DIRECT DECOMPOSITIONS OF BANACH SPACES *

By Nelson Dunford.

I wish to present in a general way two related problems which are among the important problems of spectral theory and which concern the direct decomposition of a linear vector space. I shall not attempt any general solution of these problems but shall endeavor to show how they form, in a sense, a unifying principle for such seemingly diverse problems as the existence and uniqueness of Haar measure, the general closure theorems in $L(-\infty, \infty)$ of Norbert Wiener, the mean ergodic theorem of von Neumann and Riesz, and its generalization to the case of n-parameter continuous semi-groups, and some expansion theorems in reflexive Banach spaces.

The first of these decomposition problems concerns a single continuous linear operator T in a normed linear vector space X, and consists of determining conditions on T or X or both which will insure the validity of the decomposition.

$$X = M(T) \oplus N(T) , \tag{1}$$

where $M(T)$ consists of those $x \epsilon X$ where $Tx = 0$ and $N(T)$ is the closed linear manifold (c.l.m.) determined by the range TX of T. The equation (1) means that every vector $x \epsilon X$ has a unique decomposition $x = y + z$ where $y \epsilon M(T)$ and $z \epsilon N(T)$. If we define the operators $E(T)$, $E'(T)$ by writing $y = E(T)x$, $z = E'(T)x$ it is readily seen that

* Presentado en el Segundo Congreso Nacional de Matemáticas, efectuado en Guadalajara, Jalisco, del 28 de mayo al 2 de junio de 1945.

$$(1') \qquad \begin{gathered} I = E(T) + E'(T) \, , \ E^2(T) = E(T) \, , \\ E(T)X = M(T) \, , \ E'(T)X = N(T) \, . \end{gathered}$$

Conversely if an operator $E(T)$ in X exists which satisfies $(1')$ then T satisfies (1) so that the problem is that of finding conditions on T and X which will insure the existence of a projection $E(T)$ which projects the whole space onto $M(T)$ and whose complement projects X onto $N(T)$.

The second problem is analogous and forms a generalization of the first by considering a family G of operators in X instead of a single operator. If by $M(G)$ is understood the set of all x in X for which $Tx = 0$ for every T in G and by $N(G)$ the closed linear manifold determined by the union of all the ranges TX, $T \epsilon G$, the problem is that of determining conditions on G and X such that

$$(2) \qquad X = M(G) \oplus N(G) \, .$$

As in the case of the first problem equation (2) is equivalent to the existence of a projection operator $E(G)$ which projects X onto $M(G)$ and whose complement $E'(G) = I - E(G)$ maps X onto $N(G)$. Thus the second problem is that of determining the families G of operators on the space X for which there exists an operator $E(G)$ satisfying

$$(2') \qquad \begin{gathered} I = E(G) + E'(G) \, , \ E^2(G) = E(G) \, , \\ E(G)X = M(G) \, , \ E'(G)X = N(G) \, . \end{gathered}$$

There is a lattice theoretic relation between the two problems which I should like to mention at this point and return to in more detail later. Suppose that (1) holds for each T in a family G of operators in X and that the corresponding projections commute. The product $E(T)\ E(U)$ where $T, U \ \epsilon \, G$ is readily seen to be a projection of X onto the manifold $M(T) \cdot M(U)$. If the projections in a commutative family of projections are ordered according to the customary convention of saying that $A \subset B$ if and only if $AB = A$, i. e., if and only if $AX \subset BX$, then $E(T)\ E(U)$ is the greatest lower bound (g.l.b.) of $E(T)$ and $E(U)$ and is sometimes written $E(T) \wedge E(U)$. Now the passage from (1) to (2) requires the existence of a projection $E(G)$ with $E(G)X = M(G) = \Pi\, M(T)$ where the product is taken over all T in G. Such a projection is clearly the greatest lower bound of all the projections $E(T)$ with $T \epsilon G$.

