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TWO THEOREMS ABOUT TRUTH FUNCTIONS*

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The formulas of the propositional calculus, or the logic of truth functions, are built up of statement letters 'p', 'q', 'r', ... by applying the notations which express the various truth functions: '$\bar{p}$' for negation, 'pq' for conjunction, 'p ∨ q' for alternation, 'p ⊃ q' for the conditional, 'p ≡ q' for the biconditional. These various notations can be reduced by expressing some of them in terms of the others in familiar ways.

A formula is called *valid* if it comes out true under all assignments of truth values to the letters, and *consistent* if it comes out true under some assignments. One formula is said to *imply* another if the conditional formed from the two formulas in that order is valid; or, equivalently, if every assignment of truth values to letters which makes the first formula come out true makes the second come out true. Two

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formulas are called *equivalent* if they imply each other.

Statement letters and negations of them are called *literals*. A literal or conjunction of two or more literals is called a *fundamental* formula, provided that it contains no letter twice. It will be convenient simply to disregard order in a conjunction, thus treating the fundamental formulas '$p\bar{q}r$', '$pr\bar{q}$', '$r\bar{q}p$', etc. not merely as equivalents but as one and the same formula. From this point of view the fundamental formulas which contain all and only n given statement letters are just 2^n in number. Given any n statment letters, listed in an arbitrary order $\alpha_1, \alpha_2, \ldots, \alpha_n$, the 2^n fundamental formulas containing those letters can be listed exhaustively in a convenient standard order as follows:

(1) $\ulcorner \alpha_1\alpha_2\ldots\alpha_n \urcorner$, $\ulcorner \alpha_1\alpha_2\ldots\alpha_{n-1}\bar{\alpha}_n \urcorner$, $\ulcorner \alpha_1\alpha_2\ldots\alpha_{n-2}\bar{\alpha}_{n-1}\alpha_n \urcorner$,

$\ulcorner \alpha_1\alpha_2\ldots\alpha_{n-2}\bar{\alpha}_{n-1}\bar{\alpha}_n \urcorner$, $\ulcorner \alpha_1\alpha_2\ldots\alpha_{n-3}\bar{\alpha}_{n-2}\alpha_{n-1}\alpha_n \urcorner, \ldots, \ulcorner \bar{\alpha}_1\bar{\alpha}_2\ldots\bar{\alpha}_n \urcorner$.

A fundamental formula ϕ will be said to *subsume* a fundamental formula ψ (of same or less length) if ψ is identical (disregarding permutations, as usual) with part or all of ϕ; hence if all the literals which are conjoined to form ψ are among the literals which are conjoined to form ϕ. Clearly if ϕ subsumes ψ then ϕ implies ψ.

Fundamental formulas and alternations of distinct fundamental formulas (distinct in the above sense which ignores permutations in conjunctions) are called *normal* formulas. There is a familiar routine for transforming any consistent formula into an equivalent which is normal[1]. The fundamental

[1] See e.g. my *Methods of Logic* (New York, 1950), pp.53-58.

formulas whereof a normal formula is an alternation will be called its *clauses*. (A normal formula which is not an alternation, but rather simply a fundamental formula, counts as its own clause.) It is not required that all the clauses of a normal formula contain the same letters, nor that they all be of the same length.

Order in an alternation, as in a conjunction, will be disregarded. Thus normal formulas will be treated not merely as equivalents but as one and the same formula when they have the same clauses.

There is a quick *implication criterion* for any two normal formulas Φ and Ψ such that Ψ lacks negation signs: viz., Φ implies Ψ if and only if each clause of Φ subsumes a clause of Ψ. That such subsumption is sufficient in order that Φ imply Ψ is seen as follows. Suppose each clause of Φ subsumes, and therefore implies, a clause of Ψ; then, since each clause of Ψ implies Ψ, each clause of Φ implies Ψ; and accordingly Φ implies Ψ. That the subsumption condition is also necessary is seen as follows. Suppose some clause $\ulcorner \alpha_1\alpha_2\ldots\alpha_m\bar{\beta}_1\bar{\beta}_2\ldots\bar{\beta}_n \urcorner$ $(m \geq 0,\ n \geq 0)$ of Φ subsumes no clause of Ψ. This is the same as supposing (since Ψ lacks negation signs) that every clause ψ_i of Ψ contains a letter γ_i other than $\alpha_1,\ldots,\alpha_m$. If we assign truth to $\alpha_1,\ldots,\alpha_m$ and falsity to all other letters, then $\ulcorner \alpha_1\alpha_2\ldots\alpha_m\bar{\beta}_1\bar{\beta}_2\ldots\bar{\beta}_n \urcorner$ comes out true. (Note that the β's repeat no α's, by the definition of "fundamental formula".) Therefore Φ comes out true. On the other hand ψ_i for each i comes out false under the described assignment of truth values, on account of γ_i (for remember that Ψ lacks negations signs);

so Ψ comes out false. Therefore Φ does not imply Ψ.

