*$ is a monomorphism, as is shown in the proof of (4.1). In turn, each of these short exact sequences splits under the homomorphism

$$t\colon H^q(M^2) \to H^q(M^2, M^2 - \Delta), \tag{4.2}$$

defined by $t(x \times y) = \langle y, \bar{\mu}\rangle(x \times 1) \smile U'$, where $U' = \psi(U)$, $\bar{\mu} \in H_n(M)$ is the generator, and $\langle y, \bar{\mu}\rangle$ is the Kronecker index. Indeed every $v \in H^q(M^2, M^2 - \Delta)$, with $q \geq n$, can be uniquely expressed as $v = (x \times 1) \smile U'$, for certain $x \in H^{q-n}(M)$; and on the other hand

$$ti^*(v) = t((x \times 1) \smile \bar{U}) = (x \times 1) \smile U' = v, \tag{4.3}$$

since, according to [9; p. 48], $\bar{U}$ contains the term $1 \times \mu$, where $\langle \mu, \bar{\mu}\rangle = 1$.

Then we have

$$H^*(M^2) \approx \mathrm{Im}(i^*) \oplus \mathrm{Ker}(t), \tag{4.4}$$

with $i^*\colon H^*(M^2, M^2 - \Delta) \approx \mathrm{Im}(i^*)$, $j^*\colon \mathrm{Ker}(t) \approx H^*(M^2 - \Delta)$.

Let us consider the action of the Steenrod algebra $A$ on the decomposition (4.4). Clearly, $\mathrm{Im}(i^*)$ is an $A$-module, and we need only to analyze the behaviour of $\mathrm{Ker}(t)$. If $V = \sum_{i=0}^{(n/2)} V_i$ is the total Wu class of $M$ defined in [9; p. 55], then we have

THEOREM 4.5. *For every $v \in H^*(M^2)$, the equality*

$$t\,\mathrm{Sq}\,v = \mathrm{Sq}\,t((V^{-1} \times V) \smile v)$$

*holds, where $V^{-1}$ is the inverse of $V$ in the sense of the cup-product.*

*Proof.* Obviously it is sufficient to consider an element of the form $v = x \times y$. We have

$$t\,\mathrm{Sq}\,v = t(\mathrm{Sq}\,x \times \mathrm{Sq}\,y) = \langle \mathrm{Sq}\,y, \bar{\mu}\rangle(\mathrm{Sq}\,x \times 1) \smile U';$$

but, by the definition of $V$, $\langle \mathrm{Sq}\, y, \bar{\mu}\rangle = \langle V \smile y, \bar{\mu}\rangle$ and, therefore,

$$t\, \mathrm{Sq}\, v = \langle V \smile y, \bar{\mu}\rangle(\mathrm{Sq}\, x \times 1) \smile U'.$$

Using the antiautomorphism $c\colon A \to A$ determined by $\mathrm{Sq}\, c(\mathrm{Sq}) = 1$, the above expression can be written as

$$(4.6) \qquad t\, \mathrm{Sq}\, v = \langle V \smile y, \bar{\mu}\rangle \mathrm{Sq}[(x \times 1) \smile c(\mathrm{Sq})U'].$$

Now, we will show that

$$(4.7) \qquad c(\mathrm{Sq})U' = (V^{-1} \times 1) \smile U'.$$

By definition of $\tau^*(c(\mathrm{Sq}))$, we have

$$(4.8) \qquad c(\mathrm{Sq})U = \pi^*\tau^*(c(\mathrm{Sq})) \smile U;$$

and applying Sq to both members of (4.8), we obtain

$$U = \pi^*\, \mathrm{Sq}\, \tau^*(c(\mathrm{Sq})) \smile \mathrm{Sq}\, U = \pi^*\, \mathrm{Sq}\, \tau^*(c(\mathrm{Sq})) \smile \pi^*\tau^*(\mathrm{Sq}) \smile U;$$

but $\pi^*$ and the cup-product by $U$ are isomorphisms, so

$$1 = \mathrm{Sq}\, \tau^*(c(\mathrm{Sq})) \smile \tau^*(\mathrm{Sq}),$$

and, since $\tau^*(\mathrm{Sq}) = W = \mathrm{Sq}\, V$ and Sq is an automorphism, it follows that $\tau^*(c(\mathrm{Sq})) = V^{-1}$. Substitution of this expression in (4.8) and application of $\psi$ yields (4.7). With (4.7) in (4.6), we obtain

$$t\, \mathrm{Sq}\, (v) = \langle V \smile y, \bar{\mu}\rangle \mathrm{Sq}[(x \times 1) \smile (V^{-1} \times 1) \smile U'],$$

and it is then easy to verify that

$$t[(V^{-1} \times V) \smile (x \times y)] = \langle V \smile y, \bar{\mu}\rangle(x \times 1) \smile (V^{-1} \times 1) \smile U'.$$

This finishes the proof of (4.5).

THEOREM 4.9. *The direct sum decomposition* $H^*(M^2) \approx \mathrm{Im}(i^*) \oplus \mathrm{Ker}(t)$ *is a decomposition as* $A$*-modules if and only if* $\tau \in E_0(M)$, *where* $\tau$ *is the tangent bundle of* $M$.

*Proof.* The sufficiency is an immediate consequence of (4.5), since the hypothesis implies that $V = 1$. For the necessity, if $q < n$ for every $x \in H^q(M)$, $1 \times x \in \mathrm{Ker}(t)$ and, by the hypothesis,

$$t\, \mathrm{Sq}^{n-q}(1 \times x) = t(1 \times \mathrm{Sq}^{n-q}x) = t(1 \times (V_{n-q} \smile x)) = \langle V_{n-q} \smile x, \bar{\mu}\rangle U' = 0.$$

Therefore, $V_{n-q} = 0$ and, consequently, $W = 1$.

THEOREM 4.10. *Let* $M$ *be a differentiable* $n$*-manifold;* $U \in H^n(E, E_0)$, *the fundamental class of the tangent bundle* $\tau$ *of* $M$; *and* $\bar{U} \in H^n(M^2)$, *the image of* $U$ *under* $i^*\psi$. *Then, for every stable secondary operation* $\Phi$ *of degree* $r$, *we have*

(i) $\Phi(U)$ *is defined if and only if* $\Phi(\bar{U})$ is defined;

(ii) $\Phi(\bar{U}) \neq 0$ *implies* $\Phi(U) \neq 0$;

(iii) $\Phi(\bar{U}) = (\tau^*(\Phi) \times 1) \smile \bar{U}$ *in* $H^{n+r}(M^2)/Q^{n+r}(\Phi; M^2)$; *and*
(iv) *if* $\tau \in E_0(M)$, *then*, $\Phi(U) \neq 0$ *if and only if* $\Phi(\bar{U}) \neq 0$.

*Proof.* Statement (i) is a direct consequence of (4.1). The naturality of $\Phi$ implies (ii), since $\Phi(\bar{U}) = i^*\psi\Phi(U)$. Again, (iii) is a consequence of the naturality of $\Phi$, since, by (1.5), $\Phi(U) = \pi^*\tau^*(\Phi) \smile U$ in $H^{n+r}(E, E_0)/Q^{n+r}(\Phi; E, E_0)$. With respect to (iv), since $\tau \in E_0(M)$, it follows from (4.5) that $t$ induces a homomorphism

$$t\colon H^{n+r}(M^2)/Q^{n+r}(\Phi;\, M^2) \to H^{n+r}(M^2,\, M^2 - \Delta)/Q^{n+r}(\Phi;\, M^2,\, M^2 - \Delta);$$

and we assert that $t\Phi(\bar{U}) = \Phi(U')$. In fact, using (iii), similarly to (4.3), we have

$$(4.11) \quad t\Phi(\bar{U}) = t[(\tau^*(\Phi) \times 1) \smile \bar{U}] = (\tau^*(\Phi) \times 1) \smile U' = \Phi(U').$$

But (4.11) implies (iv) since $\Phi(U') = \psi\Phi(U)$.

## 5. Normal secondary characteristic classes

Suppose $X$ is a finite $CW$-complex imbedded in a $t$-sphere $S^t$, and let $D_tX$ be a $t$-dual of $X$ in the sense of Spanier-Whitehead ([16]). Under the Alexander-Pontrjagin duality, for every operation $\theta\colon H^q(X) \to H^{q+r}(X)$, with $\theta \in A$, there corresponds its dual operation $c(\theta)\colon H^{t-q-r-1}(D_tX) \to H^{t-q-1}(D_tX)$, where $c\colon A \to A$ is the canonical antiautomorphism of $A$. The homomorphism $\theta$ is non-trivial if and only if the homomorphism $c(\theta)$ is non-trivial ([12], [16; p. 271]).

