# A REVISION OF THE THEORY OF ELASTICITY *

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The fundamental relation on which the whole theory of elasticity rests is that which connects the two main tensors of the theory, namely, the stress tensor and the strain tensor. This relationship may be derived from the principle of the conservation of energy and it is known as Hooke's Law when the two basic simplifying hypotheses of the classical theory of elasticity are granted. These are the following:

1. The strain is *infinitesimal.* By this we mean simply that the problem may be *linearized* in the sense that a sufficiently small, i. e. infinitesimal, part of a curve may be regarded as a line.

2. The initial position of the elastic medium, i. e. the position from which the strain is measured, is that in which there is no applied stress. In other words the stress tensor is granted to be zero when the strain tensor is zero.

The whole theory of structures and of the strength of materials is based on the theory of elasticity (including these simplifying hypotheses) and we can have nothting but admiration for the successes of these theories. Nevertheless the theory fails to predict the phenomena when the applied stress is large, and this is becoming more and more important in technical applications.

The first idea that comes to the mind of anyone who tries to build a theory of strains which are too large to be regarded as infinitesimal is the following. Just as a portion of a curve which is too large to be treated

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as a line may, by inserting a suitable number of points of subdivision, be regarded as a broken or polygonal line, so a strain which is too large to be treated as infinitesimal may be regarded as a succession of small or infinitesimal strains by the insertion of a suitable number of new reference positions of the medium from each of which one of the succession of infinitesimal strains is measured. Why not use the classical theory of elasticity for each of these infinitesimal strains and then by a process of integration, obtain the stress corresponding to the strain which was too large to be regarded as infinitesimal? In other words, why not apply to the problem the familiar methods of integral calculus? The first objection to this proposal arises when we look at the second of the two main simplifying hypotheses of the classical theory of elasticity: in none of the succession of small trains, save the first, is the initial position of the medium, to which the strain is referred, unstressed. We must, then, if we wish to apply the integration method, revise the theory of elasticity so as to formulate Hooke's Law in such a way as to take care of a medium which is initially stressed and to which an additional (infinitesimal) stress is applied. I propose to consider this revision with you today. In doing so I am forced to use such technical concepts as matrices, tensors, etc., but I trust that my use of these will not hide the essential simplicity of the method. I shall apply the method to two important instances of stress:

a) uniform hydrostatic pressure

b) uniform tension of an elastic cylinder.

The first of these is important in questions concerning the internal constitution of the earth, and the second is fundamental in the theory of structures. It is a sobering thought that we yet know very little about the interior of this earth on which we live; what little we know (or, rather, guess) derives largely from the theory of wave propagation (which theory is based on the classical theory of elasticity) and its application to observations on earthquakes. The pressures in the interior of the eath are so enormous that predictions based on the classical inifinitesimal theory of elasticity must be regarded with some skepticism.

Let us first briefly recall the way in wich the strain of an elastic medium is specified. Let $x = (x, y, z)$ be the rectangular Cartesian coordinates of a typical particle of the medium when in the position to which the strain is being referred, and let $\xi = (\xi, \eta, \varsigma)$ be the coordinates of this same particle when the medium is strained from this position. Denote by J the Jacobian matrix

$$J = \frac{\partial(\xi)}{\partial(x)} = \frac{\partial(\xi, \eta, \varsigma)}{\partial(x, y, z)} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} & \frac{\partial\xi}{\partial z} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z} \\ \frac{\partial\varsigma}{\partial x} & \frac{\partial\varsigma}{\partial y} & \frac{\partial\varsigma}{\partial z} \end{pmatrix}$$

and by $J^*$ the matrix obtained from J by interchanging its rows and columns. Then the strain is described by means of the symmetric matrix $J^*J$. When the displacement $x \rightarrow \xi$ is a mere rigid motion, so that no strain is involved, the matrix J is orthogonal and $J^*J$ is the unit matrix. In a non rigid displacement the strain is measured by the difference between $J^*J$ and the unit matrix; we write

$$e = \tfrac{1}{2} \, (J^*J - E); \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we term the symmetric matrix e the *strain tensor*. If we write $\xi = x + + \delta x$ and agree that the elements of the matrix $\frac{\partial}{\partial x} \delta x$ are infinitesimal, i. e., so small that their squares and higher powers are negligible, we readily find the well-known expressions of the classical theory of elasticity:

$$e_{xx} = \frac{\partial}{\partial x} \delta x; \ldots \qquad e_{yz} = \tfrac{1}{2} \left( \frac{\partial}{\partial y} \delta z + \frac{\partial}{\partial z} \delta y \right); \ldots$$

