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NON-NORMAL TRUTH-TABLES FOR THE PROPOSITIONAL CALCULUS*

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Besides the usual two-valued truth-tables for the propositional calculus, it is known that there are many characteristic[1]** systems of truth-tables (characteristic matrices[2]) in which there are more than two truth-values.

In particular, since the two-valued truth-tables constitute a two-element Boolean algebra[3], any sustem of truth-tables having the two-element Boolean algebra as a homomorphic image[3] will be a characteristic system if the designated truth-values are taken to be those which have the unit of the Boolean algebra as their image. In this way characteristic systems of truth-tables may be obtained with any number of truth-values (not less than two)[4]. Characteristic systems of truth-tables of this kind we shall call *normal in the sense of Carnap*, and

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**Véanse todas las notas al final del artículo.

all others will be called *non-normal in the sence of Carnap*[5], or *weakly non-normal.*

A characteristic system of truth-tables may also be obtained from an arbitrary Boolean algebra by taking the unit, I, of the algebra as the single designated value[6]. If the Boolean algebra has more than two elements, the resulting characteristic system of truth-tables is weakly non-normal. And we may obtain additional weakly non-normal characteristic systems by means of homomorphisms, as before, taking any system of truth-tables to which the Boolean algebra is homomorphic, and taking the designated truth-values to be those which have the unit of the Boolean algebra as image.

Characteristic systems of truth-tables obtained in this way are necessarily *regular*[7], in the sense that $p \supset q$ never has a designated truth-value when p has a designated truth-value and q a non-designated truth-value.

An example of such a characteristic system of truth-tables is provided in Tables I, at the end of this paper. Indeed, Tables I exhibit a four-element Boolean algebra, with the unit, I, of the Boolean algebra as the designated truth-value. Thus they are the simplest example of a weakly non-normal characteristic system of truth-tables for the propositional calculus (or, as we shall say briefly, of weakly non-normal truth-tables for the propositional calculus).

Tables II and III provide examples of weakly non-normal truth-tables for the propositional calculus which are of a different kind[8], since they are non-regular.

However, all three examples, Tables I, II, and III, are in a certain sense trivial, since they become normal in the sense of Carnap if (without other change) the designated truth-values are taken to be b and 1, instead of 1 alone. In fact, in each case, a simple proof that the tables are characteristics can be obtained by first showing that the tautologies are the same whether b and 1 or 1 alone are taken as the designated truth-values, and then observing that, when b and 1 are taken as designated, the tables are normal in the sense of Carnap.

Thus every example so far found of a characteristic system of truth-tables for the propositional calculus either is a Boolean algebra or reduces to such under a homomorphism. It is morover well known that the propositional calculus is *formally* a Boolean algebra, in the sense that every identically true Boolean equation becomes a theorem of the propositional calculus if the variables in the Boolean equation are replaced by (or reconstrued as) propositional variables, the signs for the Boolean complement, the Boolean product, and the Boolean sum are replaced respectively by the signs of negation, conjunction, and disjunction, the signs, 0 and 1, for the Boolean zero and unit are replaced by (e.g.) $p \sim p$ and $p \vee \sim p$ respectively, and the sign of equality, =, is replaced by the sign of material equivalence, ≡. Likewise every statement of inclusion which is identically true for Boolean algebras becomes a theorem of the propositional calculus if the same replacements are made as just described and at the same time the sign of inclusion, ⊂ , is replaced by the sign of material implication, ⊃ .

For this reason it might be natural to suppose that every characteristic system of truth-tables for the propositional calculus either is a Boolean algebra (of two or more elements) or reduces to such under a homomorphism.

And indeed this does follow if we assume that the truth-table of $\supset$ is regular (in the sense already explained). For since

$$[p \equiv q] \supset [q \equiv p]$$

and

$$[p \equiv q] \supset [[q \equiv r] \supset [p \equiv r]]$$

are theorems of the propositional calculus and therefore tautologies according to the given system of truth-tables, it follows from the regularity of the truth-table of $\supset$ that the truth-table of $\equiv$ is symmetric and transitive. I.e., if, for particular truth-values of p and q, $p \equiv q$ has a designated value, then $q \equiv p$ must have a designated value; and if, for particular truth-values of p, q, and r, both $p \equiv q$ and $q \equiv r$ have designated values, then $p \equiv r$ must have a designated value. The truth-values are thus divided into equivalence-classes, two truth-values x and y belonging to the same equivalence-class if and only if $p \equiv q$ has a designated truth-value for the values x, y of p, q. Moreover the equivalence-class to which the value of an expression P belongs will remain unchanged if the value x of one of the variables, say p, is altered in such a way as to leave unchanged the equivalence-class to which x belongs as follows from the regularity of the truth-table of $\supset$, together with the fact that

$$[p \equiv q] \supset [P \equiv Q]$$

is a theorem of the propositional calculus if Q is obtained from P by substituting q for p. Hence a homomorphism such that two truth-values have the same image if and only if they belong to the same equivalence-class will be a homomorphism of the given system of truth-tables into a Boolean algebra[9].