If $\Phi$ is a stable secondary operation associated with the relation (1.1), there exists a stable secondary operation $c(\Phi)$, associated with the relation $\sum_{k=1}^{j} c(\beta_k)c(\alpha_k) = 0$, such that, under the Alexander-Pontrjagin duality, an analogous result holds for $\Phi$ and $c(\Phi)$, as the one we have above for primary operations ([7], [11]). Clearly, all these results remain valid when $D_tX$ is replaced by an $S$-equivalent space $Y$, if the proper shift in dimension is taken into account. Applying these results to differentiable manifolds we obtain the following:

THEOREM 5.1. *Let* $\nu = (E, M, \pi)$ *be the normal bundle to a compact differentiable manifold* $M$ *immersed in* $R^t$. *Then for every* $\theta \in A$ *of degree* $r$, $\theta\colon H^q(E, E_0) \to H^{q+r}(E, E_0)$ *is non-trivial, if and only if* $c(\theta)\colon H^{t-q-r}(M) \to H^{t-q}(M)$ *is non-trivial. Similarly, for every stable secondary cohomology operation* $\Phi$ *of degree* $r$,

$$\Phi\colon K^q(\Phi; E, E_0) \to H^{q+r}(E, E_0)/Q^{q+r}(\Phi; E, E_0)$$

*is non-trivial, if and only if,*

$$c(\Phi)\colon K^{t-q-r}(c(\Phi); M) \to H^{t-q}(M)/Q^{t-q}(c(\Phi); M)$$

*is non-trivial.*

*Proof.* According to [9; p. 100] there is a natural isomorphism in positive dimensions $H^*(T(\nu)) \approx H^*(E, E_0)$, where $T(\nu)$ is the Thom space of $\nu$. Now if $\nu$, $\nu'$ are normal bundles of $M$ associated with two immersions, it follows from [4; (2.6)] that $T(\nu)$ and $T(\nu')$ are $S$-equivalent. In particular, if $\nu'$ is an imbedding from [10], we have that $T(\nu')$ is dual to the disjoint union of $M$ and a point. Therefore, for any immersion $\nu$, the Thom space $T(\nu)$ is an $S$-dual of $M \smile \infty$. Then (5.1) follows from the results mentioned at the beginning of this section.

As an application of (5.1) we obtain the following generalization of a result of Massey [8; Th. 1].

COROLLARY 5.2. *Let $M$ and $\nu$ be as in* (5.1) *and $\theta \in A$, an element of degree $n - q$. Then $\nu^*(\theta) \neq 0$ implies $\alpha(n) \leq q$, where $\alpha(n)$ is the number of non-zero terms in the dyadic expansion of $n$.*

*Proof*. According to §1, $\nu^*(\theta) \neq 0$ if and only if $\theta\colon H^t(E, E_0) \to H^{t+n-q}(E, E_0)$ is non-trivial. By (5.1), $\theta$ is non-trivial if and only if there exists $u \in H^q(M)$ such that $c(\theta)u \neq 0$. Now $c(\theta)u \neq 0$ implies that there exists an admissible monomial $\mathrm{Sq}^I$ of degree $n - q$, with $\mathrm{Sq}^I u \neq 0$. But then formula (17.5) of Serre ([15; p. 212]) implies $q \geq \alpha(n)$.

## 6. A family of secondary cohomology operations

In this section we construct and establish properties of a family of secondary cohomology operations, which will enable us to make some applications of secondary characteristic classes to the immersion problem of projective spaces.

Let $\Phi_{2j}$, with $j \geq 2$, be a family of stable secondary cohomology operations associated, according to Adams ([1]), with the relations:

$$(\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+2}\mathrm{Sq}^1 = 0, \text{ if } 2j = 4k + 2, \tag{6.1}$$

$$(\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{4k-2} + \mathrm{Sq}^{4k}\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^{4k} = 0, \text{ if } 2j = 4k. \tag{6.2}$$

Thus, $\Phi_{4k+2}$ is an operation associated with (6.1), and is defined in the intersection of the kernels of $\mathrm{Sq}^1$ and $\mathrm{Sq}^{4k}$. Similarly $\Phi_{4k}$ is associated with (6.2). Clearly the degree of $\Phi_{2j}$ is $2j$.

We now give a criterion which shows when $\Phi_{2j}$ vanishes for dimensional reasons. In order to establish this criterion, we need to restrict the natural domain of definition of $\Phi_{2j}$, which rigorously means to consider another operation. However, for simplicity, this restricted operation will be identified with $\Phi_{2j}$.

Let $A_k \subset A$ be the vector subspace of homogeneous elements of $A$ of degree $k$.

THEOREM 6.3. *Suppose $\Phi_{2j}(u)$ is defined for $u \in H^q(X)$. If $A_{2j}(u) = 0$ and $\mathrm{Sq}^3 u = 0$, then $\Phi_{2j}(u) = 0$ for $q \leq 2j - 3$.*

*Proof*. The stability of $\Phi_{2j}$ and the stability of the hypotheses imply that (6.3) holds for all $q \leq 2j - 3$ if it holds for $q = 2j - 3$. Hence we will give the proof for this particular dimension. Let $f\colon X \to K(Z_2, 2j - 3)$ be a map such that $f^*\gamma = u$, where $\gamma$ is the fundamental class of $K(Z_2, 2j - 3)$. Using the second formula of Peterson-Stein, with the notation of [3; (5.2)], we have

$$\Phi_{2j}(u) = \alpha_f\beta(\gamma)$$

in $H^{4j-3}(X)/Q^{4j-3}(\Phi_{2j}; X) + f^*H^{4j-3}(K(Z_2, 2j - 3))$, where $\alpha\beta = 0$ is the relation (6.1) or (6.2) according to the value of $2j$. The hypotheses on $u$ imply that $f^*H^{4j-3}(K(Z_2, 2j - 3)) = 0$, since all admissible monomials in $A_{2j}$ vanish on $u$ and the conditions $\mathrm{Sq}^1 u = 0$ and $\mathrm{Sq}^3 u = 0$ guarantee that all possible cup-products in the image of $f^*$ vanish. Consequently, $\Phi_{2j}(u) = \alpha_f\beta(\gamma)$, with the natural indeterminacy of $\Phi_{2j}$. Now, by inspection of the relations (6.1) and (6.2), one easily verifies that the hypotheses of [3; (6.6)] hold with $q = 2j - 3$. Then $\alpha_f\beta(\gamma) = 0$, and this finishes the proof of (6.3).

Let $I_1 \subset A$ be the two-sided ideal generated by $\mathrm{Sq}^1$.

THEOREM 6.4. *Let $u \in H^p(X)$, $v \in H^q(X)$ be classes such that $\Phi_{2j}(u)$, $\Phi_{2j}(v)$ are defined. Moreover, if $A_k(u) = 0$ for all $1 \leq k \leq 2j$, $A_{2j}(v) = 0$, and $\theta(v) = 0$ for all $\theta \in I_1$ of degree $\leq 2j$, then*

$$\Phi_{2j}(u \smile v) = u \smile \Phi_{2j}(v) + \textstyle\sum_{k=0}^{j-2} \Phi_{2j-2k}(u) \smile \mathrm{Sq}^{2k}v,$$

*modulo the total indeterminacy.*

*Proof.* Accordingly with [1; p. 80], it is sufficient to establish (6.4) for a single pair of dimensions $(p, q)$ in the stable range. We will give here only a rough indication of the proof. Let $R$ be the free Steenrod algebra with $\mathrm{Sq}^0 = 1$ and $R^+$ the ideal of positive dimensional elements. Let $\rho_{2j}$ denote (6.1) or (6.2), according to the value of $2j$, as an element of $R$. If $I$ is the ideal of relations in $R$, then $\rho_{2j} \in I$. If $\psi: R \to R \otimes R$ is the diagonal map, one can verify by direct computation that

$$\psi(\rho_{2j}) = 1 \otimes \rho_{2j} + \textstyle\sum_{k=0}^{j-2} \rho_{2j-2k} \otimes \mathrm{Sq}^{2k},$$

modulo the subspace $(I \otimes I_1 + R^+ \otimes I)_{2j}$. Now, using the functional representation of $\Phi_{2j}$ and the methods of [3], the proof of (6.4) follows from the above formula.

COROLLARY 6.5. *If $A_k(u) = 0$, $A_k(v) = 0$ for $1 \leq k \leq 2j$ then*

$$\Phi_{2j}(u \smile v) = \Phi_{2j}(u) \smile v + u \smile \Phi_{2j}(v)$$

COROLLARY 6.6. *If $A_k(u) = 0$ for $1 \leq k \leq 2j$, then for $h > 0$,*

$$\Phi_{2j}(u^h) = hu^{h-1} \smile \Phi_{2j}(u).$$

## 7. Secondary classes of complex projective spaces

In this and the next section we will work with $CP^m$, the complex projective space of real dimension $2m$, and with $CP^\infty$, the infinite dimensional complex projective space. In order to avoid confusion, let $\omega \in H^2(CP^m)$ and $w \in H^2(CP^\infty)$ denote the generators. The total Stiefel-Whitney class of $CP^m$ is given by

$W(CP^m) = (1+\omega)^{m+1}$. With (4.10) we can compute the secondary tangential classes $\tau^*(\Phi)$ by determining the action of $\Phi$ in the class $\bar{U} \in H^{2m}(CP^m \times CP^m)$. The class $\bar{U}$ is explicitly given by the Poincaré duality matrix ([9; Th. 15]). For $CP^m$ we have

$$\bar{U} = \sum_{i=0}^{m} \omega^i \times \omega^{m-i}, \tag{7.1}$$

where $\omega^j$ is the $j$th power of the generator $\omega$.

