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METRICS ON THE TORUS WITHOUT CONJUGATE POINTS*

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1. Most theorems on Riemann spaces of a predominantly geometric nature prove to be independent of the Riemannian character of the metric in the sense that after some rewording of the hypotheses they are seen to become special cases of theorems on Finsler spaces. *Of special significance for the understanding of both types of spaces are therefore those theorems where a simple extension form the Riemannian to the Finsler case is impossible.* The outstanding example in this direction is Beltrami's Theorem which characterizes the Riemann spaces with the straight lines as geodesics as the spaces with constant curvature, whereas the characterization of the Finsler spaces with this property is one of Hilbert's problems.

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M.Morse, G.Hedlund and E.Hopf[1] showed that *a torus with a Riemannian metric without conjugate points has curvature* 0, so that the metric is euclidean. The present note shows that this theorem is of the *non-extendable* type: the geodesics of a Finsler metric on the torus without conjugate points need not be the straight lines, and for any metric without conjugate points there are always many essentially different metrics with the same geodesics. In fact, there is so much arbitrariness in the choice of the metric when the geodesics are prescribed that the problem to determine all of them becomes uninteresting.

However, it turns out to be reasonable to ask *which curve systems on the torus can occur as geodesics in a metric without conjugate points.* The answer to this question --in therms of the plane as the universal covering space of the torus-- is the main result of the present paper.

2. Let the (x,y)-plane P be the universal covering space of a torus with the translations

$$(1) \qquad T(m,n):\quad x' = x + m, \quad y' = y + n, \quad m,n \text{ integers}$$

as covering transformations. A system of geodesics on the torus without conjugate points, yields a system S of curves in P with the following properties:

I. *Each curve is* S *is a topological image* $p(t) = (x(t), y(t)),\ -\infty < t < \infty$, *of the real t-axis, such that* $x^2(t) + y^2(t) \to \infty$ *for* $t \to \infty$.

[1] This Theorem was proved by M.Morse and G.Hedlund in [3] under the additional hypothesis that there are no focal points. In this general form it is due to E.Hopf [4]. Numbers in brackets refer to the references cited at the end of the paper.

II. *Any two points of* P *lie on exactly one curve of* S.

III. *The system* S *goes into itself under the translations* T(m, n).

IV. *If a curve* L *of* S *contains* q *and* qT(m, n) *then it contains all points* $qT(\nu m, \nu n)$, $\nu = \pm 1, \pm 2, \ldots$

V. *The system* S *satisfies the parallel axiom: for a given curve* L *in* S *and a given point* p *not on* L *there is exactly one curve in* S *through* p *which does not intersect* L.

The main result is

THEOREM 1. *If a metric in the plane* P *has a system* S *with properties* I *and* II *as geodesics and is invariant under the translations* T(m, n) *then* S *satisfies* III, IV *and* V.

Conversely, if in P *a system* S *of curves with properties* I *to* V *is given then a metric invariant under the* T(m, n) *exists for which the curves in* S *are the geodesics.*

First the necessity of the conditions IV and V will be proved using freely the results and concepts of [1]. Each geodesic is congruent to a euclidean straight line [1, p.79 Theorem 1]. Denote by U the unit square $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. For any T = T(m,n), where not both m and n vanish, there is a point a in U for which

$$a\, aT = \min_{p \in U} p\, pT$$

where ab denotes the distance of a and b in the given metric. If q is any point in P then a $T' = T(m', n')$ exists such that $q_0 = qT' \in U$. Then

$$a\, aT \leqslant q_0\, q_0 T = q_0 T'\, q_0 T\, T' = q_0 T' q_0\, T'T = q\, q\, T$$

hence

$$a\,a\,T = \min_{p\in P} p\,p\,T$$

(1) *For any* $T(m,n) \neq 1$ *the points* $qT(\nu m,\nu n) = qT^{\nu}$ *lie on a geodesic, if and only if* $q\,q\,T = \min_{p\in P} p\,p\,T$, see [2, Theorem (6.1)].

It follows that the points aT^{ν} lie on a geodesic L which, because of II, goes into itself under all T^{ν}, hence the points qT^{ν} lie on L and $qqT = aaT$. Moreover for any arbitrary $T' = T(m',n')$ and $a' = aT$

$$a'a'T = aT'a\,T'T = a\,T'a\,T\,T' = a\,a\,T$$

so that the points $a'T^{\nu}$ lie also on a geodesic L' which is of course the line LT'.