If, however, regularity fails in the truth-table of $\supset$, there is the possibility of obtaining a characteristic system of truth-tables for the propositional calculus of such a sort that no Boolean algebra is homomorphic to it. Such a characteristic system of truth-tables we shall call *strongly non-normal*.

Tables IV are a rather obvious example of strongly non-normal truth-tables for the propositional calculus, being so constructed that they follow the usual two-valued truth-tables with regard to the values 0 and 1, and that the value of an expression is always h for any system of values of the propositional variables that includes the value h for any system of values of the propositional variables that includes the value h for one of the variables. A large variety of more elaborate variations on this theme are evidently possible.

Perhaps more interesting as an example of strongly non-normal truth-tables for the propositional calculus are Tables V, due to Z.P. Dienes[10].

In order to see that Tables V are characteristic of the propositional calculus, notice first that the usual two-valued truth-tables are followed in the case of the values 0 and 1, and hence that no non-theorem can be a tautology. Now for an expression P, consider a system S of values of its variables that includes the value h for one or more of them. At each occurence of a variable having the value h, replace the h by 0 or 1, taking the various occurrences independently, and abandoning the requirement (as regards these variables) that the same value be assigned to different occurrences of the same variable. If on doing this in all possible ways the value of P is always 0 or always 1, then the value of P is 0 or 1, respectively, for the system S of values of its variables; but otherwise the value of P is h for the system S of values of its variables. I.e., Tables V are so constructed that this will be the case, as may readily be verified. Since h is a designated value, it follows that every tautology in the truth-values 0 and 1 remains a tautology when the additional truth-value h is admitted. Hence every theorem of the propositional calculus is a tautology (in the three truth-values, 0, h, 1).

These examples are brought together here in order to raise the question whether exist other strongly non-normal truth-tables for the propositional calculus, beyond those cited (together with direct products and other obvious elaborations); or more generally, to raise the question of a survey or characterization in some sense of the possible strongly non-normal truth-tables for the propositional calculus.

Another motive is the suggestion, which was made to me by Paco Lagerström ten years or more ago, that use may be made of non-normal truth-tables for the propositional calculus in order to extend to the functional calculi of first and higher orders, and other related systems, the method of proving independence of axioms which is familiar in the case of the propositional calculus.

In question at that time were only Boolean algebras, in the rôle of weakly non-normal truth-tables. And the remark was made in particular by Lagerström[11] that in my *Formulation of the Simple Theory of Types*[12] the independence of the axioms 9^{α} from axioms 1–8 and $10^{\alpha\beta}$ can be established by means of a complete non-atomic Boolean algebra. For this purpose, axioms 9^{α} are to be rewritten in the weaker form

$$(\exists t_{\alpha(o\alpha)}) \cdot f_{o\alpha} x_{\alpha} \supset \cdot (y_{\alpha}) [f_{o\alpha} y_{\alpha} \supset x_{\alpha} = y_{\alpha}] \supset f_{o\alpha}(t_{\alpha(o\alpha)} f_{o\alpha}),$$

since the question of independence would otherwise be trivial. The range of the variables of type o is to be the Boolean algebra in question; the range of the variables of type ι is to be the natural numbers; and the range of the variables of type $\alpha\beta$ is to be the functions from B to A, where A and B are the ranges of the variables of types α and β respectively. The universal quantifier is to correspond to the infinite Boolean product, so that the value of, say, $(x_{\iota})M$ for a given system of values of the free variables is

$$\prod_{i=0}^{\infty} \Phi(i) ,$$

where $\Phi(i)$ is the value of M for the value i of x_{ι}.

In a similar fashion, the independence of the axioms 6^{α} can be established by means of a four-element Boolean algebra (Tables I). The value of $(x_{\alpha})M$ is to be taken as 1 in case the value of M is 1 for all values of x_{α}, and as 0 in all other cases.

Footnotes.

[1] In any particular system of propositional calculus (we shall here, however, be concerned with only one such system, the classical propositional calculus), an expression of the calculus is called a *tautology* according to a particular system of truth-tables if, for every system of truth-values of its variables, the truth-value of the expression, as obtained from the truth-tables, belongs to the class of *designated* truth-values. And a system of truth-tables is called *characteristic* of a particular system of propositional calculus if the theorems of the propositional calculus are the same as the tautologies according to the truth-tables.

[2] For a system of truth-tables, the term "matrix" is becoming usual, in spite of the awkward conflict between this use of the word "matrix" and the quite different use of the same word which was introduced in *Principia Mathematica*. (The long-established use of this word in algebra is of course still a third use.)