In this section we will consider *only* $CP^m$ with $m = 2a - 1$ where $a = 2^s$ and $s \geq 1$. Then, for this case $W(CP^m) = 1$ and the tangent bundle $\tau$ of $CP^m$ belongs to $E_0(CP^m)$. If $\gamma = \omega \times 1 + 1 \times \omega$, the form of $m$ allows us to express $\bar{U}$ as a power of $\gamma$, explicitly

$$\bar{U} = \gamma^m. \tag{7.2}$$

By (4.1), for $\theta \in A$ of positive degree, we have $\theta(\bar{U}) = 0$. Then $\Phi(\bar{U})$ is defined for every stable secondary cohomology operation, and from (1.8) it follows that $\Phi(\bar{U})$ is idependent of the particular operation $\Phi$ associated with a relation of type (1.1). The computation of $\Phi(\bar{U})$ is reduced to the computation of functional cohomology operations by means of the Peterson-Stein formula ([3; (5.2)]), as follows. Let

$$f\colon CP^m \times CP^m \to CP^\infty \tag{7.3}$$

be a map such that $f^*w = \gamma$. By (7.1) and (7.2), $f^*w^m = \gamma^m = \bar{U}$ and $f^*\beta(w^m) = \beta(\bar{U}) = 0$. Therefore,

$$\Phi(\bar{U}) = \alpha_f\beta(w^m) \tag{7.4}$$

in $H^{2m+r}(CP^m \times CP^m)/Q^{2m+r}(\Phi; CP^m \times CP^m)$, since

$$f^*H^{2m+r}(CP^\infty) = 0, \text{ for all } r > 0. \tag{7.5}$$

This last statement follows from $\gamma^{m+1} = 0$. Thus, $\Phi(\bar{U})$ admits a functional representation preserving its natural indeterminacy.

Clearly, if the degree of $\Phi$ is odd, $\Phi(\bar{U}) = 0$. Then it is sufficient to consider only secondary operations of even degree. For this case, we may express $\alpha_f\beta(w^m)$ by means of the functional operations $\alpha_{kf}\beta_k(w^m)$, as indicated by the following

PROPOSITION 7.6. *Let* $\alpha\beta = 0$ *be a relation of type* (1.1), *homogeneous of degree* $r+1$ *in* $A$, *where* $r$ *is even. Then,*

$$\alpha_f\beta(w^m) = \sum_{k=1}^{j} \alpha_{kf}\beta_k(w^m)$$

*in* $H^{2m+r}(CP^m \times CP^m)/\sum \alpha_k H^{2m+t_k-1}(CP^m \times CP^m)$.

*Proof.* Each term $\alpha_{kf}\beta_k(w^m)$ is defined. In fact, $f^*\beta_k(w^m) = 0$ and, since the degree of $\alpha_k\beta_k$ is odd, $\alpha_k\beta_k(w^m) = 0$. Using the definition of a functional operation, it is easily verified that a sum of representatives of the terms in the right hand is a representative of $\alpha_f\beta(w^m)$. By (7.5), the common indeterminacy of the functional operations is the one indicated in the proposition, and this finishes the proof.

It follows from (7.6) that it is sufficient to study simple functional operations of the type $\theta_f\sigma(w^m)$, where $\theta$ and $\sigma$ are homogeneous elements of $A$ and $\theta\sigma$ of odd degree. Obviously, if the degree of $\sigma$ is odd, then $\theta_f\sigma(w^m) = 0$. Therefore we will suppose that the degree of $\sigma$ is even. Consequently, $\theta$ is of odd degree and $\theta \in I_1$, the two sided ideal generated by $\mathrm{Sq}^1$.

PROPOSITION 7.7. *Let $\theta \in A$ be an element of odd degree and hence of the form $\theta = \sum a_i \mathrm{Sq}^1 b_i$. Then, for every $\sigma \in A$ of positive degree, $\theta_f\sigma(w^m)$ is defined, and with zero indeterminacy we have*

$$\theta_f\sigma(w^m) = \sum a_i(\mathrm{Sq}_f^1 b_i\sigma(w^m)).$$

The proof is omitted since it is similar to that of (7.6).

In conclusion, (7.4), (7.6) and (7.7) reduce the calculation of $\Phi(\bar{U})$ to the computation of $\mathrm{Sq}_f^1$. In general, $\mathrm{Sq}_f^1$ can be computed using the following well known proposition, whose proof is also omitted.

PROPOSITION 7.8. *Let $f: X \to Y$ be a map and $f^*: H^q(Y; Z) \to H^q(X; Z)$ the induced homomorphism in cohomology with integer coefficients. For $u_1 \in H^q(Y; Z)$ suppose that $f^*u_1 = 2kv_1$ where $k$ is an integer and $v_1 \in H^q(X; Z)$.* If $u \in H^q(Y)$, *$v \in H^q(X)$ are the reductions modulo 2 of $u_1$ and $v_1$ respectively, then* $\mathrm{Sq}_f^1 u$ *is defined and has $kv$ as a representative.*

Specializing (7.8) to the situation of (7.3) we obtain the following:

*Proposition 7.9. With zero indeterminacy, we have* $(q \geq 1)$,

$$\mathrm{Sq}_f^1(w^{m+q}) = \gamma^{q-1} \smile (\omega^a \times \omega^a).$$

*Proof.* Let $w_1 \in H^2(CP^\infty; Z)$ and $\omega_1 \in H^2(CP^m; Z)$ be integral generators, and set $\gamma_1 = \omega_1 \times 1 + 1 \times \omega_1$. Then $f^*w_1^{m+q} = \gamma_1^{m+q} = \gamma_1^{q-1} \smile \gamma_1^{m+1}$, but $\gamma_1^{m+1} = 2\omega_1^a \times \omega_1^a \bmod 4$. Therefore, $f^*w_1^{m+q} = 2\gamma_1^{q-1} \smile (\omega_1^a \times \omega_1^a) \bmod 4$. Now, by (7.5) the indeterminacy is zero and from (7.8) the proof of (7.9) follows.

We are now in the position to compute any operation $\Phi$ in the class $\bar{U}$. In particular for the operations $\Phi_{2j}$ of §6, we have

THEOREM 7.10. *If $2j \neq m + 1$, then $\Phi_{2j}(\bar{U}) = 0$. If $2j = m + 1$, then the indeterminacy is zero and*

$$\Phi_{m+1}(\bar{U}) = (\omega^a \times 1) \smile \bar{U}.$$

*Proof.* Using (7.4), (7.6) and (7.7) we obtain

$$\Phi_{4k}(\bar{U}) = \mathrm{Sq}_f^1\mathrm{Sq}^{4k}w^m + \mathrm{Sq}^2(\mathrm{Sq}_f^1\mathrm{Sq}^{4k-2}w^m), \tag{7.11}$$

modulo $\mathrm{Sq}^{4k}H^{2m}(CP^m \times CP^m)$, if $2j = 4k$, and

$$\Phi_{4k+2}(\bar{U}) = \mathrm{Sq}^2(\mathrm{Sq}_{f}{}^1\mathrm{Sq}^{4k}w^m), \tag{7.12}$$

modulo $\mathrm{Sq}^{4k+2}H^{2m}(CP^m \times CP^m)$, if $2j = 4k + 2$.

By applying (7.9) to the last term in the right hand side of (7.11) we get

$$\mathrm{Sq}^2(\mathrm{Sq}_{f}{}^1\mathrm{Sq}^{4k-2}w^m) = \mathrm{Sq}^2(\mathrm{Sq}^1 w^{m+2k-1}) = \mathrm{Sq}^2(\gamma^{2k-2} \smile (\omega^a \times \omega^a)) = 0.$$

Again, using (7.9), the relations (7.11) and (7.12) reduce to the single relation

$$\Phi_{2j}(\bar{U}) = \gamma^{j-1} \smile (\omega^a \times \omega^a), \tag{7.13}$$

modulo $\mathrm{Sq}^{2j}H^{2m}(CP^m \times CP^m)$. Now, using (4.11) in (7.13), we obtain

$$t\Phi_{2j}(\bar{U}) = \Phi_{2j}(U') = t(\gamma^{j-1} \smile (\omega^a \times \omega^a)),$$

modulo zero, since $\mathrm{Sq}^{2j}U' = 0$. On the other hand, $i^*\Phi_{2j}(U') = \Phi_{2j}(\bar{U})$. Therefore, except for the indeterminacy, it is sufficient to show that

$$t(\gamma^{j-1} \smile (\omega^a \times \omega^a)) = \begin{cases} 0, \text{ if } 2j \neq m + 1 \\ (\omega^a \times 1) \smile U', \text{ if } 2j = m + 1 \end{cases} \tag{7.14}$$

The definition of $t$ gives

$$t(\gamma^{j-1} \smile (\omega^a \times \omega^b)) = \binom{j-1}{a-1} (\omega^j \times 1) \smile U'.$$

If $j < a$, then $\binom{j-1}{a-1} = 0$. If $j > a$ and $\binom{j-1}{a-1} \not\equiv 0 \bmod 2$, then, since $a = 2^s$, we have $j = (k + 1)a$ for some $k \geq 1$, and in this case $\omega^j = 0$. The case $j = a$ follows directly, and this establishes (7.14). Finally, we need only to show that the indeterminacy of $\Phi_{m+1}(\bar{U})$ is zero, but this follows by a straightforward calculation using the Cartan formula.

