A SIMPLIFIED PROOF OF THE REDUCTION OF
ALL MODALITIES TO 42 IN S 3.[*]
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We write N for negation, M for possibility, L for necessity, → for strict implication, = for strict equivalence.

Some very elementary proofs are omitted, others may be found in Parry's fundamental paper (J S L IV, pp.137-154), the theorems of Parry being mentioned as "P ...".

0. Definitions, and consequences.

00. Df $Lp = NMN\ p$

01. Df $Op = NM\ p$

02. Df $Yp = NMM\ p$

03. $Op = LN\ p$

04. $Yp = LLN\ p$

05. $Yp \rightarrow Op$

06. $Yp = OM\ p$

07. $Yp = LO\ p$

1. Axiom proper to S 3, and consequences.

10. $(p \rightarrow q) \rightarrow (Mp \rightarrow Mq)$

15. If X is an affirmative modality, then

$$(p \rightarrow q) \rightarrow (X\ p \rightarrow X\ q)$$

16. If X' is a negative modality, then

$$(p \rightarrow q) \rightarrow (X'\ q \rightarrow X'\ p)$$

2. First key-theorem for reductions, and consequences.

20.	LL p $\rightarrow$ LLL p	(P 32.2)
21.	LL p = LLL p	(P 32.21)
22.	MM p = MMM p	(P 32.23)
25.	Y p = YM p	From 02, by 22.
26.	Y p = LY p	From 04, by 21.

3. Second key-theorem for reductions, and consequences.

30.	0 p $\rightarrow$ 000 p	(P 32.3)
31.	Y p $\rightarrow$ 00Y p	From 30 by 10 and 06.
35.	0000 p = 00 p	(P 32.31)
36.	000Y p = 0Y p	From 35 by 06.
37.	Y000 p = Y0 p	From 35 by 07.

4. Third key-theorem for reductions, and consequences.

40. 0 p $\rightarrow$ 0YY p   (P 32.5)

41. YY p = Y0 p

(1)	YY p $\rightarrow$ 00YY p	31 subst.
(2)	LYY p $\rightarrow$ L00YY p	By 15
(3)	YY p $\rightarrow$ Y0YY p	By 26 and 07.
(4)	Y0YY p $\rightarrow$ Y0 p	From 40 by 16.
(5)	YY p $\rightarrow$ Y0 p	From (3) and (4).

    (6) YO p → YY p   From 05 by 16.
    (7) Th   From (5) and (6).
   42. YOO p = YYO p = YYY p = YOY p   By 41.
   43. YOOO p = YYOO p = YYYO p = YOYO p   42 subst.
   44. YOOY p = YYOY p = YYYY p = YOYY p   42 subst.

5. Lemmas.
   51. YOYO p = YO p
     (1) YOOO p = YO p   37.
     (2) Th   By 43.
   511. YOYY p = YY p
     (1) YOYOM p = YOM p   51 subst.
     (2) Th   By 06.
   52. OYOO p = OY p
     (1)   O p → OOO p   30.
     (2)   OLOOO p → OLO p   By 16.
     (3)   OYOO p → OY p   By 07.
     (4)   OY p → OYYY p   40 subst.
     (5)   OY p → OYOO p   By 42.
     (6) Th   From (3) and (5).
   521. OYOY p = OY p
     (1) OYOOM p = OYM p   52 subst.
     (2) Th   By 06 and 25.

6. The reduction.
   61. L being defined (in OO) by means of M and N,
we can express any proper modality by means of M and N
only.

    Two consecutive N may be cancelled and as by 22 any
three consecutive M reduce to two, we can express any modality

by a sequence of symbols being (at most) alternatively N, and
M or MM. We call such sequences *simplified modalities.*

Simplified proper modalities may be divided into 4
types:

Type A: beginning with N, ending with M,
Type B: beginning with M, ending with M,
Type C: beginning with N, ending with N,
Type D: beginning with M, ending with N.

62. Now the reduction for type A modalities, it is
clear that these may be written as sequences of O and Y
only.

621. With ONE symbol O or Y we have *two* modalities:

$$O \ p, \quad Y \ p.$$

622. With TWO symbols O or Y we might have four
modalities: OOp, OY p, YO p, YY p.

But YY p = YO p   (41)

Hence there remain only *three* distinct modalities
(three not-equivalent modalities): OO p, OY p, YO p.

623. We might have six distinto modalities with *three*
symbols, these beginning with OO, OY or YO.

But YOY p = YOO p   42.

OYY p = OYO p   By 41.

Hence *four* modalities only are left: OOO p, OOY p,
OYO p, YOO p.

624. We might have eight distinct modalities, with
FOUR symbols, these beginning with OOO, OOY, OYO, YOO. But
modalities beginning with OOO, OYO, YOO reduce to modalities
with two symbols O or Y only.

$$0000\ p = 00\ p \qquad 35.$$
$$0Y00\ p = 0Y\ p \qquad 52.$$
$$Y0Y0\ p = Y0\ p \qquad 51.$$
$$000Y\ p = 0Y\ p \qquad 36.$$
$$0Y0Y\ p = 0Y\ p \qquad 521.$$
$$Y0YY\ p = YY\ p \qquad 511.$$

And for modalities beginning with 00Y:

$$00YY\ p = 00Y0\ p \qquad By\ 41.$$

625. The only possible distinct modalities with FIVE symbols would be those beginning with 00Y0, thus 00Y00 p and 00Y0Y p, But:

$$00Y00\ p = 00Y\ p \qquad By\ 52.$$
$$00Y0Y\ p = 00Y\ p \qquad By\ 521.$$

626. Hence the distinct modalities of type A are at most ten: 0 p, Y p, 00 p, 0Y p, Y0 p, 000 p, 00Y p, 0Y0 p, Y00 p, 00Y0 p.

63. Type C modalities are the type A modalities, with N p instead of p. Type B modalities and type D modalities are the negations of type A and type C modalities respectively.

64. We have thus 40 distinct proper modalities, ten of each type, plus the two improper modalities p and N p, 42 modalities in all.

For the proof that these modalities may not be reduced further (that no further strict equivalence is provable between them), see Parry l.c.

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