However it is not necessary to adopt at the very beginnig this linearization; we can get along quite well with the exact formula $e = \tfrac{1}{2} \, (J^*J - E)$.

So much for the strain tensor; what about the *stress tensor*? This is simply a symmetric matrix which acts as a linear operator or machine. If we use it to operate on any unit vector, i.e. direction, at any point of the medium the result of this operation is simply the stress, or force per unit area, across an element of area perpendicular to the given direction. In particular if the element of area belongs to the bounding surface of the elastic medium the result of the operation by the stress tensor on the unit vector normal to the surface is the force per unit area acting on the surface of the medium. We denote the stress tensor by T (the first letter of the word tension) and we denote the elements of T by $T_{xx}, \ldots, T_{yz}, \ldots$ A particularly simple and important case occurs when T is a scalar matrix, i.e. a multiple of the unit matrix. This happens in the case of hydrostatic pressure; in this case $T = -pE$ and p is termed simply the *pressure*. The negative sign is due to the fact that the stress is a pressure rather than a tension.

When an elastic medium is strained under the action of applied forces these forces do work upon the medium and we assume that this work is stored up in the medium in the form of energy of deformation. We denote by $\varphi$ the energy of deformation per unit mass and we assume that $\varphi$ is a function of the strain tensor e. This assumption enables us to find a connection between the stress tensor and the strain tensor. We confine our attention to the case where the deformation takes place adiabatically so that we do not have to allow for heat transfer in the energy equation. Let $\varrho$ denote the density of the medium in the unstrained position x and $\varrho + \delta\varrho$ the density in the strained position $\xi = x + \delta x$. Then

$$T + \delta T = (\varrho + \delta\varrho)\, J \frac{\partial\varphi}{\partial e} J^*$$

where $\frac{\partial\varphi}{\partial e}$ is the matrix obtained by differentiating $\varphi$ with respect to each component of the strain tensor. This is the basic relation which must take the place of Hooke's Law and which is valid whether or not the strain is small and whether or not the initial position from which the strain is measured is free from stress (provided always that we are willing to accept the hypothesis that $\varphi$ is a function of e).

Let us suppose the function $\varphi$ developed as a power series in the components of the strain tensor:

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

where $\varphi_1$ is a homogeneous function of the first degree, and $\varphi_2$ a homogeneous function of the second degree, in the components of e. When $e = 0$ both J and $J^*$ reduce to the unit matrix and $\delta T$ and $\delta\varrho$ are zero so that

$$T = \varrho \frac{\partial \varphi_1}{\partial e}$$

Hence $\varphi_1 = 0$ if, and only if, $T = 0$, i.e. if the medium is initially free from stress. In this case the lowest order terms in the expression for $\delta T$ are given by setting $\varphi = \varphi_2$, $\delta\varrho = 0$, $J = J^* = E$ so that we obtain

$$T = \varrho \frac{\partial \varphi_2}{\partial e} = \frac{\partial}{\partial e} (\varrho\varphi_2)$$

This is the familiar expression of Hooke's Law: The stress $\delta T$ corresponding to an infinitesimal strain e from an unstressed position is the gradient whit respect to e of the elastic-energy-per-unit-volume $\varrho\varphi_2$. It is clear, however, that this principle is no longer valid when the initial position is a stressed one. In this case $\frac{\partial \varphi}{\partial e}$ starts off with terms which are not zero when $e = 0$ and so it is not legitimate, when setting down the first order terms in the expression $(\varrho + \delta\varrho)\, J \frac{\partial \varphi}{\partial e} J^*$, to merely set $\delta\varrho = 0$ and to replace J and J* each by the unit matrix. To see, without getting involved in technical calculations, what must be done we shall treat the simple and important case of hydrostatic pressure.