THEOREM 7.15. *Let $\tau$ be the tangent bundle of $CP^{2a-1}$, where $a = 2^s$ and $s \geq 1$. Then, with zero indeterminacy, we have $\tau^*(\Phi_{2j}) = 0$, if $j \neq a$, and $\tau^*(\Phi_{2a}) = \omega^a$.*

*Proof.* It is a direct consequence of (7.10) and (4.10).

THEOREM 7.16. *Let $\nu$ be the normal bundle to an immersion of $CP^{2a-1}$, where $a = 2^s$ and $s \geq 1$. Then, with zero indeterminacy, we have $\nu^*(\Phi_{2j}) = 0$, if $j \neq a$, and $\nu^*(\Phi_{2a}) = \omega^a$.*

*Proof.* It follows from (7.15) and (3.3).

## 8. Cohomology operations in complex projective spaces

In order to extend the preceding results on normal characteristic classes to other projective spaces, we will compute the operations $\Phi_{2j}$ in $H^*(CP^\infty)$. As a preliminary we establish the following:

LEMMA 8.1. *Suppose $H^*(X)$ is a polynomial algebra over $Z_2$ on a single generator $x$ of dimension $q = 1$ or $2$. Let $qn = 2^{t_1} + \cdots + 2^{t_s}$ be the dyadic expansion of* dim $x^n$. *Then* $A_r(x^n) \neq 0$ *if and only if* $qn + r = 2^{k_1} + \cdots + 2^{k_s}$, *where the* $k_i$ *are not necessarily distinct and* $k_i \geq t_i$, *for* $i = 1, \cdots, s$.

*Proof.* Set $J(k, t) = \mathrm{Sq}^{2^{t+k}} \cdots \mathrm{Sq}^{2^{t+1}}\mathrm{Sq}^{2^t}$, for $k \geq 0$ and $t \geq 0$, and $J(-1, t) = \mathrm{Sq}^0$, for $t \geq 0$. We will show that the condition is necessary by induction on $\alpha(qn)$, the number of non-zero terms in the dyadic expansion of $qn$. If $\alpha(qn) = 1$, then $qn = 2^t$, for some $t$, and in this case the only admissible monomials that act non-trivially on $x^n$ are of the form $J(k, t)$. Then, $r + nq = \dim J(k, t)x^n = 2^{t+k+1}$, for some $k$. Now suppose it is true for $\alpha(qn) < s$ and all $r$, and consider $\alpha(qn) = s$. Write $x^n = x^{n-2^t} \smile x^{2^t}$, where $q2^t$ is the last term of the dyadic expansion of $qn$. If $\theta \in A_r$ is such that $\theta(x^n) \neq 0$ and $\psi(\theta) = \sum \theta_i \otimes \theta_i'$ under the diagonal map then, for some $i$, $\theta_i(x^{n-2^t}) \neq 0$ and $\theta_i'(x^{2^t}) \neq 0$. Let $r_1$, $r_2$ be the degrees of $\theta$ and $\theta_i'$ respectively. Since $r = r_1 + r_2$, the induction hypotheses on $r_1 + q(n - 2^t)$ and $r_2 + q2^t$ yield the result for $r + qn$.

To prove that the condition is sufficient, with the notation of the lemma, we may clearly suppose that $k_1 \leq k_2 \leq \cdots \leq k_s$. Then it is immediate to verify that $J(k_1 - t_1 - 1, t_1) \cdots J(k_s - t_s - 1, t_s)x^n \neq 0$, and that its dimension is $qn + r$. This finishes the proof.

COROLLARY 8.2. *If* $\alpha(qn + r) > \alpha(qn)$, *then* $A_r(x^n) = 0$.

Returning to our objective, we will first compute the operations $\Phi_{2j}$ in some elements of $H^*(CP^\infty)$.

THEOREM 8.3. *Let* $w \in H^2(CP^\infty)$ *be the generator*, $a = 2^r$ *with* $r \geq 1$ *and* $h > 0$. *Then, for* $j < 2a - 1$, $\Phi_{2j}(w^{2ha})$ *is defined and, with zero indeterminacy, we have*

$$\Phi_{2a}(w^{2ha}) = hw^{(2h+1)a}, \textit{ and}$$

$$\Phi_{2j}(w^{2ha}) = 0, \textit{ if } j \leq 2a - 1 \textit{ and } j \neq a.$$

*Proof.* Since $a = 2^r$, it follows that $\mathrm{Sq}^i w^{2ha} = 0$ for $i = 1, \cdots, 4a - 1$. Then, if $j < 2a - 1$, by (6.1), (6.2) $\Phi_{2j}(w^{2ha})$ is defined and has zero indeterminacy. Applying (6.6) we have $\Phi_{2j}(w^{2ha}) = hw^{(2h-2)a} \smile \Phi_{2j}(w^{2a})$. Therefore, for the proof of (8.3) it is enough to consider the case $h = 1$. For this, choose an imbedding of $CP^{2a-1}$ in $R^{8a-4}$, and let $\nu = (E, CP^{2a-1}, \pi)$ be the normal bundle of this imbedding. The pair $(E, E_0)$ behaves like a dual of $CP^{2a-1}$ (see proof of (5.1)), and, if $U \in H^{4a-2}(E, E_0)$ is the fundamental class of $\nu$, we have $\mathrm{Sq}^i U = 0$ for all $i > 0$. On the other hand the Atiyah-James duality for complex projective spaces ([4; p. 307]) asserts that an $S$-dual of $CP^{2a-1}$ is of the form $X = CP^m/CP^{n-1}$, for some value of $n$ which depends on $a$, and $m = n + 2a - 2$. The identification map $CP^m \to X$ allows us to identify $\omega^n$ with the non-trivial element of $H^{2n}(X)$. Since the action of $A$ in $U$ is trivial, it is also trivial in $\omega^n$ as an element of $H^{2n}(X)$. Then $\mathrm{Sq}^i\omega^n = 0$, for $i = 1, \cdots, 4a - 2$. This, and the action of $\mathrm{Sq}^i$ in $H^*(CP^\infty)$, implies that the only possible values of $n$ are those of the form $n = 2ga$. On the other hand, from (7.16) we have $\Phi_{2j}(U) \neq 0$ if and only if $j = a$. Consequently, in $H^*(X)$, $\Phi_{2j}(\omega^{2ga}) \neq 0$ if and only if $j = a$. Naturality under the map $CP^m \to X$ gives the same statement in $H^*(CP^m)$. Here, applying (6.6), we obtain

$$\Phi_{2j}(\omega^{2ga}) = g\omega^{(2g-2)a} \smile \Phi_{2j}(\omega^{2a}), \text{ for all } j \geq 2. \tag{8.4}$$

From (8.4) with $j = a$, it follows that $g$ is odd and that $\Phi_{2a}(\omega^{2a}) = \omega^{3a}$. In the same form, if $j \neq 0$, then, since $g$ is odd, we have $\Phi_{2j}(\omega^{2a}) = 0$. Now, under the inclusion $CP^m \to CP^\infty$, these last statements translate into (8.3) for $h = 1$, and this ends the proof.

THEOREM 8.5. *Let $w \in H^2(CP^\infty)$ be the generator, and let $a = 2^r$, $b = 2^s$, and $c = 2^t$ with $r > s \geq t \geq 0$, $h > 0$; then, with zero indeterminacy, we have*

$$\Phi_{2(a+b-c)}(w^{(2h+1)a-b-c}) = hw^{2(ha+a-c)}.$$

*Proof.* The hypotheses of (6.4) hold for $u = w^{2ha}$ and $v = w^{a-b-c}$. In fact, since $\mathrm{Sq}^i w^{2ha} = 0$ for $0 < i < 4a$, we have $A_i(w^{2ha}) = 0$ for $1 \leq i \leq 2(a + b - c)$. Trivially $I_1(w^{a-b-c}) = 0$. To verify that $A_{2(a+b-c)}(w^{a-b-c}) = 0$, we apply (8.2), showing easily that $\alpha(4a - 4c) < \alpha(2a - 2b - 2c)$. The total indeterminacy is zero. By (6.3) we have $\Phi_{2(a+b-c)}(w^{a-b-c}) = 0$. From (8.3), if $k \neq b - c$, we have $\Phi_{2(a+b-c)-2k}(w^{2ha}) = 0$. Therefore, the product formula (6.4) reduces to

$$\Phi_{2(a+b-c)}(w^{(2h+1)a-b-c}) = \Phi_{2a}(w^{2ha}) \smile \mathrm{Sq}^{2(b-c)}w^{a-b-c},$$

and, by applying (8.3) again, we obtain (8.5).