By the *length* of a normal formula let us understand simply the number of occurrences of letters and alternation signs. A formula can have several shortest normal equivalents, all equally short. For example, the normal formulas '$p\bar{q} \vee \bar{p}q \vee pr$' and '$p\bar{q} \vee \bar{p}q \vee qr$' are equivalent and equally short, and they have no shorter normal equivalent.

An obvious expedient for shortening normal formulas is that of simply deleting clauses which subsume other clauses (thus exploiting the familiar equivalence of '$pq \vee p$' to 'p'). Now it can be proved that this expedient always eventuates in a *shortest* normal equivalent if the normal formula with which we begin lacks negation. Such is the content of the following theorem, which is one of the two from which this paper gets its title.

Theorem 1. *If a formula is normal and lacks negation and none of its clauses subsumes any other of its clauses, then it has no shorter normal equivalent.*

Proof. Suppose (i) that Ψ is a normal formula lacking negation, (ii) that no clause of Ψ subsumes another clause of Ψ, and (iii) that Φ is normal and equivalent to Ψ; to prove that Φ is no shorter than Ψ. By (i) and (iii) and the above implication criterion, each clause of Φ subsumes a clause of Ψ. Let the thus subsumed clauses of Ψ be $\psi_1, \ldots, \psi_n$. Since to subsume is to imply, each clause of Φ implies one of $\psi_1, \ldots, \psi_n$; hence Φ implies $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$. By (iii), then, Ψ implies $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$. Then, since $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$ lacks negation (by (i)), we can conclude from the implication criterion that every clause of Ψ subsumes a clause of $\ulcorner\psi_1 \vee \ldots \vee \psi_n\urcorner$. By (ii), then, Ψ has no clauses but $\psi_1, \ldots, \psi_n$. Now assign truth to all letters of ψ_i, for some i, and falsity to all other letters. By (i), ψ_i lacks negation and hence comes out true. Therefore Ψ comes out true (since truth of ψ_i assures truth of $\ulcorner\psi_1 \vee \ldots \vee \psi_n\urcorner$). Therefore, by (iii), Φ comes out true. Hence some clause ϕ of Φ comes out true. On the other hand each of $\psi_1, \ldots, \psi_n$ other than ψ_i contains a letter other than those of ψ_i, by (ii), and hence comes out false under the given assignment. Therefore ϕ, which comes out true, subsumes none of $\psi_1, \ldots, \psi_n$ other than ψ_i. So, since every clause of Φ subsumes one or another of $\psi_1, \ldots, \psi_n$, we must conclude that ϕ subsumes ψ_i and none of the others. Applying this reasoning to each choice of i, we see that each of $\psi_1, \ldots, \psi_n$ is subsumed by a clause of Φ which subsumes no others of $\psi_1, \ldots, \psi_n$. Therefore Φ is no shorter than $\ulcorner\psi_1 \vee \ldots \vee \psi_n\urcorner$, q.e.d.

Preparatory to the other theorem which it is the business of this paper to prove, viz. Theorem 2 below, let us look back to the 2^n fundamental formulas listed in (1). The alternation of the first i of those formulas will be called [i]. Clearly $[2^n]$ is valid, and hence has '$p \vee \bar{p}$' as shortest normal equivalent. On the other hand

Theorem 2. *If* $m < 2^n$ *then* [m] *has as a shortest normal equivalent a formula which lacks negation.*

Proof. For each h up to n, the first 2^{n-h} formulas of (1) exhaust the ways of distributing negation signs over $\alpha_{h+1}, \ldots, \alpha_n$ while keeping $\alpha_1, \ldots, \alpha_h$ affirmative. Clearly, therefore,

(2) $[2^{n-h}]$ is equivalent to $\ulcorner\alpha_1 \ldots \alpha_h\urcorner$

After the 2^{n-h}th formula, the series (1) repeats as from the beginning but with α_h negated. Thus, where $1 \leqslant i \leqslant 2^{n-h}$

(3) $[2^{n-h} + i]$ is $[2^{n-h}]$ in alternation with $[i]$ with α_h negated.