Since L and L' cannot intersect more than once they are either identical or do not intersect at all. If q is a point of the plane which does not lie on any $LT(m',n')$ then m' and n' can be chosen such that q lies between L and $LT(m',n')$. If $S(a,b)$ denotes the segment of the geodesic through a and b with endpoints a and b, then $\bigcup_{\nu=-\infty}^{\infty} S(qT^{\nu}, qT^{\nu+1})$ is by [1, p.119 (c)] a curve which bounds a convex region both together with L and together with L'. Hence the points qT^{ν} lie on a geodesic and it follows from (1) that $qqT = aaT$. This proves IV.

The implications of this result for the torus are sufficiently interesting to state them explicitly.

THEOREM 2. *In a metrization of a torus without conjugate* *points all geodesic one-gons are closed geodesics. There is exactly one closed geodesic in a given free homotopy class through a given point and all geodesics in the class have the same length.*

3. It is considerably more difficult to prove V. For brevity call rational a line which contains two points q and $qT(m,n)$, $(m,n) \neq (0,0)$, and hence all points $qT(\nu m, \nu n)$. It is easy to establish the parallel axiom for the rational lines by showing: if L contains the points pT^{ν}, $p \neq 1$ and q does not lie on L then the line L' containing the points qT^{ν} is the only line through q which does not intersect L.

If this were not so, the assymptote H (see [1, Chapter III, 4]) through q to one of the orientations, say L^{+}, of L would be different from L'. Let the limit circle Λ with L^{+} as central ray (l.c.) intersect L at $\bar{p}$. Then ΛT^{-1} is the limit circle with L^{+} as central ray through qT^{-1} and $\bar{p}T^{-1}$ ([1, p.200 (d)]). It was just shown that $\bar{p}\bar{p}T^{-1} = qqT^{-1}$. On the other hand H intersects ΛT^{-1} in a point f which is the unique foot of q on ΛT^{-1} ([1, p.102, Theorem 5]) and $qf = \bar{p}\bar{p}T^{-1}$. But then $qqT^{-1} = qf$ contradicts the uniqueness of the foot.

By means of a topological transformation of the torus (or the plane) on itself we can reach that the euclidean lines $x = \text{const.}$ and $y = \text{const.}$ represent geodesics. Every other geodesic has then because of II and the validity of the parallel axiom for the lines $x = \text{const.}$ and $y = \text{const.}$ a representation of the form

$$y = f(x), \quad -\infty < x < \infty$$

with $f(x)$ either strictly increasing or strictly decreasing and $|f(x)| \to \infty$ for $x \to \infty$. It will be shown that for every such line the "*slope*"

$$(2) \qquad \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

exists and is different from 0 and ∞.

Consider first the case where L is a rational line through the origin z and the point $(m,n) = zT(m,n)$, $m \neq 0$, $n \neq 0$. Then $f(\nu m) = \nu f(m)$ and if $\nu m \leqslant x \leqslant (\nu+1)m$ then because $f(x)$ is monotone

$$|f(x) - f(\nu m)| < |f[(\nu+1)m] - f(\nu m)| = f(m)$$

hence with $\theta_1 < 1$

$$\frac{f(m)}{m} = \lim_{\nu \to \pm\infty} \frac{f(\nu m)}{\nu m} = \lim_{\nu \to \pm\infty} \frac{f(\nu m) + \theta_1 f(m)}{\nu m + \theta_2 m} = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

A rational line L_κ obtained from L by the translation $T(0,\kappa)$ has the equation $y = f(x) + \kappa$ so that L_κ has the same slope as L. If L' is any line parallel to L, with the equation $y = f'(x)$, then L' lies for a suitable κ between L and L_κ so that L' also has this slope.

Now let $y = f(x)$ represent an arbitrary line. If it did not have a slope, then m and n different from 0 would exist such that

$$\liminf \frac{f(x)}{x} < \frac{m}{n} < \limsup \frac{f(x)}{x}$$

The rational line containing the points $(0, f(0))T(\nu m, \nu n)$ would then intersect L more than once, without coinciding with L.