THEOREM 8.6. *Suppose $a = 2^r$, $b = 2^s$, and $c = 2^t$ with $r > s \geq t \geq 0$, and let $\nu = (E, CP^{2a-1}, \pi)$ be the normal bundle of an immersion of $CP^{2a-1}$ in $R^{4a-2+k}$. If $U \in H^k(E, E_0)$ is the fundamental class of $\nu$ then, with zero indeterminacy, we have*

$$\Phi_{2(a+b-c)}(\pi^*\omega^{a-b-c} \smile U) = \pi^*\omega^{2(a-c)} \smile U.$$

*Proof.* As in the proof of (8.3), it is clearly sufficient to compute the operation in the corresponding dimension of an $S$-dual of $CP^{2a-1}$. This $S$-dual is of the form $X = CP^{2(ga+a-1)}/CP^{2ga-1}$, with $g$ an odd number. The fundamental class $U$ corresponds to $\omega^{2ga}$, and the element $\pi^*\omega^{a-b-c} \smile U$ corresponds to $\omega^{(2g+1)a-b-c}$. The value of $\Phi_{2(a+b-c)}$ in this class is given by (8.5), and since $g$ is odd, the result of (8.6) follows.

As an application of (8.6) we compute the dual operations $c(\Phi_{2(a+b-c)})$ on some elements of $H^*(CP^\infty)$. This may be regarded as a dual statement of (8.5). With $a$, $b$, $c$ as above, we have the following.

THEOREM 8.7. *If $w \in H^2(CP^\infty)$ is the generator, then with zero indeterminacy we have*

$$c(\Phi_{2(a+b-c)})(w^{2c-1}) = w^{a+b+c-1}.$$

*Proof.* It is a direct consequence of (5.1), (8.5) and naturality under the inclusion $CP^{2a-1} \to CP^\infty$.

*Remarks.*

(1) If $c = 1$, we obtain $c(\Phi_{2(a+b-1)})(w) = w^{a+b}$. This gives a family of stable secondary cohomology operations of arbitrarily high degree acting non-trivially on a two dimensional class. For $a = 2$ and $b = 1$, this formula specializes to $c(\Phi_4)(w) = w^3$, which is Theorem 4.4.1 of [1].

(2) If $a = 2b$ and $b = c$, we have $c(\Phi_{2a})(w^{a-1}) = w^{2a-1}$. This implies that the iteration $c(\Phi_{2^k a}) \cdots c(\Phi_{4a})c(\Phi_{2a})(w^{2a-1}) = w^{2^k a-1}$ in $H^*(CP^\infty)$, for all $k > 0$. Combining this with the action of the Steenrod algebra, it follows that any element of $H^*(CP^\infty)$ can be obtained, with zero indeterminacy, by means of an iteration of squares and the operations $c(\Phi_{2^r})$ applied to the fundamental class $w$.

## 9. Non-immersion of complex projective spaces

Using the results of the preceding section we obtain the following generalization of (7.16).

THEOREM 9.1. *Let $\nu$ be the normal bundle to an immersion of $CP^{a+b+c-1}$, where $a = 2^r$, $b = 2^s$, and $c = 2^t$ with $r > s \geq t \geq 0$. Then, with zero indeterminacy, we have $\nu^*(\Phi_{2(a+b-c)}) = \omega^{a+b-c}$.*

*Proof.* It follows directly from (5.1), (8.7) and naturality.

Similar to the use of normal Stiefel-Whitney classes, we will show how normal secondary classes can be applied to establish non-immersion results.

THEOREM 9.2. *The complex projective space $CP^n$ cannot be immersed in $R^{4n-5}$ if $n = 2^r + 2^s$ and $r > s \geq 0$.*

*Proof.* Suppose $CP^n$ admits such an immersion, and let $\nu$ be the normal bundle. Then, from (9.1) with $c = 1$, we have $\nu^*(\Phi_{2n-2}) \neq 0$. This is equivalent with $\Phi_{2n-2}(U) \neq 0$, where $U \in H^{2n-5}(E, E_0)$ is the fundamental class of $\nu$. On the other hand, $U$ satisfies the hypotheses of (6.3). In fact we need only to verify for $\theta \in A_{2n-2}$ that $\theta(U) = 0$. But, by (5.1), this is equivalent with $c(\theta)(\omega) = 0$, and this follows easily by (8.2). Consequently $\Phi_{2n-2}(U) = 0$, and the contradiction establishes (9.2).

*Remarks.* Using the associated sphere bundle we can improve (9.2) for $s > 0$, in one more unit (see proof of (12.3)). However, as in the case of $CP^{2^r}$ and Stiefel-Whitney classes, we cannot obtain directly the best possible non-immersion result. This has been obtained recently by Sanderson and Schwarzenberger (in [14]), who prove that $CP^n$, with $n = 2^r + 2^s$ and $s > 0$, cannot be immersed in $R^{4n-3}$. For the case $s = 0$, whether $CP^n$ can or cannot be immersed in $R^{4n-4}$ has not been settled.

## 10. Multiple secondary cohomology operations

In order to establish results for the real projective spaces $RP^n$ similar to those obtained for $CP^n$, we need to consider stable secondary operations which act non-trivially in the cohomology of $RP^\infty$. The operations $\Phi_{2j}$ are not adequate for this purpose because of their indeterminacy.

Adams in [2] has indicated how to construct double secondary operations associated with a pair of relations and whose indeterminacy is the diagonal indeterminacy. He has shown that these operations act non-trivially in $RP^\infty$. The construction of double operations extends automatically to $n$-tuple operations, which are associated with $n$ relations. In our applications we will consider only double and triple operations.

To simplify the notation, we describe the general construction of these operations and the formula of Peterson-Stein only in the case of double operations. The extension of these results to $n$-tuple operations is immediate. In the last part of this section we construct for $k > 0$ a family of double operations $\Psi_{8k}$ and another family of triple operations $\Theta_{8k+4}$. For the triple operations we will use the direct generalization of the results which we establish only for double operations.

We now describe the form of constructing such operations. If $G$ is an abelian group and $G \oplus G$ is the direct sum of two copies of $G$, the diagonal subgroup $\Delta G \subset G \oplus G$ is defined as the image of the homomorphism

$$\Delta: G \to G \oplus G, \tag{10.1}$$

where $\Delta(g) = (g, g)$, for all $g \in G$.

Let

$$\alpha\beta = \textstyle\sum_{k=1}^{m} \alpha_k\beta_k = 0 \text{ and} \tag{10.2}$$

$$\theta\beta = \textstyle\sum_{k=1}^{m} \theta_k\beta_k = 0 \tag{10.3}$$

be two homogeneous relations in $A$ of degrees $a + 1, b + 1$, respectively. All the $\alpha_k$, $\theta_k$, $\beta_k$ are of positive degree and, in general, some of the $\alpha_k$, $\theta_j$ may be the zero operation. Set $t_k =$ degree $\beta_k$. Associated with the relations (10.2) and (10.3), a stable secondary cohomology operation $\Psi$ is defined. For a space $X$, its domain of definition $K^q(\Psi; X)$ is the subgroup of $H^q(X)$ formed by all the elements $u$ such that $\beta(u) = 0$; i.e., $\beta_k(u) = 0$ for $k = 1, \cdots, m$. The value $\Psi(u)$ is a coset in the direct sum $H^{q+a}(X) \oplus H^{q+b}(X)$ modulo

$$Q(\Psi; X) = \textstyle\sum_{k=1}^{m}(\alpha_k \oplus \theta_k)\Delta H^{q+t_k-1}(X), \tag{10.4}$$

where $\Delta$ is defined by (10.1). Thus

$$\Psi: K^q(\Psi; X) \to H^{q+a}(X) \oplus H^{q+b}(X)/Q(\Psi; X). \tag{10.5}$$

Briefly, the construction of $\Psi$ is as follows. Let $\pi: E \to K(Z_2, q)$ be a fibre space determined by $\beta$ and which is a universal example for secondary operations associated with relations of type (10.2), (10.3). It is sufficient to construct $\Psi$ in the stable range. Then, with $q > \max(a, b)$, as in [2], we choose elements $x \in H^{q+a}(E)$ and $y \in H^{q+b}(E)$ associated with (10.2) and (10.3) respectively. Now, given $u \in H^q(X)$, let $f: X \to K(Z_2, q)$ be a characteristic map for $u$. If

$\beta(u) = 0$, there exists $h: X \to E$ such that $\pi h = f$. Define $\Psi(u) = \{h^*(x) \oplus h^*(y)\}$, where the parentheses indicate the coset of the direct sum modulo $Q(\Psi; X)$. One verifies that another choice of $h$ does not alter the coset.

Let $\Phi_1$, $\Phi_2$ be the simple secondary operations determined by $x$, $y$ respectively. The indeterminacy of the direct sum $[\Phi_1(u), \Phi_2(u)]$ is in general larger than that of $\Psi(u)$.