Since the invariants of the product of two matrices are insentitive to an interchange of the order in which the two matrices are written down, the invariants of the new stress matrix $T + \delta T$ are the same as the invariants of $(\varrho + \delta\varrho)\, J^*J \frac{\partial \varphi}{\partial e} = (\varrho + \delta\varrho)\,(E + 2e) \frac{\partial \varphi}{\partial e}$ so that these invariants are functions of e. When the stress is a hydrostatic pressure the first invariant of $T + \delta\, T$ is $-3\,(p + \delta p)$ and so $\delta p$ is a function of p and e.

For any strain we have

$$\det\,(E + 2e) = \det J^*J = \left(\frac{\varrho}{\varrho + \delta\varrho}\right)^2 = \left(\frac{v + \delta v}{v}\right)^2$$

where $v = 1/\varrho$ is the specific volume of the medium. It follows that

$$\left(\frac{\partial}{\partial e}\delta v\right)^{-1} = (\varrho + \delta\varrho)\ (E + 2e)$$

and so the invariants of $T + \delta T$ are the same as the invariants of $\frac{\partial\varphi}{\partial e}\left(\frac{\partial}{\partial e}\delta v\right)^{-1}$. For the case of an isotropic medium, i.e. one in which no direction is privileged, from the elastic point of view, over any other $\varphi$ is a function of the invariants of the strain matrix e and when e is a scalar matrix (as is the case when a uniform hydrostatic pressure is applied) this implies that $\varphi$ is a function of $\delta v$ and that.

$$T + \delta T = \frac{\partial\varphi}{\partial\delta v} .$$

In order that $T = -p$ the first order terms in $\varphi$ must be $-\frac{p}{\varrho} I_1$, where $I_1$ is the first invariant of the strain tensor; adopting the notation of the classical theory of elasticity we write the second order terms in $\varphi$ as

$$\frac{1}{\varrho}\left\{\frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2\right\}.$$

so that up to the terms of the second order

$$\varrho\varphi = \varrho\varphi_0 - pI_1 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2$$

where

$$I_1 = e_{xx} + e_{yy} + e_{zz};\ I_2 = \begin{vmatrix} e_{yy} & e_{yz} \\ e_{zy} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{zz} & e_{zx} \\ e_{xz} & e_{xx} \end{vmatrix} + \begin{vmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{vmatrix}$$

are the first two invariants of the strain tensor. Conversely it is important to observe that if the initial stress is not scalar the first order temrs in $\varphi$ are not a multiple of $I_1$ so that the medium is not elastically isotropic: in other words a non-scalar stress introduces privileged directions, which is quite natural since the principal axes of the stress tensor are privileged. On writing $e = \varepsilon E$ we have

$$\varepsilon = \tfrac{1}{2} \left\{ \left(1 + \frac{\delta v}{v}\right)^{2/3} - 1 \right\}$$

so that

$$\frac{\partial \varepsilon}{\partial \delta v} = \frac{1}{3v} \left(1 + \frac{\delta v}{v}\right)^{-1/3}$$

A simple calculation shows that

$$-\delta p = \left\{ (\lambda + \frac{2}{3}\mu) + \frac{1}{3}\varrho \right\} \frac{\delta v}{v}$$

so that

$$-v \frac{dp}{dv} = (\lambda + \frac{2}{3}\mu) + \frac{1}{3} p.$$

The quantity $-v\dfrac{dp}{dv}$ is the reciprocal of the compressibility coefficient and so we see that even when we make the drastic hypothesis that $\lambda + 2/3\,\mu$ is independent of p the compressbility coefficient depends on p, its reciprocal being a linear function of p. If we adopt the more reasonable point of view that $\lambda + \dfrac{2}{3}\mu$ varies with p and treat it, to a first approximation as a linear function of p we may write

$$-v \frac{dp}{dv} = c + \alpha p$$

where c has the dimensions of a pressure and $\alpha$ is a dimensionless constant. On integrating this equation we obtain

$$1 + \frac{\alpha}{c} p = \left(\frac{v_o}{v}\right)^{\alpha} = \left(\frac{\varrho}{\varrho_o}\right)^{\alpha}$$

where $\varrho_o$ is the density when $p = 0$. This simple formula agrees reasonably well with experimental results even to such high pressures as $10^5$ atmospheres. For lithium $\alpha$ is approximately 2 and for sodium aproximately

3. The value $\alpha = 2$ yields a famous formula proposed by Laplace connecting the density and pressure in the interior of the earth.