While a commutative family of projections in X generates a Boolean algebra of projections in X under the operations

$$A \wedge B = AB, \qquad A \vee B = (A'B')',$$

(where $A' = I - A$) in which crosscuts (g.l.b.) and unions (l.u.b.) exist for any finite number of elements, it is not in general true that to an arbitrary set of elements in such a Boolean algebra corresponds a projection in X which is the crosscut or greatest lower bound of the given set. Here and elsewhere when we speak of a projection or an operator in X we mean a continuous and linear transformation of X into all or part of itself. Thus the passage from (1) to (2) is related, in the commutative case, to the problem of determining which Boolean algebras of projections in a Banach space X may be embedded in a *complete* Boolean algebra of projections in X, i. e., in a Boolean algebra which contains crosscuts and unions of arbitrarily many of its elements. Later I shall outline one solution of this problem in the case that X is a reflexive space.

Before proceeding to the problem and the relations to analysis mentioned above I shall discuss briefly the first decomposition problem in the case where X is finite dimensional and point out some of the similarities and differences between this case and the general one. Suppose for the present then that X is finite dimensional and that T is a linear operator in X. If (1) holds then clearly we have

$$(1'') \qquad M(T) \cdot N(T) = 0.$$

Conversely (1″) implies (1). For (1″) shows that T is 1—1 on $N(T)$ and since this space is of finite dimension T is a 1—1 map of $N(T)$ into all of itself. Also since $N(T)$ is finite dimensional $TX = N(T)$ and hence for every $x \epsilon X$ there is a $z \epsilon N(T)$ with $Tx = Tz$. Therefore if $y = x - z$ we have $y \epsilon M(T)$ and hence (1″) implies (1) and the three conditions (1), (1′), (1″) are equivalent.

A fourth condition which is also equivalent to any one of these is the statement that $\lambda = 0$ is a root of multiplicity 0 or 1 of the minimal equation of T. This means that there is a polinomial $P(\lambda)$ with

$$(1''') \qquad TP(T) = 0, \qquad P(0) \neq 0.$$

To see that (1‴) is equivalent to the other forms (1), (1′) and (1″)

suppose that $(1'')$ holds. This means, as we have seen, that T has an inverse when considered as an operator in $N(\mathrm{T})$ and hence $\lambda = 0$ is not a root of its minimal polynomial where it is considered as an operator in $N(\mathrm{T})$. Thus there is a polynomial $\mathrm{P}(\lambda)$ with $\mathrm{P}(0) \neq 0$ and such that $\mathrm{P(T)E'} = 0$. Hence $\mathrm{TP(T)} = \mathrm{TP(T)E} + \mathrm{TP(T)E'} = 0$ since $\mathrm{TE} = 0$. Conversely if $(1''')$ holds for some polynomial $\mathrm{P}(\lambda)$ then since the two polynomials λ, $\mathrm{P}(\lambda)$ have no common root there are polynomials $\mathrm{R}(\lambda)$, $\mathrm{Q}(\lambda)$ with

$$\lambda\, \mathrm{R}(\lambda) + \mathrm{P}(\lambda)\, \mathrm{Q}(\lambda) \equiv 1,$$
$$\mathrm{T\,R(T)} + \mathrm{P(T)\,Q(T)} = \mathrm{I}.$$

If we multiply by T and use de fact that $\mathrm{T\,P(T)} = 0$ we see that $\mathrm{R(T)T^2} = \mathrm{T}$ and hence $\mathrm{T^2x} = 0$ implies $\mathrm{Tx} = 0$. Hence $M(\mathrm{T}) \cdot \mathrm{T}X = 0$ and since $\mathrm{T}X = N(\mathrm{T})$ we see that $(1'')$ is satisfied. Thus if X is finite dimensional $(1''')$ is equivalent to any one if the statements (1), $(1')$, and $(1'')$.