In order to establish the second formula of Peterson-Stein, we need to consider double functional cohomology operations. Consider the relations $\alpha\beta = 0$, $\theta\beta = 0$ as the composition of operations of several variables (see [3; section 2]). Given an inclusion $f: X \to Y$, with the cohomology sequence of the pair $(Y, X)$, we form the following commutative diagram:

$$\begin{array}{ccccccc}
\xrightarrow{\delta} \Delta[\oplus H^{q+t_k}(Y, X)] & \xrightarrow{j^*} & \Delta[\oplus H^{q+t_k}(Y)] & \xrightarrow{f^*} & \Delta[\oplus H^{q+t_k}(X)] \xrightarrow{\delta} \\
\downarrow \alpha \oplus \theta & & \downarrow \alpha \oplus \theta & & \downarrow \alpha \oplus \theta \\
\xrightarrow{\delta} H^c(Y, X) \oplus H^d(Y, X) & \xrightarrow{j^*} & H^c(Y) \oplus H^d(Y) & \xrightarrow{f^*} & H^c(X) \oplus H^d(X) \xrightarrow{\delta}
\end{array}$$

where $c = q + a + 1$ and $d = q + b + 1$. The horizontal rows are exact sequences and the direct sums in the upper row run over $1 \le k \le m$. The vertical operations are defined in the obvious way. Now, if $u \in H^q(Y)$, it follows that $\beta(u) \in \oplus H^{q+t_k}(Y)$ and that $(\alpha \oplus \theta)\Delta\beta(u) = 0$. Thus, if we suppose $f^*\beta(u) = 0$, in the usual form, using the above diagram, we define the functional operation.

$$(10.6) \qquad (\alpha \oplus \theta)_f \Delta\beta(u),$$

which is a coset of $H^{q+a}(X) \oplus H^{q+b}(X)$ modulo the subgroup

$$(10.7) \qquad (\alpha \oplus \theta)\Delta[\oplus H^{q+t_k-1}(X)] + f^*H^{q+a}(Y) \oplus f^*H^{q+b}(Y).$$

THEOREM 10.8. *Let $\Psi$ be a double secondary operation associated with the pair of relations $\alpha\beta = 0, \theta\beta = 0$. If $f: X \to Y$ is a map and $u \in H^q(Y)$ is such that $f^*\beta(u) = 0$, then the operations $\Psi(f^*u)$ and $(\alpha \oplus \theta)_f\Delta\beta(u)$ are defined, and we have*

$$\Psi(f^*u) = (\alpha \oplus \theta)_f \Delta\beta(u),$$

*modulo the total indeterminacy*

$$Q(\Psi; X) + f^*H^{q+a}(Y) \oplus f^*H^{q+b}(Y).$$

*Proof.* We give the proof in the universal example $\pi: E \to B$, where $B = K(Z_2, q)$ and $q$ is in the stable range. Let $F \subset E$ be the fibre over the point $z \in B$, $\pi_1: (E, F) \to (B, z)$ the map induced by $\pi$ and $j: E \to (E, F)$ the inclusion. We identify $\pi = \pi_1 j$. If $\gamma \in H^q(B)$ is the fundamental class, then $j^*\Delta\beta(\pi_1^*\gamma) = \pi^*\Delta\beta(\gamma) = 0$ and $(\alpha \oplus \theta)\Delta\beta(\gamma) = 0$. Using the naturality for functional cohomology operations ([17; 15.8]) we have

$$(10.9) \qquad (\alpha \oplus \theta)_j \Delta\beta(\pi_1^*\gamma) = (\alpha \oplus \theta)_\pi \Delta\beta(\gamma),$$

modulo the common indeterminacy, which reduces to

$$(\alpha \oplus \theta)\Delta[\oplus H^{q+t_k-1}(E)] + \pi^* H^{q+a}(B) \oplus \pi^* H^{q+b}(B), \tag{10.10}$$

because in the stable range $\pi_1{}^*$ is an isomorphism and we have $j^* H^{q+a}(E, F) = \pi^* H^{q+a}(B)$, with a similar result for $q + b$. We now form the following commutative diagram:

$$\begin{tikzcd}
\Delta[\oplus H^{q+t_k-1}(E)] \arrow[r, "i^*"] & \Delta[\oplus H^{q+t_k-1}(F)] \arrow[r, "\tau"] \arrow[dr, "\delta"'] & \Delta[\oplus H^{q+t_k}(B)] \arrow[r, "\pi^*"] \arrow[d, "\pi_1{}^*"] & \Delta[\oplus H^{q+t_k}(E)] \\
& & [\oplus H^{q+t_k}(E, F)] \arrow[ur, "j^*"'] &
\end{tikzcd} \tag{10.11}$$

where $\tau$ is the transgression. The horizontal sequence is exact and the broken sequence is that of the pair $(E, F)$. A representative of $\Psi(\pi^*\gamma)$ is constructed as follows. Since $\pi^*\Delta\beta(\gamma) = 0$, there exists $\Delta w$ such that $\tau\Delta w = \Delta\beta(\gamma)$. Form $(\alpha \oplus \theta)\Delta w$; then, since $\tau(\alpha \oplus \theta)\Delta w = (\alpha \oplus \theta)\Delta\beta(\gamma) = 0$, there exists $x \oplus y \in H^{q+a}(E) \oplus H^{q+b}(E)$ such that

$$i^*x \oplus i^*y = (\alpha \oplus \theta)\Delta w. \tag{10.12}$$

The element $x \oplus y$ is a representative of $\Psi(\pi^*\gamma)$. To show that this element is also a representative of $(\alpha \oplus \theta)_j\Delta\beta(\pi_1{}^*\gamma)$ we make the following observation; every functionalization of a cohomology operation with respect to the inclusion $j\colon E \to (E, F)$ can be constructed using the cohomology exact sequence of the pair $(E, F)$. Indeed, it can be verified that the sequence of [17; p. 978] constructed with $j$ is term by term isomorphic in a natural way with that of the pair $(E, F)$. Then, to compute $(\alpha \oplus \theta)_j\Delta\beta(\pi_1{}^*\gamma)$, we may use the broken sequence of (10.11). From the commutativity of the diagram we have $\delta\Delta w = \Delta\beta(\pi_1{}^*\gamma)$. Then, by (10.12), it follows that $x \oplus y$ is a representative of $(\alpha \oplus \theta)_j\Delta\beta(\pi_1{}^*\gamma)$; but, by (10.9), $x \oplus y$ is also a representative of $(\alpha \oplus \theta)_\pi\Delta\beta(\gamma)$. Since the indeterminacy of $\Psi(\pi^*\gamma)$ is smaller than that of $(\alpha \oplus \theta)_\pi\Delta\beta(\gamma)$, we obtain

$$\Psi(\pi^*\gamma) = (\alpha \oplus \theta)_\pi\Delta\beta(\gamma),$$

modulo the indeterminacy (10.10), and this finishes the proof.

Consider now the family of double secondary operations $\Psi_{8k}$ with $k > 0$, introduced by Adams in [2], which are associated with the following relations:

$$\mathrm{Sq}^1\mathrm{Sq}^{8k} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k}\mathrm{Sq}^1 = 0, \tag{10.13}$$

$$\mathrm{Sq}^2\mathrm{Sq}^{8k} + \mathrm{Sq}^4\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k}\mathrm{Sq}^2 + \mathrm{Sq}^{8k+1}\mathrm{Sq}^1 = 0. \tag{10.14}$$

Consider also a family of triple secondary operations $\Theta_{8k+4}$ with $k > 0$, associated with the following triple of relations:

$$\mathrm{Sq}^1\mathrm{Sq}^{8k+4} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{8k+2} + \mathrm{Sq}^{8k+4}\mathrm{Sq}^1 = 0, \tag{10.15}$$

$$\mathrm{Sq}^5\mathrm{Sq}^{8k+2} + \mathrm{Sq}^{8k+5}\mathrm{Sq}^2 = 0, \tag{10.16}$$

$$\mathrm{Sq}^4\mathrm{Sq}^{8k+4} + \mathrm{Sq}^{8k+6}\mathrm{Sq}^2 + \mathrm{Sq}^{8k+7}\mathrm{Sq}^1 = 0. \tag{10.17}$$

The relations (10.13) and (10.15) are the relations (6.2) which were used to construct the simple operations $\Phi_{4k}$. As an application of the Peterson-Stein formula (10.8) for double and triple operations, we give a criterion for $\Psi_{8k}$ and $\Theta_{8k+4}$ to vanish by dimensional reasons.

THEOREM 10.18. *Let $u \in H^q(X)$ be such that $\Psi_{8k}(u)$ is defined. If $A_{8k}(u) = 0$, $A_{8k+1}(u) = 0$, and $\mathrm{Sq}^4 u = 0$, then $\Psi_{8k}(u) = 0$ for all $q \leq 8k - 3$.*

THEOREM 10.19. *Let $u \in H^q(X)$ be such that $\Theta_{8k+4}(u)$ is defined. If $A_{8k+4}(u) = 0$, $A_{8k+6}(u) = 0$, $A_{8k+7}(u) = 0$, and $\mathrm{Sq}^6 u = 0$, then $\Theta_{8k+4}(u) = 0$ for all $q \leq 8k + 1$.*

The proofs of (10.18) and (10.19) are entirely analogous to the proof of (6.3) and for this reason are omitted.