Now let $h_1, \ldots, h_k$ be, in ascending order, the integers such that

(4) $$m = 2^{n-h_1} + 2^{n-h_2} + \ldots + 2^{n-h_k}$$

(They are all positive, since $m < 2^n$; and they are distinct. To find them, write m in binary notation and count the places to the right of each occurrence of '1'. Each of $h_1, \ldots, h_k$ is n minus one of those counts). By (4) and (3), $[m]$ is $[2^{n-h_1}]$ in alternation with $[2^{n-h_2} + \ldots + 2^{n-h_k}]$ with α_{h_1} negated. But, by (3) again, $[2^{n-h_2} + \ldots + 2^{n-h_k}]$ in turn is $[2^{n-h_2}]$ in alternation with $[2^{n-h_3} + \ldots + 2^{n-h_k}]$ with α_{h_2} negated; so $[m]$ is the alternation of $[2^{n-h_1}]$, $[2^{n-h_2}]$ with α_{h_1} negated, and $[2^{n-h_3} + \ldots + 2^{n-h_k}]$ with α_{h_1} and α_{h_2} negated. Continuing thus, we finally find that $[m]$ is the alternation of $[2^{n-h_1}]$, $[2^{n-h_2}]$ with α_{h_1} negated, $[2^{n-h_3}]$ with α_{h_1} and α_{h_2} negated, ..., and $[2^{n-h}\ ]$ with $\alpha_{h_1}, \ldots, \alpha_{h_{k-1}}$ negated. But, by (2), $[2^{n-h_1}]$ is equivalent to $\ulcorner \alpha_1 \ldots \alpha_{h_1} \urcorner$. Also, by (2), $[2^{n-h_2}]$ is equivalent to $\ulcorner \alpha_1 \ldots \alpha_{h_2} \urcorner$, and hence, since substitution for letters prserves equivalence, $[2^{n-h_2}]$ with α_{h_1} negated is equivalent to $\ulcorner \alpha_1 \ldots \alpha_{h_1 - 1} \bar{\alpha}_{h_1} \alpha_{h_1 + 1} \ldots \alpha_{h_2} \urcorner$. Continuing thus, we find $[m]$ equivalent to

$$(5)\qquad \ulcorner a_1 \ldots a_{h_1} \vee a_1 \ldots a_{h_1 - 1} \bar{a}_{h_1} a_{h_1 + 1} \ldots a_{h_2} \vee a_1 \ldots a_{h_1 - 1} \bar{a}_{h_1} a_{h_1 + 1} \ldots a_{h_2 - 1} \bar{a}_{h_2} a_{h_2 + 1} \ldots a_{h_3} \vee \ldots \vee a_1 \ldots a_{h_1 - 1} \bar{a}_{h_1} a_{h_1 + 1} \ldots a_{h_2 - 1} \bar{a}_{h_2} a_{h_2 + 1} \ldots a_{h_{k-1} - 1} \bar{a}_{h_{k-1}} a_{h_{k-1} + 1} \ldots a_{h_k} \urcorner .$$

Now the last two of the k clauses of (5) are related in the manner of 'pq' and 'p$\bar{\text{q}}$r', with $a_{h_{k-1}}$ in the rôle of 'q'; and 'pq $\vee$ p$\bar{\text{q}}$r' is equivalent by truth tables to 'pq $\vee$ pr'. Hence the occurrence of $\ulcorner \bar{a}_{h_{k-1}} \urcorner$ in (5) can be dropped. Again the last three clauses of the thus amended (5) are related in the manner of 'pq', 'p$\bar{\text{q}}$r', and 'p$\bar{\text{q}}$s', with $a_{h_{k-2}}$ in the rôle of 'q'; and 'pq $\vee$ p$\bar{\text{q}}$r $\vee$ p$\bar{\text{q}}$s' is equivalent to 'pq $\vee$ pr $\vee$ ps'. Hence the two occurrences of $\ulcorner \bar{a}_{h_{k-2}} \urcorner$ can be dropped. Continuing thus, we delete all negative literals from (5) and are left with

$$\ulcorner a_1 \ldots a_{h_1} \vee a_1 \ldots a_{h_1 - 1} a_{h_1 + 1} \ldots a_{h_2} \vee a_1 \ldots a_{h_1 - 1} a_{h_1 + 1} \ldots a_{h_2 - 1} a_{h_2 + 1} \ldots a_{h_3} \vee \ldots \vee a_1 \ldots a_{h_1 - 1} a_{h_1 + 1} \ldots a_{h_2 - 1} a_{h_2 + 1} \ldots a_{h_{k-1} - 1} a_{h_{k-1} + 1} \ldots a_{h_k} \urcorner .$$

But thus lacks negation. Moreover, none of its clauses subsumes any other of its clauses; so, by Theorem 1, there is no shorter normal equivalent.

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