Finally it might be pointed out that the existence of projection E of X onto $M(\mathrm{T})$ which commutes with T is also equivalent to (1). For if $\mathrm{E}^2 = \mathrm{E}$, $\mathrm{E}X = M(\mathrm{T})$, and $\mathrm{ET} = \mathrm{TE}$ we have $\mathrm{E'T} = (\mathrm{I}-\mathrm{E})\mathrm{T} = \mathrm{T}-\mathrm{ET} = \mathrm{T}-\mathrm{TE} = \mathrm{T}$. Thus $\mathrm{T}X = N(\mathrm{T}) \subset \mathrm{E}'X$ and hence $M(\mathrm{T}) \cdot N(\mathrm{T}) \subset \mathrm{E}X\,\mathrm{E}'X = 0$ and $(1'')$ is satisfied.

An example of an operator which does not satisfy (1) and which illustrates nicely the various conditions $(1), \ldots, (1''')$ is the following. Let X be two dimensional Euclidean space an T the operation which maps the point (ξ, η) into the point $(\eta, 0)$. Geometrically T is the result of first projecting the point (ξ, η) onto the η-axis and then rotating clockwise through an angle $\pi/2$. Clearly $\mathrm{T} \neq 0$ but $\mathrm{T}^2 = 0$ so that $(1''')$ is not satisfied. Also $M(\mathrm{T}) = N(\mathrm{T}) =$ the ξ-axis so that (1) and $(1'')$ are false. There can be no projection E of X onto $M(\mathrm{T})$ which commutes with T for otherwise we would have $0 = \mathrm{TE} = \mathrm{ET} = \mathrm{E}$ [since $\mathrm{T}X = M(\mathrm{T})$]. Since a projection satisfies (1) and a rotation satisfies (1) this example also shows that the product of two operators need not satisfy (1) even though the factors do.

In the general case the question is not so elementary. While the existence of a projection $\mathrm{E(T)}$ satisfying $(1')$ is still necessary and sufficient for (1) the condition $(1'')$ is necessary but no longer sufficient and $(1''')$ is sufficient but not necessary for (1). No non-trivial conditions which are necessary and sufficient for (1) are known to me and the methods that have been used to solve the various problems in analysis that come under the heading of (1) or (2) have varied widely with the problem as well as with the mathematician solving it. I shall show how each of the problems mentioned at the beginning, i. e., the ergodic theorem, Haar measure, the closure theorems in $L(-\infty, \infty)$, etc.., may be regarded as a problem of the type (1) or (2).

Let us first consider the simplest case of the mean ergodic theorem. This concerns a single linear operator U with $|U| = 1$ and states, under certain restrictions, that for each $x \epsilon X$ the $\lim_n 1/n \sum_{j=0}^{n-1} U^j x$ exists. Suppose that we have the decomposition (1) for the operator $T = I - U$, i. e., every $x \epsilon X$ may be written as $x = y + z$ where $y = Uy$ and $z \epsilon \overline{TX}$. Clearly $1/n \sum U^j y = y$ for all n and for an element $z_1 \epsilon TX$, say $z_1 = u - Uu$, we have $1/n \sum U^j z_1 = 1/n(u - U^n u) \to 0$. Thus the average $1/n \sum U^j$ converges to 0 on TX and hence on $\overline{TX}$ since its norm is bounded and we may say that the decomposition (1) implies the convergence of the averages $1/n \sum_{j=0}^{n-1} U^j x$ for each $x \epsilon X$. Conversely it is not difficult to show that the convergence of the averages implies the decomposition (1), so that a proof of the mean ergodic theorem for the operator U is equivalent to a proof of (1) for the operator $T = I - U$.

Next let us see how the mean ergodic theorem in the case of an n-parameter continuous group may be interpreted as a problem of type (2). For simplicity we shall take $n = 1$. The theorem is concerned with a continuous group of operators T_s, $-\infty < s < \infty$ satisfying

