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SOME FORMULAS IN THE THEORY OF SURFACES*

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Introduction

The object of this paper is to study certain integro-differential invariants of a closed orientable surface differentiably imbedded in the ordinary Euclidean space. Some identities between them will be established. As applications we shall give new proofs of some theorems of Liebmann characterising the sphere as the only closed convex surface with constant mean curvature and constant Gaussian curvature respectively.

1. Fundamental Formulas

To write down the basic formulas in the theory of surfaces we follow the classical treatment of G. Darboux and

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E. Cartan by employing the method of moving trihedrals[1]. A right-handed trihedral $Pe_1e_2e_3$ in the Euclidean space E consists of a point P and three mutually perpendicular unit vectors e_1, e_2, e_3, which form a right-handed system. With the choice of a fixed origin 0 in E, P determines and is determined by a position vector which will also be denoted by P. When a family of such trihedrals is given, depending differentiably on certain parameters, we can write

$$(1) \qquad \begin{aligned} dP &= \sum_k \omega_k e_k \,, \\ de_i &= \sum_k \omega_{ik} e_k \,, \end{aligned} \qquad i,k = 1,2,3,$$

where ω_i, ω_{ik} are linear differential forms (in these parameters) satisfying the relations

$$(2) \qquad \omega_{ik} + \omega_{ki} = 0 \,.$$

Since the exterior derivative of an exterior derivative is zero, we have

$$(3) \qquad \begin{aligned} d(dP) &= 0 \,, \\ d(de_i) &= 0 \,. \end{aligned}$$

Applying these relations to (1), we get

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[1] A modern version of this method can be found in W. Blaschke, Elementare Differentialgeometrie, Berlin 1950.

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki} , \qquad\qquad i,j,k = 1,2,3, \tag{4}$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} ,$$

where the "wedge product" is the product in the sense of exterior multiplication of Grassmann. Equations (4) are call the equations of structure of the group of proper motions in E.

Now suppose a closed orientable surface S be given in E. It determines a family of (right-handed) triehedrals $Pe_1e_2e_3$ such that $P \in S$ and e_3 is the unit vector in the direction of the outward normal of S at P. For this family of trihedrals we have

$$\omega_3 = 0 . \tag{5}$$

The first equation of (4) then gives

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0 .$$

Since ω_1, ω_2 are linearly independent, this implies the relations

$$\omega_{13} = a\omega_1 + b\omega_2 , \tag{6}$$
$$\omega_{23} = b\omega_1 + c\omega_2 .$$

The coefficients a,b,c in these linear combinations are not invariants of S. But it is easy to construct invariants from them. In particular, the mean curvature H and the Gaussian curvature K are given by the formulas

$$(7)\qquad \begin{aligned} 2H &= a + c \quad , \\ K &= ac - b^2 \quad . \end{aligned}$$

The so-called first and second fundamental forms are

$$(8)\qquad \begin{aligned} I &= dP^2 = \omega_1^2 + \omega_2^2 \quad , \\ II &= -dPde_3 = \omega_1\,\omega_{13} + \omega_2\,\omega_{23} = a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2 \quad . \end{aligned}$$

Their ratio II/I is the normal curvature. We also observe that the element of area is given by the exterior quadratic form

$$(9)\qquad d\Sigma = \omega_1 \wedge \omega_2 \quad .$$

2. Some Identities

We consider the scalar products

$$(10)\qquad y_i = Pe_i \quad .$$

Geometrically, y_i is the oriented distance from the origin 0 to the plane Pe_je_k, $j,k \neq 1,2,3$. In particular, y_3, which we shall also write as p, is the oriented distance from 0 to the tangent plane $\pi(P)$ to S at P. The purpose of this paper is to study the integrals

$$(11) \qquad A_n = \int p^n \, d\Sigma \,, \qquad B_n = \int p^n H d\Sigma \,, \qquad C_n = \int p^n K d\Sigma \,,$$

where the integrations are over S. Moreover, let P' be the foot of perpendicular from 0 to $\pi(P)$, and let d_P be the distance PP' and ρ_P the normal curvature in the direction PP'. We introduce also the integrals

$$(12) \qquad D_n = \int p^n \rho_P \, d_P^2 \, d\Sigma \,, \qquad E_n = \int p^n d_P^2 \, K d\Sigma \,,$$

where the integrations are again extended over S. The main result of this paper consists in the identities

$$(13) \qquad 2A_{n-1} + 2B_n - (n-1) D_{n-2} = 0 \,, \qquad 2B_{n-1} + 2C_n - (n-1) E_{n-2} = 0 \,.$$

Before proceeding to the proof let us notice that the integrals introduced in (11) and (12) are independent of the choice of the point 0 and are therefore invariants of the surface S. At least for small values of n, they have simple geometrical meanings. For instance, A_0 is the area A of S, B_0 is the integral of mean curvature $-M$ first considered by Minkowski in the case of convex surfaces, while C_0 is 2π times the Euler characteristic χ of S, by the Gauss-

Bonnet formula.