Let us now turn to the theory underlyin the Young's-modulus experiment. Here we have a cylinder (whose axis we choose as the z-axis) subjected to a uniform tension parallel to its axis. All components of the stress tensor save $T_{zz}$ are zero and we know that the medium is not elastically isotropic (even if it were originally isotropic, i.e. if it had no privileged directions before the application of the non-scalar stress). It is reasonable to suppose that the medium remains elastically insensitive to rotations around the axis of the cylinder, and this assumption implies that the second order terms in $\varrho\varphi$ are of the form

$$\varphi_2 = \tfrac{1}{2} \left\{(\lambda + 2\mu) I_1^2 - 4\mu I_2 + \alpha e_{zz}^2 + 4\beta\,(e_{xx}e_{yy} - e_{xy}e_{yx}) + 2\gamma e_{zz}\,(e_{xx} + e_{yy})\right\}$$

Thus five elastic constants (instead of the two constants which are sufficient in the case of an isotropic medium) are involved. A straightforward calculation yields

$$\delta\, T_{zz} = \left\{(\lambda + 2\mu + \alpha + T_{zz}) - \frac{(\lambda + \gamma)\,(\lambda + \gamma - T_{zz})}{\lambda + \mu + \beta}\right\} e_{zz}$$

and since $e_{zz} = \dfrac{\delta_z}{z}$ we obtain

$$\log \frac{z}{L} = \int_0^{T_{zz}} \frac{(\lambda + \mu + \beta)\, dT}{(\lambda + \mu + \beta)\,(\lambda + 2\mu + \alpha + T) - (\lambda + \gamma)\,(\lambda + \gamma - T)}$$

where L is the length of the cylinder when it is free from traction and $z$ is its length when the tension is $T_{zz}$. If we assume that the elastic constants are independent of $T_{zz}$ we must set $\alpha$, $\beta$, $\gamma$ all zero (since they are zero when the applied tension is zero) and we obtain the formula

$$\frac{z}{L} = \left\{1 + (1 + 2\sigma)\frac{T_{zz}}{E}\right\}^{\frac{1}{1+2\sigma}}$$

where $\sigma = \frac{\lambda}{2(\lambda + \mu)}$ is Poisson's ratio and $E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu}$ is Young's modulus. This replaces the familiar formula

$$\frac{z}{L} = 1 + \frac{T_{zz}}{E}.$$

If we make the more realistic hypothesis that the five elastic constants $\lambda$, $\mu$, $\alpha$, $\beta$, $\gamma$ vary with $T_{zz}$ and approximate them by linear functions of T the integrand of the integral for log z/L is the quotient of a linear function of T by a quadratic function of T. If this quadratic function of T has a positive root $T_m$, $z \rightarrow \infty$ as $T \rightarrow T_m$. In other words, the cylinder cannot support a tension as great as $T_m$. This may serve as a qualitative explanation of the passage under great stress of a medium from the elastic to the plastic stage. No real understanding of plastic flow can be had if one adheres to the point of view that the medium remains isotropic; it is in the very existente of privileged directions that we must search for the secret laws that govern the plastic behavior of materials.

In closing this lecture, let me express my appreciation of the high honor you have done me in asking me to address you. I like to think that this Congress is but an indication of many future efforts of cooperation between the scientists of Mexico and the scientists of the United States. In selecting for my lecture to a congress of mathematicians a topic in applied mathematics, I have been led by the hope that some amongst you may feel, as I do, that the study of nature is a congenial occupation for a mathematician.