$$T_s T_t = T_{s+t}, \; |T_s| \leqq 1 \qquad -\infty < s, t < \infty,$$

and states, under suitable restrictions, that the $\lim_{a \to \infty} A_a x$ exists for each $x \epsilon X$ where $A_a x = 1/a \int_0^a T_s x \, ds$. Let us consider the family G of all operators of the form $I - T_s$, $-\infty < s < +\infty$, so that $M(G)$ consists of all y for which $y = T_s y$, $-\infty < s < +\infty$, and $N(G)$ is the c.l.m. determined by all vectors of the form $z - T_s z$, $z \epsilon X$, $-\infty < s < +\infty$. Suppose that the decomposition (2) holds for this family G. Clearly $A_a y = y$ for $y \epsilon M(G)$ and for a vector of the form $x = z - T_s z$ we have

$$A_a x = 1/a \int_0^a (T_t z - T_{t+s} z) \, dt = 1/a \int_0^s T_t z \, dt - 1/a \int_a^{a+s} T_t z \, dt,$$

and

$$| A_a x | \leqq 2 s | x | / a$$

Thus $A_a x \to 0$ for every vector of the form $x = z - T_s z$ and since $| A_a | \leqq 1$ we must have $A_a x \to 0$ por every vector in the c.l.m. determined by such vectors i. e., for every vector in $N(G)$. Thus the decomposition (2) implies the mean ergodic teorem in the continuous case. Conversely it is not difficult to show that the existence of the $\lim_a A_a x$ for $x \in X$ implies the decomposition (2).

Next let us consider an abstract set of points S and a family Φ of functions φ which map S into all or part of itself. Suppose that there are sufficiently many $\varphi \in \Phi$ to distinguish between the points in S. That is for $s, t \in S$ there is a $\varphi \in \Phi$ with $\varphi s = t$. Let F be the family of all subsets of S and consider the problem of determining whether or not there exists uniquely a set function m which satisfies

(α) m is finitely additive and bounded on F

(β) $m(S) = 1$

(γ) $m(E) = m(\varphi^{-1} E)$, $E \subset S$, $\varphi \in \Phi$,

where by $\varphi^{-1} E$ we mean the set of all $s \in S$ such that $\varphi s \in E$. If the functions $\varphi \in \Phi$ are 1—1 maps of S into all of itself then the condition (γ) is equivalent to

(γ') $m(E) = m(\varphi E)$ $E \subset S$, $\varphi \in \Phi$.

I shall merely point out here that an affirmative answer to the question is equivalent to the statement (2) where G is constructed as follows. Let X be the Banach space of all bounded real functions defined on S with $|f| \equiv$ l.u.b. $| f(s) |$. Let $T\varphi f = g$ where $g(s) = f(\varphi s)$. Let G be the family of all $I - T\varphi$ where $\varphi \in \Phi$. Thus $M(G)$ consists of all $f \in X$ for which $f(s) = f(\varphi s)$, $\varphi \in \Phi$. Since the points in S may be distinguished by the members of Φ the manifold $M(G)$ is one dimensional and consists of the constants. Thus if (2) holds $N(G)$ is a hyperplane and there is therefore one and only one linear functional (except for constant multiples) $x^* \neq 0$ which vanishes on $N(G)$. To say that x^* vanishes on $N(G)$ is equivalent to the statement

(γ'') $x^* f = x^* T \varphi f$, $\varphi \in \Phi$, $f \in X$.

Now the linear functional on X is given by an integral with respect to a finitely additive measure which is defined and bounded on F. Thus (γ'') gives us such a measure m with $\int_S f(s)\, dm = \int_S f(\varphi s)\, dm$ for every $f \in X$ and every $\varphi \in \Phi$. By taking f the characteristic function of an arbitrary set $E \subset S$ we obtain (γ). Since x^* does not vanish on $M(G)$ we have $x^* f_0 \neq 0$ where f_0 is the constant function $f(s) \equiv 1$ and so it may be assumed (by multiplying x^* by a constant if necessary) that

$$1 = x^* f_0 = \int_S dm = m(S)$$

and hence (α), (β), (γ) are all satisfied. Conversely if a unique measure m exists which satisfies (α), (β), (γ) then clearly a unique x^* exists which satisfies (γ''). This means that the number of linearly independent x^* for which $x^* N(G) = 0$ is 1. Hence $N(G)$ 'is a hyperplane. In view of (β) we have for the function $f_0(s) \equiv 1$ and the functional $x^* f = \int_S f(s)\, dm$

$$1 = m(S) = \int_S f_0(s)\, dm = x^*(f_0)$$

and so $f_0 \bar{\in} N(G)$. For an arbitrary $f \in X$ we may write

$$f = (x^* f) f_0 + [\, f - (x^* f) f_0 \,]$$

which establishes the decomposition (2).