The definition (2) of the slope implies

(3) *Lines with different slopes intersect.*

(4) *There is a line with a given slipe* $\mu \neq 0, \infty$ *through a given point* $p = (x_0, y_0)$

If $\mu = \frac{n}{m}$ then the line containing the points $pT(\nu m, \nu n)$ satisfies (4). If μ is irrational[2], choose a sequence of rational numbers ρ_ν, $\nu = 1, 2, \ldots$, which increase and tend to μ and a rational number $\rho_0 > \mu$. Denote by L_i a rational line through p with slope ρ_i. Then L_{i+1} lies between L_i and L_0, hence L_i tends to a limit line L through p. If $y = f_i(x)$ and $y = f(x)$ represent L_i and L respectively, then $f_0(x) > f(x) > f_i(x)$ for $x > x_0$ and $i \geqslant 1$; hence the slope of L is at least μ, and at most ρ_0. Since $\rho_0 > \mu$ was arbitrary L has slope μ.

Statements (3) and (4) show that the parallel axiom follows from:

(5) *There is at most one line with a given slope through a given point* p.

For rational μ this follows from the fact that the parallel axiom holds for rational lines. For if $\mu = n/m$ and if the line L containing the points $pT(\nu m, \nu n)$ has the equation $y = f(x)$ then any other line L′ through $p = (x_0, y_0)$ has an equation $y = f'(x)$ with $f'(x) > f(x)$ for $x > x_0$ say. Because L

[2] The following argument applies also to $\nu = 0$ or ∞.

is parallel to $LT(0,1)$, which has the equation $y = f(x)+1$, the line L' must intersect $LT(0,1)$ for some $x' > x_0$.

Then for a suitable $\nu > 0$ the point $pT(\nu m, \nu n)\ T(0,1)$ lies on $LT(0,1)$ and between L' and L. The line L'' through p and $pT(\nu m, \nu n+1)$ has slope $\frac{\nu n+1}{\nu m} > \mu$ and the slope of L' cannot be smaller than the slope of L''.

Let μ now be irrational and assume for an indirect proof that there are two different lines L, K through p with slope μ. We may assume that p is the origin and that the two lines have equations of the form

$$L : y = f(x), \quad K : y = g(x) \quad \text{with} \quad g(x) > f(x) \quad \text{for} \quad x > 0.$$

Then for integral $n > 0$

$$(6) \qquad 0 < g(n) - f(n) < 1$$

because otherwise the segment S_n connecting $(n, f(n))$ to $(n, g(n))$ would contain a point of the form (n, m) with integral m, and the rational line L through p and (n, m) would lie between L and K. By the first part of this proof the slope L would be smaller, and the slope of K would be greater, than n/m. Because the distance is invariant under the $T(m,n)$ it follows from [1, p.103, Theorem 6] that a $\delta > 0$ exists such that

$$g(n) - f(n) > \delta \quad \text{for} \quad n \geqslant 1 .$$

For a given integral $\kappa \geqslant 3$ determine the integer m_κ by

$$(7) \qquad m_\kappa \delta \geqslant \kappa + 1 > (m_\kappa - 1) \delta \ .$$

Then $\bigcup_{i=1}^{m_\kappa} S_i$ contains $\kappa + 1$ points which represent the same point on the torus, or two ordinates differ by integers. We distinguish two cases

a) There are for some κ four points p_i no three of which lie on the same geodesic. A familiar argument from elliptic functions shows that the convex closure of these 4 points in terms of S would then contain a "period parallelogram" Q whose sides are formed by segments of curves in S. Since the domain bounded by $y = f(x)$ and $y = g(x)$ for $x > 0$ is convex Q would lie in this domain, on the other hand Q would contain a point equivalent to p that is of the form (m, n) which was already seen to be impossible.

b) At least κ of the $\kappa + 1$ points lie on a geodesic H^κ. Then H^κ is rational and has a rational slope ρ_κ. Since no two of the κ points lie on the same S_i the abscisas n_i^κ of the κ points are different. Let $n_i^\kappa < n_{i+1}^\kappa$. Then $n_\kappa^\kappa - n_1^\kappa > \kappa - 1$ hence because of (7) $n_1^\kappa / n_\kappa^\kappa \leqslant 1 - (\kappa - 1)/m_\kappa \leqslant 1 - \delta/4$. Since

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{f(x_1)}{x_1} + \frac{(x_2/x_1)\ [f(x_1)/x_1 - f(x_2)/x_2]}{1 - x_2/x_1}$$

it follows that for $x_1 \to \infty$ and $0 < x_2/x_1 \leqslant \theta < 1$

$$\lim_{x_1 \to \infty} \frac{f(x_1)}{x_1} = \lim_{x_1 \to \infty} \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

where x_2 may, or may not, be bounded.