To prove the formulas (13) we derive first from (10) and (1):

$$dy_1 = \omega_1 + \omega_{12}\, y_2 + \omega_{13}\, y_3 ,$$

(14) $$dy_2 = \omega_2 + \omega_{21}\, y_1 + \omega_{23}\, y_3 ,$$

$$dy_3 = \quad - \omega_{13}\, y_1 - \omega_{23}\, y_2 .$$

From (14) and (4) we then find

$$d(y_1\, \omega_2 - y_2\, \omega_1) = 2(1 + pH)\, \omega_1 \wedge \omega_2 ,$$

$$d(y_1 \omega_{23} - y_2 \omega_{13}) = 2(H + pK)\, \omega_1 \wedge \omega_2 .$$

It follows that

$$d\{p^{n-1}(y_1\omega_2 - y_2\omega_1)\} = \{2p^{n-1}(1+pH) - (n-1)p^{n-2}$$
$$(ay_1^2 + 2by_1y_2 + cy_2^2)\}\, d\Sigma ,$$

(15)

$$d\{p^{n-1}(y_1\omega_{23} - y_2\omega_{13})\} = \{2p^{n-1}(H+pK) - (n-1)p^{n-2}$$
$$(y_1^2 + y_2^2)\, K\}\, d\Sigma .$$

On the other hand, we have

$$d_p^2 = y_1^2 + y_2^2 ,$$

(16)

$$d_p^2\, \rho_p = ay_1^2 + 2by_1y_2 + cy_2^2 .$$

Since the linear differential forms $p^{n-1}(y_1\omega_2 - y_2\omega_1)$ and $p^{n-1}(y_1\omega_{23} - y_2\omega_{13})$ remain invariant under a rotation of the vectors e_1, e_2, they are defined on S. Applying Stokes's Theorem to (15), we find therefore that the integrals over S of the right-hand sides of (15) are zero. These give precisely the identities (13).

For $n = 1$ formulas (13) give

$$A_0 + B_1 = 0 ,$$

$$B_0 + C_1 = 0 ,$$

or

$$(17) \qquad \begin{aligned} \int pH d\Sigma + A &= 0 , \\ \int pK d\Sigma - M &= 0 . \end{aligned}$$

The second formula in (17) is a well-known formula of Minkowski, which he established for convex surfaces. According to our derivation it is valid for a closed surface of arbitrary genus.

The quadratic polynomials in x defined by

$$(18) \qquad \Delta_n(x) \equiv \int p^n \begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix} d\Sigma = C_n - 2xB_n + x^2 A_n ,$$

seem to deserve some interest. For $n = 0, 1$ we have

$$(19) \qquad \begin{aligned} \Delta_0(x) &= 2\pi X + 2xM + x^2 A , \\ \Delta_1(x) &= M + 2xA + x^2 A_1 . \end{aligned}$$

If S is convex, A_1 is clearly 3 times the volume V bounded by S. Since $X = 2$, the conditions that the discriminants of $\Delta_0(x)$, $\Delta_1(x)$ are non-negative can be written

$$M^2 - 4\pi A \geqslant 0 ,$$

(20)

$$A^2 - 3MV \geqslant 0 .$$

These are precisely the well-known "isoperimetric inequalities". It seems therefore reasonable to conjecture that

$$B_n^2 - A_n C_n \geqslant 0 ,$$

but a proof appears to be difficult. This conjecture is supported moreover by the fact that the discriminant of the integrand in (18) is

$$p^{2n} (H^2-K) ,$$

which is everywhere $\geqslant 0$.

3. Applications

We shall apply the above results to the cases that H and K are respectively constant. Suppose H be constant. Then

$$B_n = HA_n ,$$

and we have

$$\Delta_n(H) = C_n - H^2 A_n$$

From (17) we get

$$A_0 + HA_1 = 0 , \quad HA_0 + C_1 = 0 .$$

It follows that

$$\int p \begin{vmatrix} a-H & b \\ b & c-H \end{vmatrix} d\Sigma = \Delta_1(H) = C_1 - H^2 A_1 = 0 .$$

Now we have

$$\begin{vmatrix} a-H & b \\ b & c-H \end{vmatrix} = -\tfrac{1}{4} (a-c)^2 - b^2 \leqslant 0 .$$

If, therefore, there is a point O such that $p < 0$ for all points P of S, the integrand in $\Delta_1(H)$ will keep the same sign and the relation $\Delta_1(H) = 0$ is possible only when

$$a = c, \quad b = 0 ,$$

that is, only when every point of S is an umbilic. This leads to the theorem:

A closed surface of constnat mean curvature is a sphere, if there is a point lying on the negative sides of all tangent planes of S. *In particular, a closed convex surface of constant mean curvature is a sphere.*

Consider next the case that K is constant, from which we shall derive the theorem of Liebmann that S is a sphere. In fact, we have

$$C_n = KA_n$$

Since K has to be positive, we choose

$$R = +\sqrt{K} > 0 .$$

From (18) we get, be applying the formulas (17),

$$\Delta_0(R) = 2R^2 (A_0 + RA_1) ,$$

$$\Delta_1(R) = 2R (A_0 + RA_1) .$$

It follows that

(21) $$\Delta_0(R) = R\Delta_1(R) .$$

Since S is convex, 0 can be chosen to be an interior point of S, so that $p < 0$. On the other hand, we know that

$$\begin{vmatrix} a-R & b \\ b & c-R \end{vmatrix} \leqslant 0 ,$$

and that the equality sign holds only when $R = a = c$, $b = 0$. It follows that

$$\Delta_0(R) \leqslant 0 , \quad \Delta_1(R) \geqslant 0 .$$

Hence the equation (21) is possible only when

$$\Delta_0(R) = \Delta_1(R) = 0 ,$$

that is, only when

$$R = a = c , \quad b = 0 .$$

This proves that S is a sphere.

Added. August 14, 1952.

On p.34 line 14, immediatly after formula (13), read:

"Before proceeding to the proof let us notice that the integrals introduced in (11) and (12) are, for some values of n, independent of the choice of the point 0 and are therefore invariants of the surface S. This is true for $n = 0, 1$, and the corresponding invariants have simple geometrical meanings. For instance, A_0 is the area A of S, B_0 is the integral of mean curvature $-M$ first considered by Minkowski in the case of convex surfaces, while C_0 is 2π times the Euler characteristic χ of S, by the Gauss-Bonnet formula.

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