An example quite analogous to the preceding concerns a bicompact Hausdorff space S and a family Φ of continuous functions φ which map S into all or part of itself. Suppose as before that there are sufficiently many $\varphi \in \Phi$ to distinguish between points in S. The question is that of determining when there exists uniquely a completely additive measure m defined for at least the Borel sets in S with $m(S) = 1$ and $m(E) = m(\varphi^{-1} E)$ for every $\varphi \in \Phi$ and every Borel set $E \subset S$. It may be shown in a fashion similar to that of the preceding example that the answer is in the affirmative if and only if (2) holds where the family G is constructed as follows. Let X be the Banach space of all real continuous functions on S. Let $T\varphi f = g$ be defined by the relation $f(\varphi S) = g(S)$ and let G consist of all operators of the form $I - T\varphi$ where $\varphi \in \Phi$. The chief difference between this example and the last is the representation of the linear functional on X. In case S is a metric space the general linear functional on X is given by a Lebesgue integral $\int_S f(s)\, dm$ with respect to a completely additive measure m which is defined for all Borel sets in S. This result was proved by Saks and has been generalized by Kakutani to the case where S is a bicompact Hausdorff space. In case S is a group and Φ consists of all functions of the form $\varphi s = a s$ where $a \in S$ it is seen therefore that the problem of establishing the existence and uniqueness of the Haar measure is equivalent to proving the decomposition (2). It should be pointed out in connection with the two preceding examples that in the case where G is a family of commutative operators (as it will be if, for example, S is an Abelian group) it is quite elementary to show that an invariant measure exists. The existence of such a measure in either case was a consequence of the fact that $N(G) \neq X$. Since $M(G)$ contains all constant functions, to show the existence of an invariant measure it suffices therefore to show that $M(G)\, N(G) = 0$. This may be done as follows.

Let $y \in M(G)$ and $z = (I-U_1)x_1 + \ldots + (I-U_s)x_s$ be an arbitrary element of $\Sigma\, T X$ where the Σ is taken over all $T \in G$. Let $(n, U) = 1/n \sum_{j=0}^{n-1} U^j$, $W_n = (n, U_1)\, (n, U_2) \ldots (n, U_s)$, so that

$$W_n\,(y + z) = y + W_n z$$

Since $(n, U)\,(I-U) = 1/n\,(I-U^n)$ we see that $|W_n z| \leqq k/n$ and since $|W_n| \leqq 1$ we have

$$|y + W_n z| \leqq |y + z|.$$

Letting $n \to \infty$ we have

$$|y| \leqq |y + z|.$$

From this it follows that the same inequality holds for $z \in N(G)$, and hence $M(G) \cdot N(G) = 0$.

Finally let us consider the Wiener closure theorem in $L(-\infty, \infty)$ which asserts that linear combinations of the translations $f(x+\lambda)$ of a function which is integrable in the sense of Lebesgue on $-\infty < x < \infty$, are dense in this space $L(-\infty, \infty)$ of such functions if and only if the Fourier transform $F(f, u)$ of f has no real zeros. This theorem is equivalent to the decomposition (1) where T is defined as follows. Suppose

$f \epsilon L(-\infty, \infty)$ and that its Fourier transform has no real zeros. Let $Tg = h$ be a transformation in $X = L(-\infty, \infty)$ defined by placing $h = f * g$ where $f * g$ is the convolution of f and g. Let us recall that the Fourier transform has the properties

(i) $F(h*g, u) = F(h, u) \cdot F(g, u)$,

(ii) $F(h, u) = 0$, $-\infty < u < \infty$ if and only if $h = 0$.