[3] In this paper, whenever a Boolean algebra is spoken of as being a system of truth-tables for the propositional calculus or as being homomorphic to such a system of truth-tables, it is to be understood that negation, conjunction, and disjunction are represented respectively by the Boolean complement, the Boolean product, and the Boolean sum. The Boolean representatives of (material) implication and equivalence are then obtained by rewritting, $P \supset Q$ and $P \equiv Q$ as $\sim P \vee Q$ and $PQ \vee \sim P \sim Q$ respectively.

[4]Some examples of this kind are given, e.g., by K.Schröter in *Zentralblatt für Mathematik und ihre Grenzgebiete*, vol. 37 (1951), p. 4.

[5]This is a minor modification of the terminology of Carnap, who (in his *Formalization of Logic*, Cambridge, Mass., 1943) speaks rather of normal and non-normal true interpretations where any system of truth-tables will provide an interpretation of the propositional calculus, which will be a true interpretation if only every theorem is a tautology.

[6]This was perhaps first pointed out explicitly by B.A.Bernstein (in a different terminology) in *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 390 and 592.

[7]We adopt this term from McKinsey and Tarski - see *The Journal of Symbolic Logic*, vol. 13 (1948), p. 11.

[8]Tables III are a simplified version of tables given in exercise 19.11 of the forthcoming revised edition of the writer's *Introduction to Mathematical Logic, Part I*. (The latter tables, unlike Tales III, are so arranged that symmetry fails in the truth-table of ≡.)

[9]Here we make use of the fact that a Boolean algebra can be characterized by conditions which have exclusively the form of identically true equations of the algebra, together with the conditions that the complement, sum, and product exist and that there are at least two elements. (Some of Huntington's systems of postulates for Boolean algebras are, for example, substantiall, in this form.) Also use is made of the fact that the propositional calculus is formally a Boolean algebra, in the sense explained above.

[10]In *The Journal of Symbolic Logic*, vol. 14 (1949), pp. 95-97. The remark that these truth-tables are characteristic for the propositional calculus, and that no Boolean algebra is homomorphic to them, was made by Church and Rescher, ibid., vol. 15 (1940), pp. 69-70.

[11]It has never been published.

[12]*The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56-68. See errata, ibid, vol. 6, p. iv.

p	q	p ⊃ q	pq	p ∨ q	p ≡ q	~p
0	0	1	0	0	1	1
0	a	1	0	a	b	
0	b	1	0	b	a	
0	1	1	0	1	0	
a	0	b	0	a	b	b
a	a	1	a	a	1	
a	b	b	0	1	0	
a	1	1	a	1	a	
b	0	a	0	b	a	a
b	a	a	0	1	0	
b	b	1	b	b	1	
b	1	1	b	1	b	
1	0	0	0	1	0	0
1	a	a	a	1	a	
1	b	b	b	1	b	
1	1	1	1	1	1	

Table I. Designated value 1.

p	q	$p \supset q$	pq	$p \vee q$	$p \equiv q$	$\sim p$
0	0	1	0	0	1	1
0	b	1	0	1	0	
0	1	1	0	1	0	
b	0	0	0	1	0	0
b	b	1	1	1	1	
b	1	1	1	1	1	
1	0	0	0	1	0	0
1	b	1	1	1	1	
1	1	1	1	1	1	

Tables II. Designated value 1.

p	q	$p \supset q$	pq	$p \vee q$	$p \equiv q$	$\sim p$
0	0	1	0	0	1	1
0	a	1	0	0	1	
0	b	1	0	1	0	
0	1	1	0	1	0	
a	0	1	0	0	1	b
a	a	1	a	a	1	
a	b	b	0	1	0	
a	1	1	0	1	0	
b	0	0	0	1	0	a
b	a	a	0	1	0	
b	b	1	b	b	1	
b	1	1	1	1	1	
1	0	0	0	1	0	0
1	a	0	0	1	0	
1	b	1	1	1	1	
1	1	1	1	1	1	

Tables III. Designated value 1.

p	q	p ⊃ q	pq	p ∨ q	p ≡ q	~p
0	0	1	0	0	1	1
0	h	h	h	h	h	
0	1	1	0	1	0	
h	0	h	h	h	h	h
h	h	h	h	h	h	
h	1	h	h	h	h	
1	0	0	0	1	0	0
1	h	h	h	h	h	
1	1	1	1	1	1	

Tables IV. Designated values h and 1

p	q	p ⊃ q	pq	p ∨ q	p ≡ q	~p
0	0	1	0	0	1	1
0	h	1	0	h	h	
0	1	1	0	1	0	
h	0	h	0	h	h	h
h	h	h	h	h	h	
h	1	1	h	1	h	
1	0	0	0	1	0	0
1	h	h	h	1	h	
1	1	1	1	1	1	

Tables V. Designated values h and 1.