## 11. Cohomology operations in real projective spaces

We will compute the operations $\Psi_{8k}$ and $\Theta_{8k+4}$ in $H^*(RP^\infty)$. If $x \in H^1(RP^\infty)$ is the generator, the cases of interest are those in which $\Psi_{8k}(x^m)$ is defined and its indeterminacy,

$$Q(\Psi_{8k}\,;\, RP^\infty) \subset H^{m+8k}(RP^\infty) \oplus H^{m+8k+1}(RP^\infty),$$

determined by (10.4), is generated by the element $[x^{m+8k}, x^{m+8k+1}]$—in symbols

$$Q(\Psi_{8k}\,;\, RP^\infty) = \{[x^{m+8k}, x^{m+8k+1}]\}. \tag{11.1}$$

Similarly, we consider the cases in which $\Theta_{8k+4}(x^m)$ is defined and its indeterminacy,

$$Q(\Theta_{8k+4}\,;\, RP^\infty) \subset H^{m+8k+4}(RP^\infty) \oplus H^{m+8k+6}(RP^\infty) \oplus H^{m+8k+7}(RP^\infty),$$

is generated by the elements $[x^{m+8k+4}, x^{m+8k+6}, 0]$ and $[x^{m+8k+4}, 0, x^{m+8k+7}]$. That is,

$$Q(\Theta_{8k+4}\,;\, RP^\infty) = \{[x^{m+8k+4}, x^{m+8k+6}, 0], [x^{m+8k+4}, 0, x^{m+8k+7}]\}. \tag{11.2}$$

A direct verification establishes the following two propositions.

PROPOSITION 11.3. *The operation $\Psi_{8k}(x^m)$ is defined and its indeterminacy is (11.1) if and only if $m = 8n$ and $\binom{n}{k} \equiv 0$ modulo 2.*

PROPOSITION 11.4. *The operation $\Theta_{8k+4}(x^m)$ is defined and its indeterminacy is (11.2) if and only if $m = 8n + 4$ and $\binom{n}{k} \equiv 0$ modulo 2.*

Following Adams ([2]), we compute $\Psi_{8k}(x^m)$ and $\Theta_{8k+4}(x^m)$, using (8.5) and the naturality of the operations with respect to maps.

THEOREM 11.5. *Let $a = 2^r$, $b = 2^s$, and $c = 2^t$, where $r > s \geq t \geq 2$, or $r \geq 2$ and $s = t = 1$; then, with indeterminacy of the form (11.1), for all $h > 0$ we have*

$$\Psi_{2(a+b-c)}(x^{(2h+1)2a-2b-2c}) = h[x^{4(ha+a-c)}, 0].$$

*Proof.* The possible values of $a$, $b$, $c$, assure that the conditions of (11.3) are satisfied; then the indeterminacy is of the form (11.1).

Let $f\colon RP^\infty \to CP^\infty$ be a characteristic map for the generator of $H^2(RP^\infty; Z)$. Clearly, with $Z_2$ coefficients we have $f^*w = x^2$, and $f^*\colon H^q(CP^\infty) \to H^q(RP^\infty)$ is an isomorphism for $q$ even. Let $u = w^{(2h+1)a-b-c}$ and $d = a + b - c$. In $H^*(CP^\infty)$, by (8.5), with zero indeterminacy, we have

$$\Psi_{2d}(u) = [\Phi_{2d}(u), 0] = h[w^{2(ha+a-c)}, 0].$$

Applying $f^*$ and using the naturality of $\Psi_{2d}$ the result follows.

THEOREM 11.6. *Let $a = 2^r$, $b = 2^s$, with $r > s \geq 2$; then, with indeterminacy of the form* (11.2), *for all $h > 0$, we have*

$$\Theta_{2(a+b-2)}(x^{(2h+1)2a-2b-4}) = h[x^{4(ha+a-2)}, 0, 0].$$

*Proof.* As before, using (11.4) we verify that the indeterminacy is as in (11.2).

Let $\Phi_{8k+4}$, $\Phi_{8k+6}'$, and $\Phi_{8k+7}$ be simple secondary cohomology operations associated with the relations (10.15), (10.16), and (10.17) respectively. Let $u = w^{(2h+1)a-b-2}$ and $d = a + b - 2$. In $H^*(CP^\infty)$, with zero indeterminacy, we have

$$\Theta_{2d}(u) = [\Phi_{2d}(u), \Phi_{2d+2}'(u), \Phi_{2d+3}(u)].$$

The third component is zero, since $\Phi_{2d+3}$ is of odd degree. For the second component, let $\Phi_{8k+5}$ be a simple secondary operation associated with the relation $\mathrm{Sq}^4\mathrm{Sq}^{8k+2} + \mathrm{Sq}^{8k+4}\mathrm{Sq}^2 = 0$. Using the method of [1; p. 75], it follows that $\Phi_{8k+6}' = \mathrm{Sq}^1\,\Phi_{8k+5}$, modulo primary operations. With (8.2) we check easily that $A_{2d+2}(u) = 0$. Then $\Phi_{2d+2}'(u) = \mathrm{Sq}^1\,\Phi_{2d+1}(u) = 0$. Therefore, by (8.5),

$$\Theta_{2d}(u) = [\Phi_{2d}(u), 0, 0] = h[w^{2(ha+a-2)}, 0, 0].$$

With $f^*$ as above, the result follows from the naturality of $\Theta_{2d}$.

## 12. Non-immersion of real projective spaces

We now give some applications of the previous results to the immersion problem of $RP^n$. We may introduce double and triple characteristic classes associated with the operations $\Psi_{8k}$, $\Theta_{8k+4}$; however, for simplicity, we proceed directly, using the Atiyah-James duality for $RP^n$.

THEOREM 12.1 *Let $\nu = (E, RP^n, \pi)$ be the normal bundle to an immersion of $RP^n$ in $R^{n+k}$, where $n = 2^r + 7$ and $r \geq 3$. If $U \in H^k(E, E_0)$ is the fundamental class of $\nu$, then $A_{2^r}(U) = 0$, $A_{2^r+1}(U) = 0$, $\mathrm{Sq}^4\,U = 0$, and $\Psi_{2^r}(U) \neq 0$.*

*Proof.* If $X$ is an $S$-dual of $RP^n$, as in the proof of (8.3), it is sufficient to show that the conclusions of the theorem hold for the first non-vanishing class of $X$. The result of Atiyah ([4; p. 307]) allows us to take $X$ as a reduced projective space. Explicitly, $X = RP^{2N-2}/RP^{2N-n-2}$, where $N$ is a conveniently large integer, depending on $n$. The first non-trivial element of $H^*(X)$ appears in dimension $2^N - 2^r - 8 = (2^{N-r} - 1)2^r - 8 = (2h + 1)2^r - 8$, with $h = 2^{N-r-1} - 1$. Then, under the natural projection $RP^{2^N-2} \to X$, this element may be identified with $x^{(2h+1)2^r-8}$. Applying the criterion (8.2) to this class, we obtain the part of the theorem concerning the action of the Steenrod algebra on the class $U$. Using (11.5) with $a = 2^{r-1}$, $b = 2$, and $c = 2$, we obtain $\Psi_{2^r}(x^{(2h+1)2^r-8}) \neq 0$. Therefore, $\Psi_{2^r}(U) \neq 0$, and this ends the proof.

THEOREM 12.2. *Let* $\nu = (E, RP^n, \pi)$ *be the normal bundle to an immersion of* $RP^n$ *in* $R^{n+k}$, *where* $n = 2^r + 2^s + 3$ *and* $r > s \geq 3$. *If* $U \in H^k(E, E_0)$ *is the fundamental class of* $\nu$, *then* $A_{n-7}(U) = 0$, $A_{n-5}(U) = 0$, $A_{n-4}(U) = 0$, $\mathrm{Sq}^6 U = 0$, *and* $\Theta_{2^r+2^s-4}(U) \neq 0$.

*Proof.* As in the proof of (12.1), if $X$ is an $S$-dual of $RP^n$, it is sufficient to verify the conclusions in the first non-vanishing class of $X$. In this case, $X = RP^{2^N-2}/RP^{2^N-2^r-2^s-5}$ and the first non-vanishing class of $H^*(X)$ is of dimension $(2h + 1)2^r - 2^s - 4$, where $h = 2^{N-r-1} - 1$. We identify this element with $x^{(2h+1)2^r-2^s-4)}$. As before, applying (8.2) to this class, we obtain the conclusion of the theorem regarding the action of the Steenrod algebra on the class $U$. From (11.6), with $a = 2^{r-1}$ and $b = 2^{s-1}$, we obtain $\Theta_{2^r+2^s-4}(x^{(2h+1)2^r-2^s-4}) \neq 0$, and this ends the proof.

The conclusions of (12.1) and (12.2) together with (10.18) and (10.19) immediately give us non-immersion results. However, by considering the associated sphere bundle, we can improve these results in one more unit.

THEOREM 12.3. $RP^n$ *cannot be immersed in* $R^{2n-9}$ *if* $n = 4k + 3$, *where* $k = 2^r + 2^s$ *and* $r > s \geq 0$.