Also we note that if the range TX of T is dense in X, i. e., if the functions $f*g$, as g varies, are dense in $L(-\infty, \infty)$ then the linear combinations $\sum_{i=1}^{n} a_i f(x + \lambda_i)$ are dense in $L(-\infty, \infty)$. This is an elementary consequence of the continuity of T and of the fact that step functions are dense in $L(-\infty, \infty)$. Now suppose that the decomposition (1) holds. Equations (i) and (ii) show that $M(T) = 0$ and hence $N(T) = L(-\infty, \infty)$. This means that the range TX and hence the linear combinations of the translation of f are dense in $L(-\infty, \infty)$. Similar arguments show that conversely Wiener's closure theorem implies the decomposition (1) for any $f \epsilon L(-\infty, \infty)$ whose Fourier transform $F(f, u)$ has no real zeros.

Wiener's general closure theorem in $L(-\infty, \infty)$ concerns a class Σ of functions in L and asserts that the linear manifold determined by the translations of all the functions $f \epsilon \Sigma$ is dense in L if and only if there is no real zero in common to all the Fourier transforms of functions in Σ. This theorem we mention as a final example of a theorem illustrating the decomposition (2). The construction and proof are the same as for the preceding example with the exception that here we have the family G of all operators $f*g$ where f ranges over Σ.

I should like now to indicate one solution of the problem of passing from the decomposition (1) to the decomposition (2). This problem, as was mentioned earlier, may be stated for the case of commutative operators as the question — which Boolean algebras of projections in a Banach space X may be embedded in a complete Boolean algebra of projections in X?

Suppose that X is a reflexive Banach space and that B is a Boolean algebra of projections in X which satisfies the condition

$$|E| \leqq M, \qquad E \epsilon B$$

$f \epsilon L(-\infty, \infty)$ and that its Fourier transform has no real zeros. Let $Tg = h$ be a transformation in $X = L(-\infty, \infty)$ defined by placing $h = f^* g$ where $f^* g$ is the convolution of f and g. Let us recall that the Fourier transform has the properties

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Also we note that if the range TX of T is dense in X, i. e., if the functions f^*g, as g varies, are dense in $L(-\infty, \infty)$ then the linear combinations $\sum_{i=1}^{n} a_i f(x + \lambda_i)$ are dense in $L(-\infty, \infty)$. This is an elementary consequence of the continuity of T and of the fact that step functions are dense in $L(-\infty, \infty)$. Now suppose that the decomposition (1) holds. Equations (i) and (ii) show that $M(T) = 0$ and hence $N(T) = L(-\infty, \infty)$. This means that the range TX and hence the linear combinations of the translation of f are dense in $L(-\infty, \infty)$. Similar arguments show that conversely Wiener's closure theorem implies the decomposition (1) for any $f \epsilon L(-\infty, \infty)$ whose Fourier transform $F(f, u)$ has no real zeros.

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Suppose that X is a reflexive Banach space and that B is a Boolean algebra of projections in X which satisfies the condition

$$|E| \leqq M, \qquad E \epsilon B$$

where M is some constant dependent upon B. Such a Boolean algebra we shall call a *bounded Boolean algebra of projections with bound* M. Now consider an arbitrary set $\sigma \subset B$ and let $y \epsilon M(\sigma)$ and $z = E_1 x_1 + \ldots + E_s x_s \epsilon \sum_{E \epsilon \sigma} EX = \sigma X$. If $W = E_1' \ldots E_s'$ where $E' = I - E$ then $Wz = 0$ and $Wy = y$ so that $|y| = |W(y+z)| \leqq M\,|y+z|$. Since $N(\sigma)$ is by definition the closure of σX this last inequality shows that $M(\sigma) \cdot N(\sigma) = 0$. If B^* is the Boolean algebra of all adjoints E^* where $E \epsilon B$ and if σ^* consists of all $E^* \epsilon B^*$ corresponding to an $E \epsilon \sigma$ then we have similarly that $M(\sigma^*) \cdot N(\sigma^*) = 0$. Using the reflexive property of X some elementary calculations show, however, that $M(\sigma)^{\perp} = N(\sigma^*)$, $N(\sigma)^{\perp} = M(\sigma^*)$ where the symbol $\Gamma^{\perp}$ stands for the set of linear functionals which vanish on Γ. Since

