ON THE CONSTRUCTION OF LINEAR HOMOGENEOUS CONTINUA

By Richard Arens. *

A linear homogeneous continuum (LHC) has been defined by G. D. Birkhoff as any set of elements which

1. is simply ordered

2. provides a limit for any monotonely increasing (or decreasing) sequence

3. is isomorphic to any closed sub interval of itself, the endpoints of the subinterval being distitnct.

A very interesting example of such a continuum has been constructed by Vázquez and Zubieta. [1]

It is seen at once that any closed interval of real numbers is an LHC.

The following result is useful in constructing additional examples of LHC sets.

Theorem: If L is an LHC, then L^{ω} is again an LHC. [2] L^{ω} has a cardinal number equal to that of L. Moreover, a family of disjoint open intervals of L^{ω} can be found which has the cardinal of L.

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* Recibido para su publicación en septiembre de 1944.

(1) R. Vázquez y F. Zubieta: "Los continuos lineales homogéneos de George D. Birkhoff", este Boletín, Vol. I, Núm. 2, pp. 1-14.

(2) If L is an ordered set, and λ is an ordinal number, L^{λ} shall mean the class of all sequenses in L of order type λ, these sequences being ordered lexicographically.

Proof:

1. If $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ belong to L^ω, where $x_i, y_i \varepsilon L$ then $x < y$ if $x_1 < y_1$, or if, for some n, $x_1 = y_1$, $x_2 = y_2, \ldots,$ $x_{n-1} = y_{n-1}$ and $x_n < y_n$. This clearly orders L^ω.

2. Suppose that $z^1, z^2, z^3, \ldots$ is an increasing sequence of elements of L^ω.

Suppose $z^i = (z^i_1, z^i_2, z^i_3, \ldots)$ where $z^i_k \varepsilon L$. Furthermore let n be the least integer for which $z^1_n\ z^2_n, z^3_n, \ldots$ represents a strictly increasing sequence of elements of L. If no such integer exists, set

$$n = \infty. \text{ Now define}$$

$$z_k = \lim_{i \to \infty} z^i_k \quad k < n,$$

and if $n \neq \infty$, let

$$z_n = \lim_{i \to \infty} z^i_n, \ z_{n+1} = z_{n+2} = \ldots = 0,$$

(0 will be used to denote the least element of L. 1 will denote the highest element. Their existence follows from hypothesis 3 for L).

It is then easily seen that

$$\lim_{i \to \infty} z^i = (z_1, z_2, z_3, \ldots) \varepsilon L^\omega.$$

The case of a decreasing sequence is similar.

3. Finally, suppose

$$x = (x_1, x_2, x_3, \ldots) < (y_1, y_2, y_3, \ldots) = y \varepsilon L^\omega.$$

We shall exhibit on isomorphism φ transforming the closed interval [x, y] in an order preserving manner, onto L^ω.

Let m be the first integer for wich

$$x_m < y_m.$$

Define an increasing subsequence of the integers $f_1, f_2, f_3, \ldots$ so that

$$y_{f_1}, y_{f_2}, y_{f_3}, \ldots$$

are (in order) those y_i that are not equal to 0. Evidently $f_1 = m$. If the sequence terminates, let f_v be the last element; if not, set $v = \infty$

Similarly, define $g_1, g_2, g_3, \ldots$ and μ, such that

$$x_{g_i} \neq 1 \qquad 1 \leqq i < \mu, \text{ and, if } \mu \neq \infty,$$

$$x_{g_\mu} \neq 1.$$

Now we can suppose that there is a set of distinct elements $A_1, A_2, \ldots$ $B_1, B_2, \ldots$ in L such that

$$\lim_{i\to\infty} A_i = 0, \lim_{i\to\infty} B_i = 1$$

and

$$\ldots < A_{n+1} < A_n < \ldots < A_1 < B_1 < \ldots < B_n < B_{n+1} < \ldots$$

Now take an isomorphism φ_1 mapping $[x_m, y_m]$ onto $[A_1, B_1]$. Also take φ_n mapping $[x_{g_n}, 1]$ onto $[A_n, A_{n-1}]$, $n = 2, 3, \ldots < \mu$

If $\mu \neq \infty$, let φ_μ map $[x_{g_\mu}, 1]$ onto $[0, A_{\mu-1}]$.
Define Ψ_n to map $[0, y_{f_n}]$ onto $[B_{n-1}, B_n]$, $n = 2, 3, \ldots < v$.
If $v \neq \infty$, let Ψ_v map $[0, y_{f_v}]$ onto $[B_{v-1}, 1]$.

We next define φ. Suppose $x \leqq z \leqq y$. Then let $\varphi(z) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots)$. Then, either, for some n:

Case A

$$z_i = x_i \qquad (i = m, m+1, m+2, \ldots < g_n)$$

$$x_{g_n} < z_{g_n} \leqq 1, \text{ or}$$

Case B

$$z_i = y_i \qquad (i = m, m+1, m+2, \ldots < f_n)$$

$$0 \leqq z_{f_n} < y_{f_n}, \text{ or}$$

Last case

$$x_m < z_m < y_m.$$

In the last case, let $\bar{z}_1 = \varphi_1(z_m)$, $\bar{z}_2 = \varphi_2(z_{m+1})$, $\bar{z}_3 = \varphi_3(z_{m+2}), \ldots$
In case A, let $\bar{z}_1 = \varphi_n(z_{g_n})$, $\bar{z}_2 = z_{g_n+1}$, $\bar{z}_3 = z_{g_n+2} \ldots$
In case B, let $\bar{z}_1 = \Psi_n(z_{f_n})$, $\bar{z}_2 = z_{f_n+1}$, $\bar{z}_3 = z_{f_n+2}, \ldots$
also, let $\varphi(x) = (0, 0, \ldots)$, $\varphi(y) = (1, 1, \ldots)$.

It is easily seen that φ is order preserving. Furthermore, any element $(\bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots) \varepsilon L^{\omega}$ is the image of some one element in $[x, y]$, which can be easily found by first observing where $\bar{z}_1$ is in the covering
$\ldots < A_n < \ldots < A_1 < B_1 < \ldots < B_n < \ldots$
Now, for any $x \varepsilon L$, let W_x be then open interval $[(x, 0, 0, \ldots), (x, 1, 1, \ldots)]$.
The W_x are disjoint, and their cardinal is that of L.

Now the cardinal of any LHC is at least C, the cardinal of the continuum. (1) Hence that of L^{ω} is $c^{\lambda_0} = C$.

q. e. d.

This theorem raises some interesting questions:

If we let $L = R_o$, where R_o is a closed interval of real numbers (distinct endpoints), and set $R_n = R^{\omega}_{n-1}$, we obtain a sequence of LHC:

$$R_o, R_1, R_2, R_3, \ldots$$

of which the first is certainly distinct from all the others.

Could it be that the others are all isomorphic? If this is not so, they are probably all distinct.

Mrs. G. D. Mostow has observed that if R_o be a real interval, then, for example, if $\lambda = \omega^2 + \omega$, R^{λ}_{o} is not an LHC.
Is it possible to find a λ such that R^{λ}_{o} is an LHC while λ is not an ordinal of the type ω^n? (It is easily seen that $R^{\omega^n}_{o} = R_n$).

The theorem shows that there is no linear homogeneous continuum L such that $R_o = L^{\omega}$.

Finally, one might ask whether any LHC exist whose cardinal is not C.

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(1) F. Hausdorff, Mengenlehre, 3rd edition (1935), p. 54.