Hence it follows in the present case from $n_1^\kappa/n_\kappa^\kappa \leq 1 - \delta/4$ that

$$\lim_{\kappa\to\infty} \frac{f(n_\kappa^\kappa) - f(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} = \lim_{\kappa\to\infty} \frac{g(n_\kappa^\kappa) - g(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} = \mu$$

and therefore from (6) that also

$$\lim_{\kappa\to\infty} \frac{f(n_\kappa^\kappa) - g(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} = \lim_{\kappa\to\infty} \frac{g(n_\kappa^\kappa) - f(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} = \mu \quad .$$

On the other hand

$$(\kappa-1) \min_j (n_j^\kappa - n_{j-1}^\kappa) \leq n_\kappa - n_1 \leq m_\kappa - 1$$

and by (7)

$$\min_j (n_j^\kappa - n_{j-1}^\kappa) \leq (m_\kappa - 1)/(\kappa - 1) < 2/\delta \quad .$$

Therefore the denominator of the slope ρ_κ of H^κ (if reduced) cannot surpass $2/\delta$. Since μ is irrational there is an $\epsilon > 0$, independent of κ, such that $|\rho_\kappa - \mu| > \epsilon$. But if $y = h(x)$ represents H^κ, since the κ points lie between L and K,

$$\frac{g(n_\kappa^\kappa) - f(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} > \frac{h(n_\kappa^\kappa) - h(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa} = \rho_\kappa \quad \frac{f(n_\kappa^\kappa) - g(n_1^\kappa)}{n_\kappa^\kappa - n_1^\kappa}$$

which in conjunction with (8) contradicts $|\rho_\kappa - \mu| > \epsilon$. This completes the proof of V.

4. We turn now to the *second part of Theorem 1.*[3] Conditions I and II alone guarantee that the curves in S have all the usual continuity and intersection properties, compare [1, Chapter III, § 3]. We may therefore again assume that the curves x = const. and y = const. represent curves of S, because of the parallel axiom all other curves again have representations of the form $y = f(x)$ with $f(x)$ strictly monotone and $|f(x)| \to \infty$ for $|x| \to \infty$.

Take any pair m, n with $m > 0$ and denote by L_p the curve in S containing the points $pT(\nu m, \nu n)$. For any p, q the curves L_p and L_q are either parallel or identical. Let $y = f_p(x)$ represent L_p (since $n = 0$ is admitted, $f_p(x)$ may be constant). That $T(m,n)$ carries L_p into itself implies $f_p(x+m) = f_p(x) + n$ for all x, therefore $f_p(x) - f_q(x)$ is periodic with period m and the area

$$d_{m,n}(p,q) = \int_{x_0}^{x_0+m} |f_p(x) - f_q(x)|\, dx$$

of the "parallelogram" Q bounded by L_p, L_q and $x = x_0$, $x = x_0 + m$ is independent of x_0. An arbitrary translation $T' = T(m', n')$ carries Q into a parallelogram which has the same relation to pT' and qT' as Q has to p and q. But T' leaves area invariant, hence

$$(9) \qquad d_{m,n}(p,q) = d_{m,n}(pT', qT')$$

[3] The idea of the following proof is taken from a similar proof in [1, Chapter III, § 3]. There the reader will find details which he might miss in the present discussion.

clearly $d_{m,n}(p,q) = d_{m,n}(q,p)$ and

(10) $$d_{m,n}(p,q) = 0 \quad \text{if and only if} \quad L_p = L_q$$

The arbitrariness of x_0 yields

(11) $d_{m,n}(p,q) + d_{m,n}(q,r) = d_{m,n}(p,r)$ *if and only if the line* L_q *lies in the closed strip bounded by* L_p *and* L_r .