$$M(\sigma^*)\ N(\sigma^*) = M(\sigma)^{\perp} \cdot N(\sigma)^{\perp} = 0$$

we see that no non-zero linear functional vanishes on both

$$M(\sigma) \text{ and } N(\sigma).$$

Hence the fundamental Hahn-Banach extension theorem shows that $M(\sigma) \oplus N(\sigma)$ is dense in X. The inequality $|y| \leqq M\,|y+z|$ derived above allows us therefore to state the fundamental decomposition

$$M(\sigma) \oplus N(\sigma) = X.$$

The projection $E(\sigma)$ of X onto $M(\sigma)$ is clearly the greatest lower bound of all E in σ, i.e., $E(\sigma) = \wedge \sigma$. A number of results may be readily obtained from this decomposition among them being the

Teorem *Every bounded Boolean algebra of projections in a reflexive Banach space X may be embedded in a complete Boolean algebra of projections in X.*

Beside the method outlined above for the completion of a Boolean algebra there are two other methods leading to the same result. One method is as follows. Consider an arbitrary directed set of elements α. Consider a function of α whose value E_α is an element of B. Then the family of all E. H. Moore limits of the type

$$\lim_{\alpha} E_\alpha x = Ex,\ x \varepsilon X$$

constitute the same completion of B. One may also define a topology in the ring of all operators on X by defining a neighborhood of such an operator T as the set of all operators U such that

$$|Tx_i - Ux_i| < \varepsilon, \quad i = 1, \ldots, n$$

where $\varepsilon > 0$ and $x_1, \ldots, x_n$ are the quantities determining the neighborhood. In terms of this topology the closure of B is again the complete Boolean algebra of projections determined by B.

I should like to conclude by mentioning two corollaries of the theorem. The first is a theorem due to Kantorovitch and G. Birkhoff, and it asserts that in a space with finite Lebesgue measure the family of measurable sets is complete under the order relation $A \subset B$ if and only if measure $(A - B) = 0$. The second application is concerned with integral equation theory. Suppose that T is a linear operator in a reflexive space X whose resolvent $R(\lambda, T)$ is a meromorphic function of $1/\lambda$. Suppose that the projections $E(C) = \frac{1}{2}\pi i \int_C R(\lambda, T)\, d\lambda$ are bounded as the contour C varies over the resolvent set of T. Under these conditions it follows that for every $x \in X$ the series $\sum_i E_{\lambda_i} x$ is unconditionally convergent. Here we have used the symbol E_{λ_i} for the projection $E(C)$ where C is a contour containing λ_i but no other spectral point of T in its interior. The relation between this statement and the theorem is seen as follows. Since $E_{\lambda_i} E_{\lambda_j} = 0$, $i \neq j$ it is seen that the union

$$\vee(E_{\lambda_1}, \ldots, E_{\lambda_n}) = \underset{1 \leqq i \leqq n}{\text{l.u.b.}} \; E_{\lambda_i} = E_{\lambda_1} + \ldots + E_{\lambda_n}$$

By the theorem we have a projection E in X with

$$E = \underset{1 \leqq n < \infty}{\text{l.u.b}} \; E_{\lambda_n} = \underset{1 \leqq n < \infty}{\text{l.u.b}} \; (E_{\lambda_1} + \ldots + E_{\lambda_n}).$$

This projection E may be shown to be (as indicated earlier) the E. H. Moore limit

$$Ex = \lim_{\pi} \sum_{i \in \pi} E_{\lambda_i} x$$

where π is an arbitrary finite set of integers. Thus the series $\sum_{i=1}^{\infty} E_{\lambda_i} x$ is unconditionally convergent.

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