*Proof.* Suppose that $RP^n$ admits an immersion in $R^{2n-9}$, and let $\nu = (E, RP^n, \pi)$ be the normal bundle to this immersion. If $U \in H^{n-9}(E, E_0)$ is the fundamental class of $\nu$, then, by (12.1) and (12.2), we have

$$(12.4) \qquad \Psi_{n-7}(U) \neq 0, \quad \text{if} \quad s = 0,$$

$$(12.5) \qquad \Theta_{n-7}(U) \neq 0, \quad \text{if} \quad s > 0.$$

Now from $\nu^*(\mathrm{Sq}^1) = 0$ it follows that the bundle $\nu$ is orientable. The Euler class $X(\nu) \in H^{n-9}(RP^n; Z)$, if different from zero, is of order 2. Its reduction modulo 2 is $\nu^*(\mathrm{Sq}^{n-9}) = 0$. Consequently, $X(\nu) = 0$. Therefore, the integral and the modulo 2 cohomology exact sequences of the pair $(E, E_0)$ break into short exact sequences (see [9; p. 60]):

$$(12.6) \quad 0 \longrightarrow H^q(E; Z) \xrightarrow{i^*} H^q(E_0; Z) \xrightarrow{\delta} H^{q+1}(E, E_0; Z) \longrightarrow 0,$$

$$(12.7) \quad 0 \longrightarrow H^q(E) \xrightarrow{i^*} H^q(E_0) \xrightarrow{\delta} H^{q+1}(E, E_0) \longrightarrow 0.$$

If $q = n - 10$, then $H^q(E; Z) = 0$, and from (12.6) there exists $u_1 \in H^{n-10}(E_0; Z)$ such that $\delta u_1 = U_1$, where $U_1 \in H^{n-9}(E, E_0; Z)$, is the integral fundamental class of $\nu$. Then, if $u$, $U$ are the reductions modulo 2 of $u_1$, $U_1$ respectively, we have $\delta u = U$.

We will verify that $u$ fulfills the conditions of (10.18) if $s = 0$ and those of (10.19) if $s > 0$. By (12.1) and (12.2) we already know that the conditions regarding the action of $A$ are automatically satisfied by $U$.

In both cases we have $\mathrm{Sq}^1 u = 0$ and $\mathrm{Sq}^2 u = 0$. The first statement follows from the fact that $u$ is the modulo 2 reduction of an integral class. For the second statement, if we suppose $\mathrm{Sq}^2 u \neq 0$, since $\delta \mathrm{Sq}^2 u = \mathrm{Sq}^2 U = 0$, by (12.7) we have $i^*\pi^*x^{n-8} = \mathrm{Sq}^2 u$; and the application of $\mathrm{Sq}^2$ to this last equality gives $i^*\pi^*x^{n-6} = \mathrm{Sq}^2\mathrm{Sq}^2 u = \mathrm{Sq}^3\mathrm{Sq}^1 u = 0$, which is a contradiction. Also $\mathrm{Sq}^{n-7} u = 0$ and $\mathrm{Sq}^{n-9} = 0$, since the dimension of $u$ is $n - 10$. Therefore, $\Psi_{n-7}(u)$ is defined for $s = 0$, and $\Theta_{n-7}(u)$ is defined for $s > 0$.

To verify the remaining conditions we consider the two cases separately. If $s = 0$, we need to show that $\mathrm{Sq}^4 u = 0$, $A_{n-7}(u) = 0$, and $A_{n-6}(u) = 0$. Suppose $\mathrm{Sq}^4 u \neq 0$; then, since $\delta \mathrm{Sq}^4 u = \mathrm{Sq}^4 U = 0$, by (12.7) we have $i^*\pi^*x^{n-6} = \mathrm{Sq}^4 U$, and, applying $\mathrm{Sq}^1$ to this equality, we obtain $i^*\pi^*x^{n-5} = \mathrm{Sq}^5 u = \mathrm{Sq}^4\mathrm{Sq}^1 u + \mathrm{Sq}^2\mathrm{Sq}^3 u = 0$, which is a contradiction. The verification of the other two conditions uses different arguments for $n = 15$ and for $n > 15$. If $n = 15$, we have $\dim u = 5$ and, since $\mathrm{Sq}^1 u = 0$ and $\mathrm{Sq}^2 u = 0$, this implies that $A_8(u) = 0$ and $A_9(u) = 0$. If $n > 15$, we have $H^q(E) \approx H^q(RP^n) = 0$ for $q = 2n - 16, 2n - 17$, and, since $A_{n-6}(U) = 0$ and $A_{n-7}(U) = 0$, from (12.7) it follows that $A_{n-6}(u) = 0$ and $A_{n-7}(u) = 0$.

Now from $\delta\Psi_{n-7}(u) = \Psi_{n-7}(U)$ and (12.4) it follows that $\Psi_{n-7}(u) \neq 0$, but this contradicts (10.18) and establishes the theorem for the case $s = 0$.

If $s > 0$, we need to show that $\mathrm{Sq}^6 u = 0$, $A_{n-4}(u) = 0$, $A_{n-5}(u) = 0$, and $A_{n-7}(u) = 0$. Suppose $\mathrm{Sq}^6 u \neq 0$; then, since $\delta \mathrm{Sq}^6 u = \mathrm{Sq}^6 U = 0$, by (12.7) we have $i^*\pi^*x^{n-4} = \mathrm{Sq}^6 u$, and the application of $\mathrm{Sq}^2$ to this equality gives $i^*\pi^*x^{n-2} = \mathrm{Sq}^2\mathrm{Sq}^6 u = \mathrm{Sq}^7\mathrm{Sq}^1 u = 0$, which is a contradiction. Now, since $H^q(E) \approx H^q(RP^n) = 0$ for $q = 2n - 14, 2n - 15, 2n - 17$, $A_{n-4}(u) = 0$, $A_{n-5}(u) = 0$, and $A_{n-7}(u) = 0$ follow from (12.7) and the corresponding results for $U$, as in the case $s = 0$.

Finally, from $\delta\Theta_{n-7}(u) = \Theta_{n-7}(U)$ and (12.5) it follows that $\Theta_{n-7}(u) \neq 0$; but this contradicts (10.19) and establishes the theorem for the case $s > 0$, and this ends the proof.

*Remark.* Accordingly with recent results of Sanderson ([13]), if $n = 4k + 3$ and $k$ is not a power of 2, then $RP^n$ can be immersed in $R^{2n-8}$. Consequntly, this result combined with (12.3) settles the immersion problem for the case $k = 2^r + 2^s$, $r > s \geq 0$.

THEOREM 12.8. *$RP^n$ cannot be immersed in $R^{n+[n/2]}$ for all $n \geq 32$.*

*Proof.* The result is established by showing that it holds for all $n$ with $2^r \leq n \leq 2^{r+1} - 1$ and all $r \geq 5$. Clearly, if $RP^m$ cannot be immersed in $R^{m+k}$, then $RP^n$ cannot be immersed in $R^{m+k}$ for all $n \geq m$. If $2^r \leq n \leq 2^r + 6$, the result follows using Stiefel-Whitney classes. If $2^r + 7 \leq n \leq 2^{r+1} - 2$, the result follows by the application of (12.3) to the cases $2^r + 2^s + 3 \leq n \leq 2^r + 2^{s+1} + 3$ with $s = 2, \cdots, r - 2$ and to $2^r + 2^{r-1} + 3 \leq n \leq 2^{r+1} - 2$. Finally, the case $n = 2^{r+1} - 1$ follows from the results of James ([6]).

Centro de Investigación del IPN, México, D.F.

References

[1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math., **72** (1960), 20–104.

[2] J. F. Adams, *Vector fields on spheres*, Topology, **1** (1962), 63–65.

[3] J. Adem, *Sobre operaciones cohomológicas secundarias*, Bol. Soc. Mat. Mexicana, **7** (1962), 95–110.

[4] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc., **11** (1961), 291–310.

[5] A. Dold, *Über Fasernweise Homotopieäquivalenz von Faserräumen*, Math. Zeit., **62** (1955), 111–36.

[6] I. M. James, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc., **69** (1963), 231–38.

[7] C. R. F. Maunder, *Cohomology operations of the $n^{th}$ kind*, Proc. London Math. Soc., **13** (1963), 125–54.

[8] W. S. Massey, *On the Stiefel-Whitney classes of a manifold*, Amer. J. Math., **82** (1960), 92–102.

[9] J. Milnor, Lectures on characteristic classes (mimeographed), Princeton University, 1958.

[10] J. Milnor and E. Spanier, *Two remarks on fibre homotopy type*, Pacific J. Math., **10** (1960), 585–90.

[11] F. P. Peterson and N. Stein, *The dual of a secondary cohomology operation*, Illinois J. Math., **4** (1960), 397–404.

[12] F. P. Peterson, *Some non-embedding problems*, Bol. Soc. Mat. Mexicana, 2 (1957), 9–15.

[13] B. J. Sanderson, *Immersions and embeddings of projective spaces*, Proc. London Math. Soc., **16** (1964), 135–53.

[14] B. J. Sanderson and R. L. E. Schwarzenberger, *Non-immersion theorems for differentiable manifolds*, Proc. Camb. Phil. Soc., **59** (1963), 319–22.

[15] J. P. Serre, *Cohomologie mod 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv., **27** (1953), 198–232.

[16] E. H. Spanier, *Duality and the suspension category*, "Symposium internacional de topología algebracia (México, D. F., 1958).

[17] N. E. Steenrod, *Cohomology invariants of mappings*, Ann. of Math., **50** (1949), 954–88.

[18] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Sci. Ecole Norm. Sup., **69** (1952), 109–81.