(12) $d_{m,n}(p,q) + d_{m,n}(q,r) > d_{m,n}(p,r)$ *if* L_q *does not lie in this strip.*

Let δ be the difference of the ordinates of p and q and determine the integer κ by $\kappa - 1 \leqslant |\delta| < \kappa$. Then $T(0,\pm\kappa)$ carries L_p into a line L_r for which L_q (if different from L_p) lies between L_p and L_r. Then

(13) $$d_{m,n}(p,q) < d_{m,n}(p,r) = \kappa d_{m,n}(p,pT(0,1)) \leqslant (|\delta|+1)\lambda_{m,n}$$

where $\lambda_{m,n}$ depends only on m and n. A distance which satisfies our requirements will be

(14) $$pq = \Sigma' \, d_{m,n}(p,q) \, \lambda_{m,n}^{-1} \, 2^{-m-|n|} \quad ;$$

where the prime indicates that the summation is extended over all pairs m,n with $m > 0$ and all n, but such that $n/m \neq n'/m'$ for different pairs m,n and m',n'.

If p and q are given and have ordinate difference δ

then by (13) $d_m(p,q)\lambda^{-1}_{m,n} < |\delta| + 1$ for all m,n, so that pq is always finite. (9) shows that pq is invariant under all $T(m',n')$, and (11) and (12) imply that pq satisfies the triangle inequality. $pp = 0$ by (20), and $pq = qp > 0$ for $p \neq q$ follows from $d_{m,n}(p,q)$ $d_{m,n}(q,p)$ and from (10) because for suitable m,n the lines L_p and L_q (in the previous notation) will be different.

Thus pq *satisfies the axioms for a metric space.* To see that the curves in S are the geodesics it must be shown: that for three different points p,q,r

(15) $pq + qr = pr$ *if* q *lies on the segment* σ *of the curve of* S *through* p *and* r.

(16) $pq + qr > pr$ *if* q *does not lie on* σ.

If q lies on σ, then for any m,n the line L_q will either contain p and r or L_q lies between L_p and L_r. Hence it follows from (10) and (11) that (15) holds.

If finally q does not lie on σ, let L be the curve in S through two arbitrary interior points q' and q'' of the segments (in the sense of S) from q to p and r respectively. Then L separates σ from q. If L contains for suitable m,n with $m > 0$ the points $q'T(\nu m, \nu n)$, then (16) follows from (12). If L does not have this property (that is either a line $x = \text{const}$ or not rational) then the parallel axiom implies the existence of $m > 0$ and n such that the line L' containing the points $q'T(\nu m, \nu n)$ is so close to L that it also separates σ, and therefore p and r from q. Then (16) follows again from (12).

That the distance pq is equivalent to the euclidean distance is easily derived from either the analytic definition of pq or the geometric properties of S. The finite compactness of pq follows from its invariance under $T(m',n')$.

5. A few examples will conclude this paper. The construction of the distance pq in the preceding section happens to yield a Minkowski metric if the curves in S are the euclidean lines $ax + by + c = 0$. This is however, accidental because other functions $d_{m,n}(p,q)$ than the area could have been used. For instance, if $p_i = (x_i, y_i)$ then

$$p_1p_2 = [(x_1-x_2)^2 + (y_1-y_2)^2]^{\frac{1}{2}} + |7y_1+\sin 2\pi y_1 - 7y_2 - \sin 2\pi y_2|$$

yields a metric for which the euclidean lines are the geodesics, because $7y + \sin 2\pi y$ increases monotonically. Moreover, this metric is invariant under the $T(m,n)$. Instead $7y + \sin 2\pi y$ many other functions could have been used, a similarly formed term in the x_i could have been added, the euclidean distance occurring in the definition of p_1p_2 could have been replaced by an arbitrary Minkowski distance. The euclidean distance can also be modified in less obvious ways. This elucidates a point made in the introduction: *there is so much choice that the problem to determine all metrics which belong to a given system of curves becomes uninteresting.*

One might ask whether conditions I, II and III do not imply either IV or V. The example 1) in [1, p. 105] shows that this is not the case.

Finally we give an example which confirms the assertion of the introduction that *the curves of a system* S *satisfying* *conditions* I *to* IV *need not be the straight lines.* This means the following: in general, no topological mapping of P on another plane P′ exists under which the system S goes into the system of the euclidean line in P′. An obviously necessary (and actually also sufficient) condition for such a mapping to exist is that the *Theorem of Desargues* holds for the curves in S. Systems which satisfy I, II, and V but not Desargues' Theorem are well known, but a system satisfying III and IV as well has not come to the author's attention.

To construct such a system S, we first define certain functions $f_t(x)$ in the interval $0 \leq x \leq 1$. Put

$$f_1(x) = x, \quad \text{and for integral } n > 1$$

$$f_n(x) = \begin{cases} a_n x & \text{for } 0 \leq x \leq \frac{1}{2} \\ b_n(x-\frac{1}{2}) + \frac{1}{2} a_n & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

where

$$a_n = 2n - 1 - 2c_n, \quad b_n = 1 + 2c_n, \quad c_n = \sum_{\nu=1}^{n} 10^{-\nu}$$

then $f_n(1) = n$ and $f'_{n+1}(x) - f'_n(x) > 0$ for $x \neq \frac{1}{2}$ so that $f_{n+1}(x) - f_n(x)$ increases. Morover put

$$f_t(x) = (n+1-t)\, f_n(x) + (t-n)\, f_{n+1}(x) \quad \text{if } n < t < n+1 .$$

Then it is easily seen that $f_{t_2}(x) - f_{t_1}(x)$ increases for $1 < t_1 < t_2$, and that $f_t(1) = t$.

Now define $g_t^b(x)$ for all x, all $t < 1$ and all b by

$$g_t^b(x) = f_t(x-m) + mt + b \quad \text{for} \quad m \leq x \leq m + 1$$

Since

$$(g_{t_2}^{b_2}(x) - g_{t_1}^{b_1}(x))' = f'_{t_2}(x-m) - f'_{t_1}(x-m)$$

the difference $g_{t_2}^{b_2}(x) - g_{t_1}^{b_1}(x)$ increases for $t_2 > t_1$. Hence the two curves $y = g_{t_1}^{b_1}(x)$ and $y = g_{t_2}^{b_2}(x)$ intersect at most once. Morover $y = g_t^b(x)$ has slope t in the sense of (2).

The system S is defined as consisting of all curves $y = g_t^b(x)$, all lines $y = mx + b$ with $m < 1$ and the lines $x = \text{const}$. Because $(g_t^b(x))' \geq 1$ each line in S intersects each $y = g_t^b(x)$ exactly once.

Through every point of the plane there is exactly one line with a given slope (2). Hence the parallel axiom holds. It is easily verified that two distinct points of the plane lie on exactly (and not only at most) one curve in S.

To show that the system S has property IV it suffices to prove the following: If $g_t^b(x') = y'$ and $g_t^b(x'+m) = y' + n$, where m and n are integers,, then $g_t^b(x'+\nu m) = y' + \nu n$. Determine the integer k by $k \leq x' < k + 1$. Then putting $f_t(x'-k) + b = W$,

$$y' = g_t^b(x') = W + kt, \quad g_t^b(x'+m) = W + (k+m)t = W + kt + n$$

hence $t = n/m$. Moreover

$$g_t^b(x'+\nu m) = W + (k+\nu m)t = y' + \nu mt = y' + \nu n.$$

That the Theorem of Desargues does not hold is seen as in the other well known examples of non-Desarguesian systems:

Two triangles which are in the Desarguesian relation in the ordinary sense are placed so that all but one of the lines entering the theorem have slope less than 1 and hence are also curves of S. The last line L has slope > 1. The curve $y = g_t^b(x)$ through two of the three points of the Desarguesian configuration on L will in general not contain the third point.

The system S goes into itself not only under the T(m,n) but under all translations $x' = x+m$, $y' = y+b$ where m is an integer, but b is an arbitrary real number. It is easily verified that the metric constructed in section 4 is invariant under all these translations. Thus:

THEOREM 3. *There are metrizations of the torus without conjugate points which have a one-parameter group of motions and for which the geodesics are not the straight lines.*

The system S is so constructed that it also exhibits another phenomenon, which is surprising at first sight. The curves $y = g_n^o(x)$ of S approach for $n \to \infty$ the curve $x = 0$, but there is an $\epsilon > 0$ independent of n and a circular disk with radius ϵ (whose center depends on n) such that $g_n^o(x)$ does not enter this disk.

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